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1

Coordinate System

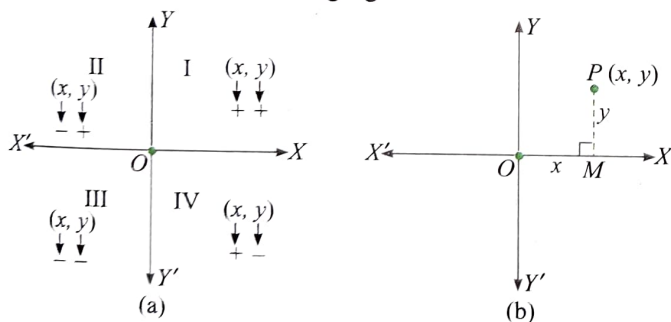
INTRODUCTION

The coordination of algebra and geometry is coordinate geometry. Historically, coordinates were introduced to help geometry. And so well did they do this job that the very identity of geometry was changed. The word 'geometry' today generally means coordinate geometry.

In coordinate geometry, all the properties of geometrical figures are studied with the help of algebraic equations. The object of coordinate geometry is to use some known facts about a curve in order to obtain its equation and then deduce other properties of the curve from the equation so obtained. For this purpose, we require a coordinate system. There are various types of coordinate systems present in two dimensions, e.g., rectangular, oblique, polar, triangular system, etc. Here, we will only discuss rectangular coordinate system in detail.

RECTANGULAR (OR CARTESIAN) COORDINATE SYSTEM

Let $X'OX$ and $Y'OY$ be two fixed straight lines at right angle. $X'OX$ is called x -axis or horizontal axis, $Y'OY$ is called y -axis or vertical axis and the point of intersection of both the axes i.e., O is called the origin. Two axes divide the plane into four regions, called quadrants. Name of the quadrant and sign of coordinates (x, y) are shown in the following figures.



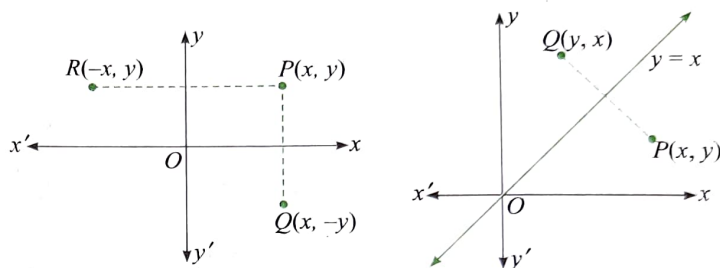
Consider point $P(x, y)$ in first quadrant. From any point P , a line is drawn parallel to OY . The directed line $OM = x$ and $MP = y$. Here, OM is abscissa and MP is ordinate of the point P . The abscissa OM and the ordinate MP together written as (x, y) are called coordinates of point P . Here, (x, y) is an **ordered pair** of real numbers x and y , which determine the position of point P .

Since x and y can be negative or positive depending upon the quadrant in which the point $P(x, y)$ lies, its distance from x -axis is $|y|$ and that from y -axis is $|x|$.

Since $X'OX$ is perpendicular to $Y'OY$, this system of representation is called **rectangular (or orthogonal) coordinate system**.

When the axes $X'OX$ and $Y'OY$ are not at right angles, they are said to be **oblique axes**.

When point $P(x, y)$ is reflected in x -axis its image is $Q(x, -y)$ and when it is reflected in y -axis its image is $R(-x, y)$.



The equation of angle bisector of first and third quadrants is $y = x$. The image of point $P(x, y)$ when it is reflected in the line $y = x$ is $Q(y, x)$, i.e., coordinates are interchanged.

Translation of Point

When a point or figure on a coordinate plane is moved by sliding it to the right, left, up or down, the movement is called a translation. Following are the coordinates of new point when point $P(x, y)$ is translated.

Translation of k units to the right $\rightarrow (x + k, y)$

Translation of k units to the left $\rightarrow (x - k, y)$

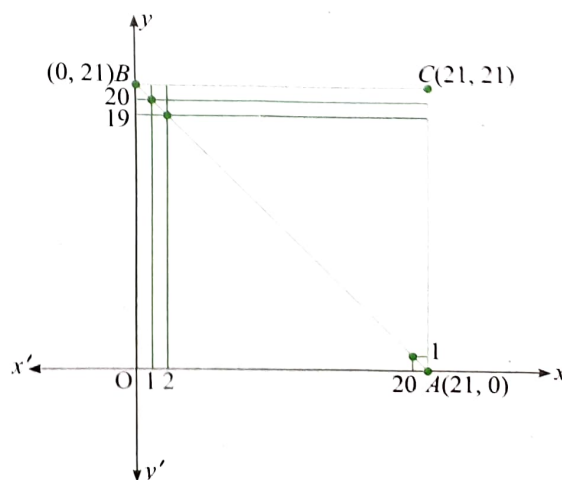
Translation of k units up $\rightarrow (x, y + k)$

Translation of k units down $\rightarrow (x, y - k)$

ILLUSTRATION 1.1

Find the number of integral points (both the coordinates should be integer) exactly in the interior of the triangle with vertices $(0, 0)$ $(0, 21)$ and $(21, 0)$.

Sol.



1.2 Coordinate Geometry

Total number of points within the square $OACB = 20 \times 20 = 400$

There are 20 points on the line AB which are $(1, 20), (2, 19), \dots, (20, 1)$.

$\therefore$ Points within $\triangle OAB$ and $\triangle ABC = 400 - 20 = 380$

By symmetry, points within $\triangle OAB = \frac{380}{2} = 190$

ILLUSTRATION 1.2

The point $(4, 1)$ undergoes the following three transformations successively:

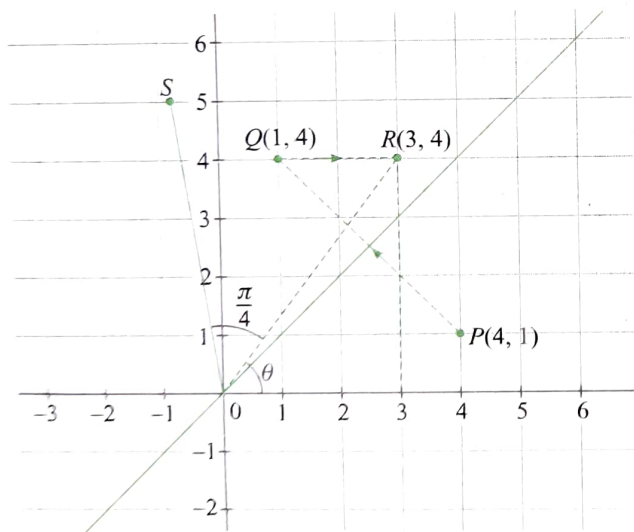
- Reflection about the line $y = x$
- Translation through a distance 2 units along the positive direction of x -axis
- Rotation through an angle of $\pi/4$ about the origin in the counter clockwise direction.

Find the final position of the point.

Sol. Reflection of $P(4, 1)$ about the line $y = x$ is $Q(1, 4)$.

On translation of $Q(1, 4)$ through a distance of 2 units along positive direction of x -axis, the point moves to $R(1 + 2, 4)$ or $R(3, 4)$.

On rotation about origin through an angle of $\pi/4$, the point R takes the position S such that $OR = OS = 5$.



From the figure, $\cos \theta = \frac{3}{5}$ and $\sin \theta = \frac{4}{5}$

From trigonometric concepts, coordinates of point S are:

$$\left(5 \cos \left(\frac{\pi}{4} + \theta \right), 5 \sin \left(\frac{\pi}{4} + \theta \right) \right)$$

$$\begin{aligned} \text{Now, } 5 \cos \left(\frac{\pi}{4} + \theta \right) &= 5 \left(\cos \frac{\pi}{4} \cos \theta - \sin \frac{\pi}{4} \sin \theta \right) \\ &= 5 \left(\frac{1}{\sqrt{2}} \cdot \frac{3}{5} - \frac{1}{\sqrt{2}} \cdot \frac{4}{5} \right) = -\frac{1}{\sqrt{2}} \end{aligned}$$

$$\text{and } 5 \sin \left(\frac{\pi}{4} + \theta \right) = 5 \left(\sin \frac{\pi}{4} \cos \theta + \sin \theta \cos \frac{\pi}{4} \right)$$

$$= 5 \left(\frac{1}{\sqrt{2}} \cdot \frac{3}{5} + \frac{1}{\sqrt{2}} \cdot \frac{4}{5} \right) = \frac{7}{\sqrt{2}}$$

Coordinates of S are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}} \right)$.

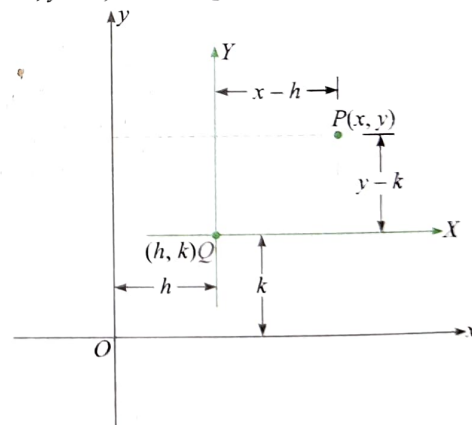
TRANSFORMATION OF AXES

As stated earlier, coordinate systems are essential for studying the equations of curves using the methods of analytic geometry. To use the method of coordinate geometry, the axes are placed at a convenient position with respect to the curve under consideration. If the curve (parabola, ellipse, hyperbola, etc.) is not situated conveniently with respect to the axes, the coordinate system should be changed to place the curve at a convenient and familiar location and orientation. The process of making this change is called transformation of coordinates. There are basically two types of transformations, shifting of origin (translation of axes) and rotation of axes.

Shifting of Origin/Translation of Axes

The shifting (translation) of the coordinate axes is done without rotation so that each axis remains parallel to its original position.

Let point P has coordinates (x, y) in original coordinates system i.e. in xy -coordinate system. If the axes are translated a distance h to the right and a distance k upward, the origin takes new position as $Q(h, k)$. Here coordinates (h, k) are with respect to original coordinate system. Let the new axes define the plane as XY -coordinate system. The coordinate of point P are now $(X, Y) \equiv (x - h, y - k)$ with respect to XY -coordinate system.



Thus $X = x - h$ and $Y = y - k$
or equivalently

$$x = X + h \text{ and } y = Y + k$$

The coordinates $(0, 0)$ of the old origin referred to the new axes are $(-h, -k)$.

If the equation of the curve with respect to original coordinate system is $f(x, y) = 0$, then its equation with respect to new coordinate system will be $f(x + h, y + k) = 0$, i.e. in equation $f(x, y) = 0$, replace x by $x + h$ and y by $y + k$.

ILLUSTRATION 1.3

At what point should the origin be shifted if the coordinates of a point $(4, 5)$ become $(-3, 9)$?

Sol. Let (h, k) be the point to which the origin is shifted. Then,
 $x = 4, y = 5, X = -3, Y = 9$

$$\begin{aligned}\therefore x &= X + h \text{ and } y = Y + k \\ \text{or } 4 &= -3 + h \text{ and } 5 = 9 + k \\ \text{or } h &= 7 \text{ and } k = -4\end{aligned}$$

Hence, the origin must be shifted to $(7, -4)$.

ILLUSTRATION 1.4

If the origin is shifted to the point $(1, -2)$ without the rotation of the axes, what do the following equations become?

- (i) $2x^2 + y^2 - 4x + 4y = 0$
 (ii) $y^2 - 4x + 4y + 8 = 0$

Sol.

- (i) Substituting $x = X + 1$ and $y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$, we get
 $2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0$
 or $2X^2 + Y^2 = 6$
 (ii) Substituting $x = X + 1$ and $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get
 $(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0$
 or $Y^2 = 4X$

ILLUSTRATION 1.5

Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain a term in y and the constant term.

Sol. Let the origin be shifted to (h, k) . Then,
 $x = X + h$ and $y = Y + k$

Substituting $x = X + h$ and $y = Y + k$ in the equation $y^2 + 4y + 8x - 2 = 0$, we get

$$\begin{aligned}(Y+k)^2 + 4(Y+k) + 8(X+h) - 2 &= 0 \\ \text{or } Y^2 + (4+2k)Y + 8X + (k^2 + 4k + 8h - 2) &= 0\end{aligned}$$

For this equation to be free from the term containing Y and the constant term, we must have

$$\begin{aligned}4 + 2k &= 0 \text{ and } k^2 + 4k + 8h - 2 = 0 \\ \text{or } k &= -2 \text{ and } h = \frac{3}{4}\end{aligned}$$

Hence, the origin is shifted at the point $(3/4, -2)$.

ILLUSTRATION 1.6

The equation of a curve referred to the new axes, which are parallel to original axes, and origin as $(4, 5)$ is $X^2 + Y^2 = 36$. Find the equation referred to the original axes.

Sol. With the given notation, we have

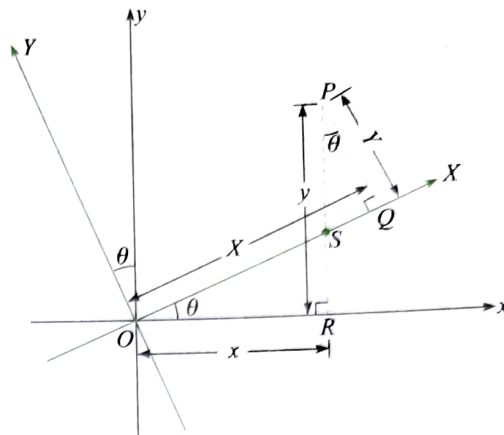
$$\begin{aligned}x &= X + 4, y = Y + 5 \\ \text{or } X &= x - 4, Y = y - 5 \\ \text{Therefore, the required equation is} \\ (x-4)^2 + (y-5)^2 &= 36 \\ \text{or } x^2 + y^2 - 8x - 10y + 5 &= 0\end{aligned}$$

which is the equation referred to the original axes.

Rotation of Axes about Origin

Another type of transformation of axes is rotation of axes. Consider point $P(x, y)$ in original coordinate system i.e., xy -coordinate system. Let these axes be rotated by an angle θ in

anticlockwise direction about origin to form new XY -coordinate system. Let us find the coordinates of point P with respect to new coordinate system.



In the figure,

$$SR = x \tan \theta, OS = x \sec \theta$$

$$\therefore PS = y - x \tan \theta$$

$$\text{and } SQ = x - x \sec \theta$$

Now in triangle PQS ,

$$\sin \theta = \frac{SQ}{PS} = \frac{X - x \sec \theta}{y - x \tan \theta}$$

$$\Rightarrow X = y \sin \theta - x \frac{\sin^2 \theta}{\cos \theta} + \frac{x}{\cos \theta}$$

$$= y \sin \theta + x \left(\frac{1 - \sin^2 \theta}{\cos \theta} \right)$$

$$\Rightarrow X = x \cos \theta + y \sin \theta \quad \dots(1)$$

$$\text{Also, } \cos \theta = \frac{PQ}{PS} = \frac{Y}{y - x \tan \theta}$$

$$\Rightarrow Y = -x \sin \theta + y \cos \theta \quad \dots(2)$$

From (1) and (2),

$$\Rightarrow x = X \cos \theta - Y \sin \theta$$

$$y = X \sin \theta + Y \cos \theta$$

The conversion of coordinates from one system to other system can be represented as follows:

	x	y
X	$\cos \theta$	$\sin \theta$
Y	$-\sin \theta$	$\cos \theta$

ILLUSTRATION 1.7

The axes are rotated through an angle of $\pi/3$ in the anticlockwise direction with respect to $(0, 0)$. Find the coordinates of point $(4, 2)$ (w.r.t. old coordinate system) in the new coordinates system.

Sol. Here, $(x, y) = (4, 2)$

$$\text{and } \theta = \pi/3$$

$$\therefore X = x \cos \theta + y \sin \theta$$

$$= 4 \cos \frac{\pi}{3} + 2 \sin \frac{\pi}{3}$$

$$= 4 \times \frac{1}{2} + 2 \times \frac{\sqrt{3}}{2} = 2 + \sqrt{3}$$

and $Y = -x \sin \theta + y \cos \theta$

$$= -4 \sin \frac{\pi}{3} + 2 \cos \frac{\pi}{3}$$

$$= -4 \times \frac{\sqrt{3}}{2} + 2 \times \frac{1}{2} = -2\sqrt{3} + 1$$

ILLUSTRATION 1.8

The equation of a curve referred to a given system of axes is $3x^2 + 2xy + 3y^2 = 10$. Find its equation if the axes are rotated about the origin through an angle of 45° .

Sol. We have

$$x = X \cos \theta - Y \sin \theta$$

$$= X \cos 45^\circ - Y \sin 45^\circ = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}$$

$$y = X \sin \theta + Y \cos \theta$$

$$= X \sin 45^\circ + Y \cos 45^\circ = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

Hence, the equation $3x^2 + 2xy + 3y^2 = 10$ transforms to

$$3\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) + 3\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right)^2 = 10$$

or $2X^2 + Y^2 = 5$

ILLUSTRATION 1.9

If θ is an angle by which axes are rotated about origin and equation $ax^2 + 2hxy + by^2 = 0$ does not contain xy term in the new system, then prove that $\tan 2\theta = \frac{2h}{a-b}$.

Sol. Clearly, $h \neq 0$.

Rotating the axes through an angle θ , we have

$$x = X \cos \theta - Y \sin \theta$$

$$y = X \sin \theta + Y \cos \theta$$

$$\therefore f(x, y) = ax^2 + 2hxy + by^2$$

$$\begin{aligned} &= a(X \cos \theta - Y \sin \theta)^2 + 2h(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) + b(X \sin \theta + Y \cos \theta)^2 \\ &= (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) X^2 \\ &\quad + 2[(b-a) \cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta)] XY \\ &\quad + (a \sin^2 \theta - 2h \cos \theta \sin \theta + b \cos^2 \theta) Y^2 \\ &= F(X, Y) \quad (\text{say}) \end{aligned}$$

Now, in $F(X, Y)$, we require that the coefficient of the XY -term is zero.

$$\therefore 2[(b-a) \cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta)] = 0$$

$$\Rightarrow (a-b) \sin 2\theta = 2h \cos 2\theta$$

$$\Rightarrow \tan 2\theta = \frac{2h}{a-b}$$

CONCEPT APPLICATION EXERCISE 1.1

- What is the minimum area of a triangle with integral vertices?
- What is length of the projection of line segment joining points (2, 3) and (7, 5) on x -axis.
- Point $P(-2, 3)$ goes through following transformations in succession:
 - reflection in line $y = x$
 - translation of 4 units to the right
 - translation of 5 units up
 - reflection in y -axis
 Find the coordinates of final position of the point.
- Find the equation to which the equation $x^2 + 7xy - 2y^2 + 17x - 26y - 60 = 0$ is transformed if the origin is shifted to the point (2, -3), the axes remaining parallel to the original axis.
- Without rotating the original coordinate axes, to which point should origin be transferred, so that the equation $x^2 + y^2 - 4x + 6y - 7 = 0$ is changed to an equation which contains no term of first degree?
- Given the equation $4x^2 + 2\sqrt{3}xy + 2y^2 = 1$. Through what angle should the axes be rotated so that the term xy is removed from the transformed equation?

ANSWERS

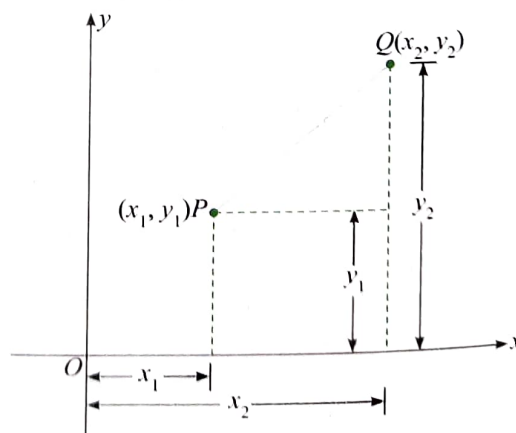
- | | | |
|-------------------------------|------------|--------------------|
| 1. $1/2$ sq. units | 2. 5 units | 3. (-7, 3) |
| 4. $X^2 + 7XY - 2Y^2 - 4 = 0$ | 5. (2, -3) | 6. $\pi/6, 2\pi/3$ |

DISTANCE FORMULA AND AREA OF POLYGON**DISTANCE FORMULA**

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{Distance of } (x_1, y_1) \text{ from origin} = \sqrt{x_1^2 + y_1^2}$$



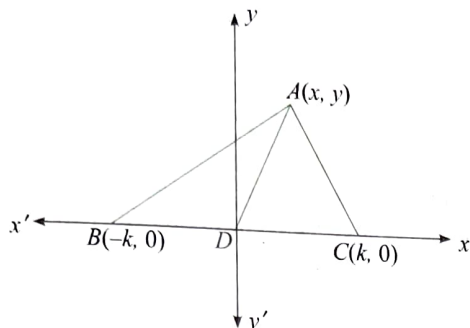
Note:

- Distance between points $(x_1, 0)$ and $(x_2, 0)$ is $|x_1 - x_2|$.
- Distance between points $(0, y_1)$ and $(0, y_2)$ is $|y_1 - y_2|$.

ILLUSTRATION 1.10

Using distance formula, prove the Apollonius' theorem that is in $\triangle ABC$, $AB^2 + AC^2 = 2(AD^2 + BD^2)$, where D is the middle point of BC .

Sol. Let point B and C lie on the x -axis such that $BD = DC$, where $D(0, 0)$ is midpoint of BC . Let point A be (x, y) .

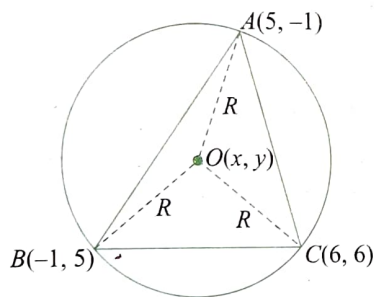


$BD = k$. So, point B is $(-k, 0)$ and point C is $(k, 0)$.
Now, $AB^2 + AC^2 = (x + k)^2 + (y - 0)^2 + (x - k)^2 + (y - 0)^2$
 $= x^2 + k^2 + 2xk + y^2 + x^2 + k^2 - 2xk + y^2$
 $= 2x^2 + 2y^2 + 2k^2 = 2(x^2 + y^2 + k^2)$
And $2(AD^2 + BD^2) = 2[(x - 0)^2 + (y - 0)^2 + k^2]$
 $= 2(x^2 + y^2 + k^2)$
Therefore, $AB^2 + AC^2 = 2(AD^2 + BD^2)$

ILLUSTRATION 1.11

Find the coordinates of the circumcenter of the triangle whose vertices are $A(5, -1)$, $B(-1, 5)$, and $C(6, 6)$. Find its radius also.

Sol.



Let the circumcenter be $O(x, y)$. Then
 $(OA)^2 = (OB)^2 = (OC)^2 = (\text{Radius})^2 = R^2$
or $(x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$
 $= (x - 6)^2 + (y - 6)^2$

Taking first two relations, we get
 $(x - 5)^2 + (y + 1)^2 = (x + 1)^2 + (y - 5)^2$
or $x = y$

Taking last two relations, we get
 $(x + 1)^2 + (y - 5)^2 = (x - 6)^2 + (y - 6)^2$
or $(x + 1)^2 + (x - 5)^2 = (x - 6)^2 + (x - 6)^2$
or $2x^2 - 8x + 26 = 2x^2 - 24x + 72$
or $x = \frac{23}{8}$
or Circumcenter $\equiv (23/8, 23/8)$
or $R^2 = (x - 5)^2 + (y + 1)^2 = (OA)^2$
 $= \left[\frac{23}{8} - 5\right]^2 + \left[\frac{23}{8} + 1\right]^2$

[From (2)]

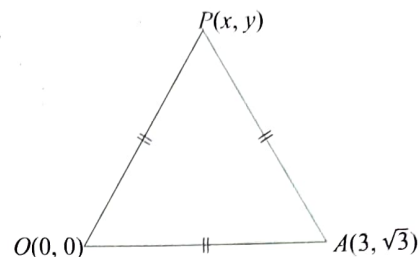
$$= \frac{17^2}{64} + \frac{31^2}{64} = \frac{1250}{64}$$

or Radius = $\frac{25\sqrt{2}}{8}$ units

ILLUSTRATION 1.12

Two points $O(0, 0)$ and $A(3, \sqrt{3})$ with another point P form an equilateral triangle. Find the coordinates of P .

Sol.



Let the coordinates of P be (x, y) .

$\therefore OA = OP = AP$
or $OA^2 = OP^2 = AP^2$
 $\therefore 12 = x^2 + y^2 = (x - 3)^2 + (y - \sqrt{3})^2$
 $\Rightarrow 3x + \sqrt{3}y = 6$
 $\Rightarrow x = 2 - \frac{y}{\sqrt{3}}$... (1)

$\therefore \left(2 - \frac{y}{\sqrt{3}}\right)^2 + y^2 = 12$

$\Rightarrow y^2 - \sqrt{3}y - 6 = 0$

$\Rightarrow (y - 2\sqrt{3})(y + \sqrt{3}) = 0$

$\Rightarrow y = 2\sqrt{3}$ or $-\sqrt{3}$

From (1), when $y = 2\sqrt{3}$, $x = 0$ and

when $y = -\sqrt{3}$, $x = 3$

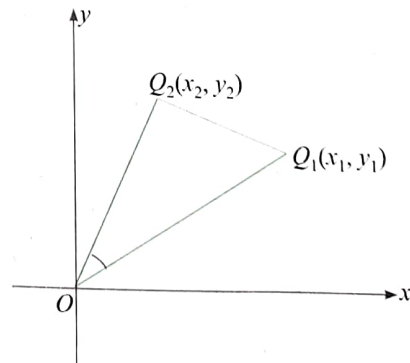
Hence, the coordinates of P are:

$(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

ILLUSTRATION 1.13

If the coordinates of any two points Q_1 and Q_2 are (x_1, y_1) and (x_2, y_2) , respectively, then prove that $OQ_1 \times OQ_2 \cos(\angle Q_1 O Q_2) = x_1 x_2 + y_1 y_2$, where O is the origin.

Sol.



In $\triangle OQ_1Q_2$, using cosine rule, we get

$$(Q_1Q_2)^2 = (OQ_1)^2 + (OQ_2)^2 - OQ_1 \cdot OQ_2 \cdot \cos(\angle Q_1 O Q_2)$$

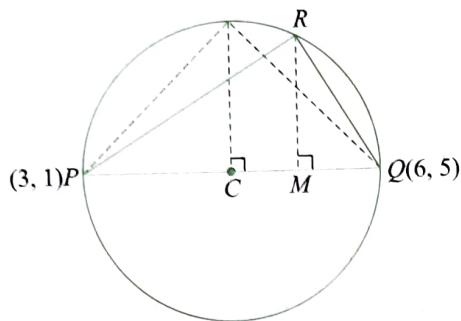
$$\therefore (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2OQ_1 \cdot OQ_2 \cdot \cos(\angle Q_1 O Q_2)$$

$$\Rightarrow OQ_1 \cdot OQ_2 \cos(\angle Q_1 O Q_2) = x_1 x_2 + y_1 y_2$$

ILLUSTRATION 1.14

$P(3, 1)$, $Q(6, 5)$ and $R(x, y)$ are three points such that PRQ is a right angle and the area of ΔRQP is 7 sq. unit. Find the number of such points R .

Sol. Clearly, R lies on the circle with P and Q as end points of diameter.



$$\text{Now, } PQ = \sqrt{(6-3)^2 + (5-1)^2} = 5$$

$$\therefore \text{Radius, } r = 2.5$$

$$\text{Now, area of triangle } PQR = \frac{1}{2} RM \cdot PQ = 7 \quad (\text{Given})$$

$$\Rightarrow RM = \frac{14}{5} = 2.8,$$

which is not possible as RM cannot be more than radius.

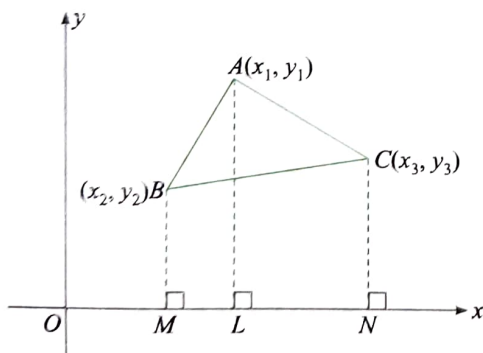
Hence, no such triangle is possible.

AREA OF A POLYGON**Area of Triangle**

The area of a triangle the coordinates of whose vertices are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is given by

$$\frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

Proof: Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.



Draw AL , BM and CN perpendiculars from A , B and C , respectively, on the x -axis.

Clearly, $ABML$, $ALNC$ and $BMNC$ are all trapeziums.

We have,

$$\begin{aligned} \text{Area of } \Delta ABC &= \text{Area of trapezium } ABML \\ &+ \text{Area of trapezium } ALNC \\ &- \text{Area of trapezium } BMNC \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} (BM + AL)(ML) + \frac{1}{2} (AL + CN)(LN) \\ &\quad - \frac{1}{2} (BM + CN)(MN) \\ &= \frac{1}{2} (y_2 + y_1)(x_1 - x_2) + \frac{1}{2} (y_1 + y_3)(x_3 - x_1) \\ &\quad - \frac{1}{2} (y_2 + y_3)(x_3 - x_2) \\ &= \frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \\ &= \frac{1}{2} [(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)] \end{aligned}$$

If the points A , B and C are plotted in the two-dimensional plane and the three points are taken in the anticlockwise sense then the area calculated of the triangle ABC will be positive while if the points are taken in clockwise sense then the area calculated will be negative. But, if the points are taken arbitrarily, then the area calculated may be positive or negative, the numerical value being the same in all the cases. To avoid this sign problem, we put modulus sign on area.

If three points A , B and C are collinear, then the area of triangle will be zero.

Area can also be found using determinant method. The determinant of order 2 is defined as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The determinant of order 3 is defined as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

In the above figure, area of triangle ABC can be put in the form of determinant of order 3 as

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Here, we consider absolute value of above determinant as area is positive.

Area of n -sided Polygon

The area of polygon whose vertices are (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , ..., (x_n, y_n) taken in cyclic order is given by

$$= \frac{1}{2} |(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \dots + (x_ny_1 - x_1y_n)|$$

Stair Method to Find Area of Polygon

Let the given n points (x_1, y_1) , (x_2, y_2) , ..., (x_n, y_n) be in cyclic order on coordinate plane. Write the coordinates of points one below the another and repeat the coordinates at the bottom which were taken at the top. Multiply values at the start and the end of the arrows.

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

Product of downward arrow is taken with positive sign and product of upward arrow is taken with negative sign.

Thus, area of polygon

$$= \frac{1}{2} \{ (x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + \dots + (x_n y_1 - x_1 y_n) \}$$

ILLUSTRATION 1.15

Find the area of a triangle having vertices $A(3, 2)$, $B(11, 8)$, and $C(8, 12)$.

Sol. Let $A \equiv (x_1, y_1) \equiv (3, 2)$, $B(x_2, y_2) \equiv (11, 8)$, and $C \equiv (x_3, y_3) \equiv (8, 12)$. Then

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} | \{ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \} | \\ &= \frac{1}{2} | \{ 3(8 - 12) + 11(12 - 2) + 8(2 - 8) \} | \\ &= \frac{1}{2} | \{ -12 + 110 - 48 \} | \\ &= 25 \text{ sq. units} \end{aligned}$$

ILLUSTRATION 1.16

Prove that the area of the triangle whose vertices are $(t, t - 2)$, $(t + 2, t + 2)$, and $(t + 3, t)$ is independent of t .

Sol. Let $A \equiv (x_1, y_1) \equiv (t, t - 2)$, $B \equiv (x_2, y_2) \equiv (t + 2, t + 2)$, and $C \equiv (x_3, y_3) \equiv (t + 3, t)$ be the vertices of the given triangle. Then

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} | \{ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \} | \\ &= \frac{1}{2} | \{ t(t + 2 - t) + (t + 2)(t - t + 2) + (t + 3)(t - 2 - t + 2) \} | \\ &= \frac{1}{2} | \{ 2t + 2t + 4 - 4t - 12 \} | = | -4 | = 4 \text{ sq. units} \end{aligned}$$

Clearly, the area of $\triangle ABC$ is independent of t .

ILLUSTRATION 1.17

Find the area of the quadrilateral $ABCD$ having vertices $A(1, 1)$, $B(7, -3)$, $C(12, 2)$, and $D(7, 21)$.

Sol. Here, given points are in cyclic order. Therefore,

$$\begin{aligned} \text{Area} &= \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 7 & -3 \\ 12 & 2 \\ 7 & 21 \\ 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} | (-3 - 7) + (14 + 36) + (252 - 14) + (7 - 21) | \\ &= 132 \text{ sq. units} \end{aligned}$$

ILLUSTRATION 1.18

For what value of k are the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ collinear?

Sol. Let the three given points be $A \equiv (x_1, y_1) \equiv (k, 2 - 2k)$, $B \equiv (x_2, y_2) \equiv (-k + 1, 2k)$, and $C \equiv (x_3, y_3) \equiv (-4 - k, 6 - 2k)$.

If the given points are collinear, then $\Delta = 0$, i.e.,

$$\begin{aligned} x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) &= 0 \\ \text{or } k(2k - 6 + 2k) + (-k + 1)(6 - 2k - 2 + 2k) &+ (-4 - k)(2 - 2k - 2k) = 0 \end{aligned}$$

$$\text{or } k(4k - 6) - 4(k - 1) + (4 + k)(4k - 2) = 0$$

$$\text{or } 4k^2 - 6k - 4k + 4 + 4k^2 + 14k - 8 = 0$$

$$\text{or } 8k^2 + 4k - 4 = 0$$

$$\text{or } 2k^2 + k - 1 = 0$$

$$\text{or } (2k - 1)(k + 1) = 0$$

$$\text{i.e., } k = \frac{1}{2} \text{ or } -1$$

But for $t = 1/2$, the first two points are coincident. Hence, $k = -1$.

ILLUSTRATION 1.19

The coordinates of two points A and B are $(3, 4)$ and $(5, -2)$, respectively. Find the coordinates of points P if $PA = PB$ and area of $\triangle PAB$ is 10 sq. unit.

Sol. Let the coordinates of point P be (x, y) .

Given that $PA = PB$

$$\text{or } PA^2 = PB^2$$

$$\Rightarrow (x - 3)^2 + (y - 4)^2 = (x - 5)^2 + (y + 2)^2$$

$$\Rightarrow x - 3y - 1 = 0 \quad (1)$$

Now, area of $\triangle PAB = 10$

$$\Rightarrow \frac{1}{2} \begin{vmatrix} x & y \\ 3 & 4 \\ 5 & -2 \end{vmatrix} = 10$$

$$\Rightarrow 6x + 2y - 26 = \pm 20$$

$$\Rightarrow 3x + y - 23 = 0 \quad (2)$$

$$\text{or } 3x + y - 3 = 0 \quad (3)$$

Solving (1) and (2), we get $x = 7, y = 2$.

Solving (1) and (3), we get $x = 1, y = 0$.

Thus, the coordinates of P can be $(7, 2)$ or $(1, 0)$.

ILLUSTRATION 1.20

If the vertices of triangle have rational coordinates, then prove that the triangle cannot be equilateral.

Sol. Let the vertices of triangle ABC are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Since vertices have rational coordinates, consider $x_i, y_i \in \mathbb{Q}$ for $i = 1, 2, 3$.

$$\begin{aligned} \text{Area of triangle, } \Delta &= \frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) \\ &\quad + (x_3 y_1 - x_1 y_3)] \quad (1) \end{aligned}$$

Clearly, this value will be rational.

Also, area of equilateral triangle, $\Delta = \frac{\sqrt{3}}{4} a^2$, where a is side length.

$$a = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\therefore \Delta = \frac{\sqrt{3}}{4} [(x_2 - x_1)^2 + (y_2 - y_1)^2] \quad (2)$$

Clearly, this value is irrational.
Thus, we have contradiction.
So, triangle cannot be equilateral.

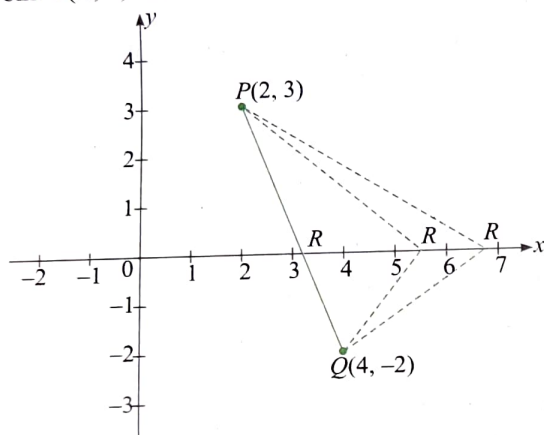
ILLUSTRATION 1.21

Given three points $P(2, 3)$, $Q(4, -2)$ and $R(\alpha, 0)$.

- Find the value of α if $PR + RQ$ is minimum
- Find the value of α if $|PR - RQ|$ is maximum.

Sol.

- Point $R(\alpha, 0)$ lies on x -axis.



Now, from triangular inequality, $PR + RQ \geq PQ$.

Thus, minimum value of $PR + RQ$ is PQ , which occurs when points P , Q and R are collinear.

$$\Rightarrow \begin{vmatrix} 2 & 3 \\ 4 & -2 \\ \alpha & 0 \end{vmatrix} = 0$$

$$\Rightarrow -4 - 12 + 2\alpha + 3\alpha = 0$$

$$\Rightarrow \alpha = 16/5$$

(ii)

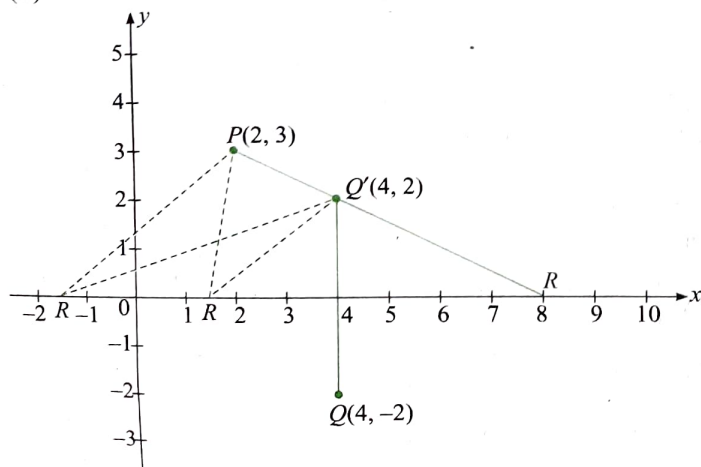


Image of Q in x -axis is $Q'(4, 2)$.

Now, $RQ = RQ'$

$$\therefore |PR - RQ| = |PR - RQ'|$$

From triangular inequality,

$$|PR - RQ'| \leq PQ'$$

$\therefore |PR - RQ'|_{\max} = PQ'$, when P , Q' and R are collinear

$$\therefore \begin{vmatrix} 2 & 3 \\ 4 & 2 \\ \alpha & 0 \end{vmatrix} = 0$$

$$\Rightarrow 4 - 12 - 2\alpha + 3\alpha = 0$$

$$\Rightarrow \alpha = 8$$

ILLUSTRATION 1.22

If $A\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$, $B\left(\frac{-3}{\sqrt{2}}, \sqrt{2}\right)$, $C\left(\frac{-3}{\sqrt{2}}, -\sqrt{2}\right)$ and $D(3\cos \theta, 2\sin \theta)$ are four points, then find the maximum area of quadrilateral $ABCD$, where $\theta \in (3\pi/2, 2\pi)$.

Sol. Area of quadrilateral $ABCD$ is maximum when area of ACD is maximum.

Area of triangle ACD ,

$$\Delta_1 = \frac{1}{2} \begin{vmatrix} \frac{3}{\sqrt{2}} & \sqrt{2} \\ \frac{-3}{\sqrt{2}} & -\sqrt{2} \\ 3\sin \theta & 2\cos \theta \end{vmatrix}$$

$$= |-3\sqrt{2} \cos \theta + 3\sqrt{2} \sin \theta|$$

$$\therefore \Delta_1(\max.) = 6, \text{ when } \theta = \frac{7\pi}{4} \quad (\text{as } \theta \in (3\pi/2, 2\pi))$$

Maximum area is 12 sq. units (as $ABCD$ is a rectangle).

CONCEPT APPLICATION EXERCISE 1.2

- Show that the distance between the points $P(a \cos \alpha, a \sin \alpha)$ and $Q(a \cos \beta, a \sin \beta)$ is $2a \sin \frac{\alpha - \beta}{2}$.
- In each of the following, check how the points A , B and C are situated.
 - $A(-2, 2)$, $B(8, -2)$, $C(-4, -3)$
 - $A(-a, -b)$, $B(a, b)$, $C(a^2, ab)$, $a > 1$
 - $A(4, 0)$, $B(-1, -1)$, $C(3, 5)$
- If the points $(1, 1)$, $(0, \sec^2 \theta)$ and $(\operatorname{cosec}^2 \theta, 0)$ are collinear, then find the values of θ .
- Find the area of the regular hexagon ends of whose diagonals are $(2, 4)$ and $(6, 7)$.
- Let $ABCD$ be a rectangle and P be any point in its plane. Show that $PA^2 + PC^2 = PB^2 + PD^2$ using coordinate geometry.
- Find the length of altitude through A of the triangle ABC , where $A \equiv (-3, 0)$, $B \equiv (4, -1)$ and $C \equiv (5, 2)$.

7. Find the area of the pentagon whose vertices are $A(1, 1)$, $B(7, 21)$, $C(12, 2)$, $D(7, -3)$, and $E(0, -3)$.
8. Four points $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$, and $D(x, 2x)$ are given in such a way that $\frac{(\text{Area of } \triangle DBC)}{(\text{Area of } \triangle ABC)} = 1/2$. Find x .

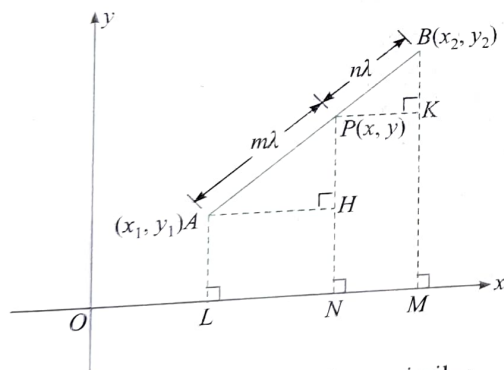
ANSWERS

2. (i) ABC is triangle right angled at A
 (ii) collinear points
 (iii) ABC is right angled isosceles triangle
3. $\theta \in R - \left\{ \frac{n\pi}{2}, n \in Z \right\}$ 4. $\frac{75\sqrt{3}}{8}$ sq. unit
6. $\frac{22}{\sqrt{10}}$ 7. 146 sq. units 8. $\frac{11}{6}$

SECTION FORMULA

INTERNAL DIVISION

Consider two different points $A(x_1, y_1)$ and $B(x_2, y_2)$ in a plane. If point $P(x, y)$ lies on the line segment AB somewhere between A and B , then point P is said to be dividing AB internally. Now, if $\frac{AP}{BP} = \frac{m}{n}$, then point P divides AB internally in the ratio $m : n$. Let us find the coordinates of point P .



In the figure, triangles AHP and PKB are similar.

$$\text{Therefore, } \frac{AP}{BP} = \frac{AH}{PK} = \frac{PH}{BK}$$

$$\Rightarrow \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$\text{Now, } \frac{m}{n} = \frac{x - x_1}{x_2 - x}$$

$$\Rightarrow mx_2 - mx = nx - nx_1$$

$$\Rightarrow x = \frac{mx_2 + nx_1}{m + n}$$

$$\text{Also, } \frac{m}{n} = \frac{y - y_1}{y_2 - y}$$

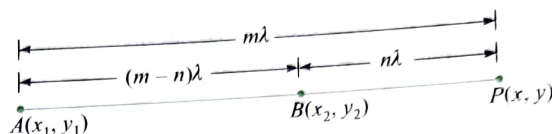
$$\Rightarrow my_2 - my = ny - ny_1$$

$$\Rightarrow y = \frac{my_2 + ny_1}{m + n}$$

Thus, the coordinates of P are $\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n} \right)$.

EXTERNAL DIVISION

If point P lies on the line joining A and B , but not between them, such that $\frac{AP}{BP} = \frac{m}{n}$, then point P is said to be dividing AB externally in the ratio $m : n$. In this case, let us find the coordinates of P .



$$\text{We have } \frac{AP}{BP} = \frac{m}{n}$$

$$\Rightarrow \frac{AP - BP}{BP} = \frac{m - n}{n}$$

$$\Rightarrow \frac{AB}{BP} = \frac{m - n}{n}$$

Thus, point B divides AP internally in the ratio $(m - n) : n$

$$\therefore \left(\frac{(m - n)x + nx_1}{(m - n) + n}, \frac{(m - n)y + ny_1}{(m - n) + n} \right) \equiv (x_2, y_2)$$

$$\Rightarrow x_2 = \frac{(m - n)x + nx_1}{m}$$

$$\Rightarrow x = \frac{mx_2 - nx_1}{m - n}$$

$$\text{Similarly, we get } y = \frac{my_2 - ny_1}{m - n}$$

Thus, coordinates of point P dividing AB externally in the ratio $m : n$ are $\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n} \right)$.

Note:

- The coordinates of midpoint of AB are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

- The coordinates of point which trisects AB are

$$\left(\frac{x_1 + 2x_2}{3}, \frac{y_1 + 2y_2}{3} \right) \text{ or } \left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3} \right)$$

- To show that quadrilateral $ABCD$ is a parallelogram, it is sufficient to show that coordinates of midpoints of diagonals AC and BD are the same.

- If ratio $\frac{m}{n} = \lambda$, then coordinates of point P dividing AB in the ratio $\lambda : 1$ are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1} \right)$.

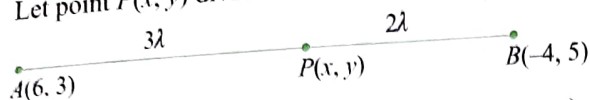
- If $\lambda > 0$, then P divides AB internally and if $\lambda < 0$, then P divides AB externally.

ILLUSTRATION 1.23

Find the coordinates of point which divides the line segments joining the points $(6, 3)$ and $(-4, 5)$ in the ratio $3 : 2$ (i) internally and (ii) externally.

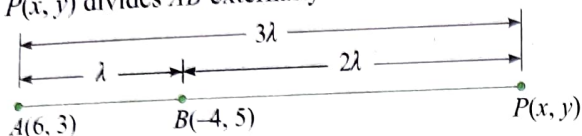
Sol.

- (i) Given points are $A(6, 3)$ and $B(-4, 5)$.
Let point $P(x, y)$ divide AB internally in the ratio $3 : 2$.



$$\therefore (x, y) \equiv \left(\frac{3(-4) + 2(6)}{3+2}, \frac{3(5) + 2(3)}{3+2} \right) \equiv \left(0, \frac{21}{5} \right)$$

- (ii) $P(x, y)$ divides AB externally in the ratio $3 : 2$.



$$\therefore (x, y) \equiv \left(\frac{3(-4) - 2(6)}{3-2}, \frac{3(5) - 2(3)}{3-2} \right) \equiv (-24, 9)$$

Alternatively, $\frac{AB}{BP} = \frac{1}{2}$ (From the figure)

$$\therefore (-4, 5) \equiv \left(\frac{2(6) + 1(x)}{2+1}, \frac{2(3) + 1(y)}{2+1} \right) \equiv \left(\frac{12+x}{3}, \frac{6+y}{3} \right)$$

$$\therefore (x, y) \equiv (-24, 9)$$

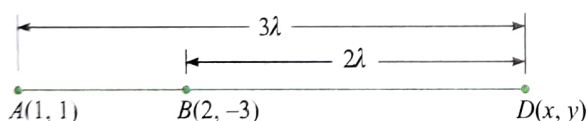
ILLUSTRATION 1.24

$A(1, 1)$ and $B(2, -3)$ are two points and D is a point on AB produced towards B such that $AD = 3AB$. Find the coordinates of D .

Sol. We have $\frac{AD}{AB} = 3$

$$\therefore \frac{AD}{AD - AB} = \frac{3}{3-1}$$

$$\Rightarrow \frac{AD}{BD} = \frac{3}{2}$$



$$\therefore (x, y) \equiv \left(\frac{3(2) - 2(1)}{3-2}, \frac{3(-3) - 2(1)}{3-2} \right) \equiv (4, -11)$$

ILLUSTRATION 1.25

Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points $(1, 3)$ and $(2, 7)$.

Sol. Suppose the line $3x + y - 9 = 0$ divides the line segment joining $A(1, 3)$ and $B(2, 7)$ in the ratio $k : 1$ at point C . Then, the coordinates of C are

$$\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1} \right)$$

But C lies on $3x + y - 9 = 0$. Therefore,

$$3 \left(\frac{2k+1}{k+1} \right) + \frac{7k+3}{k+1} - 9 = 0$$

$$\text{or } 6k + 3 + 7k + 3 - 9k - 9 = 0$$

$$\text{or } k = \frac{3}{4}$$

So, the required ratio is $3 : 4$ internally.

ILLUSTRATION 1.26

Prove that the points $(-2, -1)$, $(1, 0)$, $(4, 3)$, and $(1, 2)$ are the vertices of a parallelogram. Is it a rectangle?

Sol. Given points are in cyclic order.

Let $A \equiv (-2, -1)$, $B \equiv (1, 0)$, $C \equiv (4, 3)$ and $D \equiv (1, 2)$.
Then, the coordinates of the midpoint of AC are

$$\left(\frac{-2+4}{2}, \frac{-1+3}{2} \right) \equiv (1, 1)$$

The coordinates of the midpoint of BD are

$$\left(\frac{1+1}{2}, \frac{0+2}{2} \right) \equiv (1, 1)$$

Thus, AC and BD have the same midpoint.

Hence, $ABCD$ is a parallelogram.

Now, we shall see whether $ABCD$ is a rectangle or not.

We have

$$AC = \sqrt{\{4 - (-2)\}^2 + \{3 - (-1)\}^2} = 2\sqrt{13}$$

$$\text{and } BD = \sqrt{(1-1)^2 + (0-2)^2} = 2$$

Clearly, $AC \neq BD$. So, $ABCD$ is not a rectangle.

ILLUSTRATION 1.27

Let $A_1, A_2, A_3, \dots, A_n$ are n points in a plane whose coordinates are $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, respectively. A_1A_2 is bisected at the point P_1 , P_1A_3 is divided in the ratio $1 : 2$ at P_2 , P_2A_4 is divided in the ratio $1 : 3$ at P_3 , P_3A_5 is divided in the ratio $1 : 4$ at P_4 and so on until all n points are exhausted. Find the coordinates of the final point so obtained.

Sol. P_1 is midpoint of A_1A_2 .

$$\therefore P_1 \equiv \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right)$$

P_2 divides P_1A_3 in $1 : 2$.

$$\begin{aligned} \therefore P_2 &\equiv \left(\frac{2 \left(\frac{x_1+x_2}{2} \right) + x_3}{2+1}, \frac{2 \left(\frac{y_1+y_2}{2} \right) + y_3}{2+1} \right) \\ &\equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3} \right) \end{aligned}$$

Now P_3 divides P_2A_4 in $1 : 3$

$$\begin{aligned} \therefore P_3 &\equiv \left(\frac{3 \cdot \frac{x_1+x_2+x_3}{3} + x_4}{3+1}, \frac{3 \cdot \frac{y_1+y_2+y_3}{3} + y_4}{3+1} \right) \\ &\equiv \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4} \right) \end{aligned}$$

Proceeding in this manner, we get

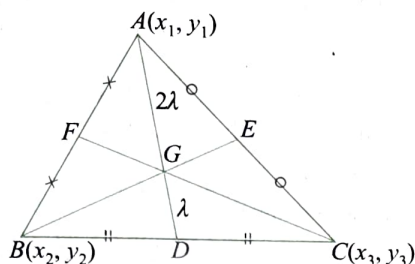
$$P_n \equiv \left(\frac{x_1+x_2+x_3+\dots+x_n}{n}, \frac{y_1+y_2+y_3+\dots+y_n}{n} \right)$$

COORDINATES OF DIFFERENT CENTRES OF TRIANGLE**CENTROID**

The point of concurrency of the medians of a triangle is called the centroid of the triangle and always lies inside the triangle. Let us get the coordinates of centroid in terms of coordinates of the vertices of triangle.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle whose medians are AD , BE and CF .

So, D , E and F are, respectively, the midpoints of BC , CA and AB .



$$\therefore D \equiv \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right)$$

Also, centroid (G) divides median AD in the ratio $2 : 1$.

$$\begin{aligned} \therefore G &\equiv \left(\frac{1(x_1) + 2\left(\frac{x_2 + x_3}{2}\right)}{1 + 2}, \frac{1(y_1) + 2\left(\frac{y_2 + y_3}{2}\right)}{1 + 2} \right) \\ &\equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right) \end{aligned}$$

Note:

- Centroid of triangle and that of triangle formed by midpoints of sides is same.
- The three medians of a triangle divide the triangle into six equal areas.
- In a triangle, the line segments connecting the midpoints of sides are parallel to the opposite sides and the four triangles created by these line segments are congruent and so have equal areas.

ILLUSTRATION 1.28

If vertex A of triangle ABC is $(3, 5)$ and centroid is $(-1, 2)$, then find the midpoint of side BC .

Sol. Let midpoint of side BC be D .

So, AD is median and centroid $G(-1, 2)$ lies on it such that $AG : GD = 2 : 1$.

Thus, D divides AG externally in the ratio $3 : 1$.

$$\therefore D \equiv \left(\frac{3(-1) - 1(3)}{3 - 1}, \frac{3(2) - 1(5)}{3 - 1} \right) \equiv \left(-3, \frac{1}{2} \right)$$

ILLUSTRATION 1.29

Let $O(0, 0)$, $P(3, 4)$, and $Q(6, 0)$ be the vertices of triangle OPQ . Find the point R inside the triangle OPQ such that the triangles OPR , PQR , OQR are of equal areas.

Sol. As discussed earlier, three medians of a triangle divide the triangle into six equal areas.

So, point R must be centroid.

$$\text{Therefore, } R \equiv \left(\frac{3 + 6 + 0}{3}, \frac{4 + 0 + 0}{3} \right) \equiv \left(3, \frac{4}{3} \right)$$

CIRCUMCENTRE

The circumcentre of a triangle is the centre of the circumscribing circle of the triangle. Also, it is the point where perpendicular bisectors of sides of triangle are concurrent. Circumcentre of acute angled triangle lies inside the triangle and that of obtuse angled triangle lies outside the triangle. Circumcentre of right angled triangle is midpoint of hypotenuse.

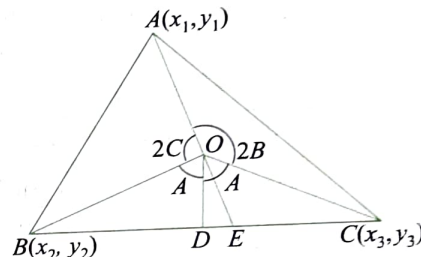
Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle ABC , then coordinates of circumcentre are:

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

Proof:

Line joining A and the circumcenter (O) meets BC at E .

The foot of perpendicular from O on BC is D . Therefore,



$$\begin{aligned} \frac{BE}{EC} &= \frac{\frac{1}{2} BE \times OD}{\frac{1}{2} EC \times OD} = \frac{\text{Area of } \triangle OBE}{\text{Area of } \triangle OEC} \\ &= \frac{\frac{1}{2} OB \times OE \times \sin \angle BOE}{\frac{1}{2} OC \times OE \times \sin \angle COE} \\ &= \frac{\frac{1}{2} R \times OE \times \sin(\pi - 2C)}{\frac{1}{2} R \times OE \times \sin(\pi - 2B)} \\ &= \frac{\sin 2C}{\sin 2B} \end{aligned}$$

Therefore, the coordinates of D are

$$\left(\frac{x_2 \sin 2B + x_3 \sin 2C}{\sin 2B + \sin 2C}, \frac{y_2 \sin 2B + y_3 \sin 2C}{\sin 2B + \sin 2C} \right)$$

Now,

$$\frac{\sin 2B + \sin 2C}{\sin 2A} = \frac{2 \sin(B + C) \cos(B - C)}{2 \sin A \cos A} = \frac{\cos(B - C)}{\cos A}$$

In $\triangle ODE$, $OD = OB \cos A = R \cos A$. Also,

$$\begin{aligned} \angle DOE &= \angle DOC - \angle EOC \\ &= A - (\pi - 2B) \\ &= \pi - B - C - \pi + 2B \\ &= B - C \end{aligned}$$

Now, in $\triangle ODE$,

$$OE = \frac{OD}{\cos(B - C)} = \frac{R \cos A}{\cos(B - C)}$$

$$\text{or } \frac{AO}{OE} = \frac{R}{R \cos A} = \frac{\cos(B-C)}{\cos A}$$

$$\text{Thus, } \frac{AO}{OE} = \frac{\sin 2B + \sin 2C}{\sin 2A}$$

The coordinates of O are

$$\left(\frac{(\sin 2B + \sin 2C) \frac{x_2 \sin 2B + x_3 \sin 2C}{(\sin 2B + \sin 2C)} + x_1 \sin 2A}{(\sin 2B + \sin 2C) + \sin 2A}, \right. \\ \left. \frac{(\sin 2B + \sin 2C) \frac{y_2 \sin 2B + y_3 \sin 2C}{(\sin 2B + \sin 2C)} + y_1 \sin 2A}{(\sin 2B + \sin 2C) + \sin 2A} \right)$$

$$\text{or } \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \right. \\ \left. \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

ILLUSTRATION 1.30

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of triangle ABC and $x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2$, then show that $x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C = y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C = 0$.

Sol. Since $x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2$, circumcentre of ΔABC is $(0, 0)$.

Now, coordinates of circumcentre are

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \right. \\ \left. \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right) \equiv (0, 0)$$

$$\text{Therefore, } x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C \\ = y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C = 0$$

ILLUSTRATION 1.31

If ΔABC having vertices $A(a \cos \theta_1, a \sin \theta_1)$, $B(a \cos \theta_2, a \sin \theta_2)$, and $C(a \cos \theta_3, a \sin \theta_3)$ is equilateral, then prove that $\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = \sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 0$.

Sol. The distance of given vertices $A(a \cos \theta_1, a \sin \theta_1)$, $B(a \cos \theta_2, a \sin \theta_2)$, and $C(a \cos \theta_3, a \sin \theta_3)$ from the origin $(0, 0)$ is a .

Hence, the circumcenter of the triangle is $(0, 0)$.

Also, in an equilateral triangle, the centroid coincides with the circumcenter. We have

$$\frac{a \cos \theta_1 + a \cos \theta_2 + a \cos \theta_3}{3} = 0$$

$$\text{and } \frac{a \sin \theta_1 + a \sin \theta_2 + a \sin \theta_3}{3} = 0$$

$$\text{or } \cos \theta_1 + \cos \theta_2 + \cos \theta_3 = \sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 0$$

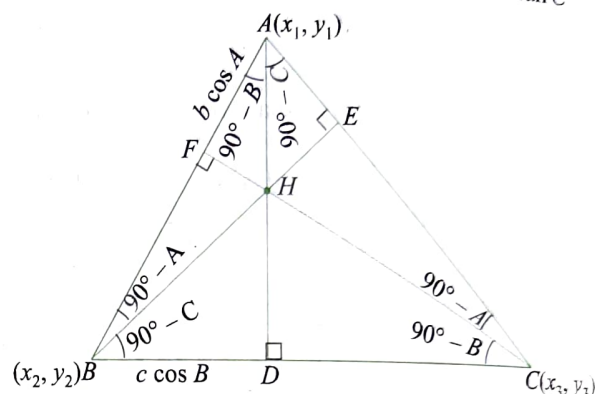
ORTHOCENTRE

The point of concurrency of altitudes of a triangle is called the orthocentre of the triangle. Orthocentre of acute angled triangle lies inside the triangle and that of obtuse angled triangle lies outside the triangle. Orthocentre of right angled triangle is the vertex where right angle occurs.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle ABC , then coordinates of orthocentre are:

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \right. \\ \left. \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

Proof:



First, we get the coordinates of point D for which we required the ratio $BD : DC$.

In ΔHDB ,

$$\tan(90^\circ - C) = \frac{HD}{BD} \Rightarrow BD = HD \tan C$$

Similarly, in ΔHDC , $CD = HD \tan B$

$$\Rightarrow \frac{BD}{CD} = \frac{\tan C}{\tan B}$$

$$\text{So, } D \equiv \left(\frac{x_2 \tan B + x_3 \tan C}{\tan B + \tan C}, \frac{y_2 \tan B + y_3 \tan C}{\tan B + \tan C} \right)$$

To get the coordinates of orthocentre H , we need the ratio $AH : HD$.

We must have

$$\frac{AH}{HD} = \frac{\tan B + \tan C}{\tan A} \\ = \frac{\sin(B+C) \cos A}{\cos B \cos C \sin A} = \frac{\cos A}{\cos B \cos C}$$

Now, let us find AH and HD .

In ΔAFH ,

$$\cos(90^\circ - B) = \frac{AF}{AH} \Rightarrow AH = \frac{b \cos A}{\sin B}$$

In ΔHDB ,

$$\tan(90^\circ - C) = \frac{HD}{BD} \\ \Rightarrow HD = (\cot C)(c \cos B) = c \cos B \frac{\cos C}{\sin C}$$

$$\text{Thus, } \frac{AH}{HD} = \frac{b \cos A}{\sin B} \cdot \frac{\sin C}{c \cos B \cos C} \\ = \frac{b \sin C}{c \sin B} \cdot \frac{\cos A}{\cos B \cos C} \\ = \frac{\cos A}{\cos B \cos C} \quad (\text{Using sine rule } b \sin C = c \sin B)$$

∴ Coordinates of H are

$$\left(\frac{(\tan B + \tan C) \frac{x_2 \tan B + x_3 \tan C}{\tan B + \tan C} + x_1 \tan A}{(\tan B + \tan C) + \tan A}, \right.$$

$$\left. \frac{(\tan B + \tan C) \frac{y_2 \tan B + y_3 \tan C}{\tan B + \tan C} + y_1 \tan A}{(\tan B + \tan C) + \tan A} \right)$$

or $\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \right.$

$$\left. \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

In the figure, we observe that vertex A is the orthocentre of triangle HBC . Similarly, B and C are orthocentres of triangles HAC and HAB , respectively.

Note:

- In a triangle other than equilateral triangle, orthocentre (H), centroid (G) and circumcentre (O) are collinear with ratio $HG : GO = 2 : 1$.
- If vertices of triangle are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) and circumcentre is $(0, 0)$, then orthocentre is $(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$.

ILLUSTRATION 1.32

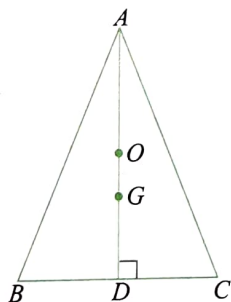
Find the orthocenter of the triangle whose vertices are $(0, 0)$, $(3, 0)$, and $(0, 4)$.

Sol. This is a right-angled (at the origin) triangle. Therefore, the orthocenter is $(0, 0)$.

ILLUSTRATION 1.33

The circumcentre and centroid of a triangle are $(3, 4)$ and $(6, 8)$, respectively. If one of the vertices of the triangle is $(0, 0)$, then what is the type of triangle?

Sol.



We have vertex $A(0, 0)$, circumcentre $O(3, 4)$ and centroid $G(6, 8)$. Point O lies on the perpendicular bisector (OD) of side BC and point G lies on the median AD .

Now, A , O and G are collinear.

So, median AD and perpendicular bisector (OD) of BC coincide.

Thus, AD is the perpendicular bisector of BC .

Therefore, triangle ABC is isosceles.

ILLUSTRATION 1.34

If the circumcentre of a triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$, then find the orthocentre.

Sol. Circumcentre is $O(0, 0)$.

$$\text{Centroid } G \equiv \left(\frac{(a+1)^2}{2}, \frac{(a-1)^2}{2} \right)$$

(Midpoint of given points)

Points H , G and O are collinear with $HG : GO = 2 : 1$.

So, using section formula, we have orthocentre as H

$$\left(\frac{3(a+1)^2}{2}, \frac{3(a-1)^2}{2} \right).$$

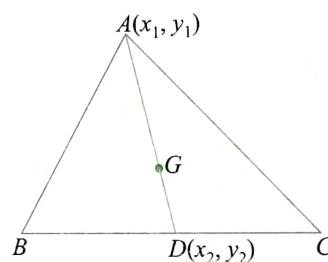
ILLUSTRATION 1.35

Orthocentre and circumcentre of $\triangle ABC$ are (a, b) and (c, d) , respectively. If the coordinates of the vertex A are (x_1, y_1) , then find the coordinates of the middle point of BC .

Sol. Given are the orthocentre $H(a, b)$ and the circumcentre $O(c, d)$.



$$\therefore \text{Centroid, } G \equiv \left(\frac{2c+a}{2+1}, \frac{2d+b}{2+1} \right) \equiv \left(\frac{2c+a}{3}, \frac{2d+b}{3} \right)$$



Now, centroid G divides AD in the ratio $2 : 1$, where D is the midpoint of BC .

$$\therefore \frac{2c+a}{3} = \frac{2x_2+x_1}{3}$$

$$\text{and } \frac{2d+b}{3} = \frac{2y_2+y_1}{3}$$

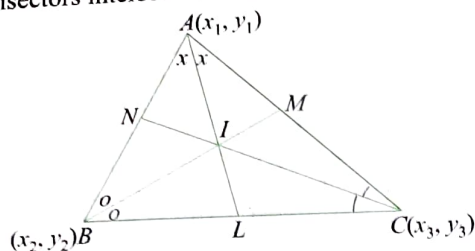
$$\therefore x_2 = \frac{2c+a-x_1}{2}, y_2 = \frac{2d+b-y_1}{2}$$

INCENTRE

The point of concurrency of internal angle bisectors of a triangle is called the incentre of the triangle. Also, it is the centre of the circle which is inscribed in the triangle and touches all the three sides of the triangle. This circle is the largest circle contained in the triangle. Let us get the coordinates of the incentre.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC such that $BC = a$, $CA = b$ and $AB = c$. Also, let AL , BM and CN be, respectively, the internal bisectors of the angles A , B and C .

These bisectors intersect at incentre I .



To find the coordinates of incentre, we use angle bisector theorem, that is 'angle bisector divides the opposite side into segments, the ratio of which is equal to the ratio of other two sides of triangle'.

$$\therefore \frac{BL}{CL} = \frac{BA}{CA} = \frac{c}{b} \quad (1)$$

$$\text{So, } L \equiv \left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c} \right)$$

To, get the coordinates of I , we should get the ratio $AI : IL$.

$$\text{In triangle } ABL, \frac{AI}{IL} = \frac{AB}{BL} \quad (2)$$

$$\text{Now, } \frac{BL}{LC} = \frac{c}{b}$$

$$\Rightarrow \frac{BL}{BL + LC} = \frac{c}{c+b}$$

$$\Rightarrow \frac{BL}{BC} = \frac{c}{c+b}$$

$$\Rightarrow BL = \frac{ac}{c+b}$$

So, from (2), we have

$$\frac{AI}{IL} = \frac{c}{\frac{ac}{c+b}} = \frac{c+b}{a}$$

$$\therefore I \equiv \left(\frac{ax_1 + (b+c) \left(\frac{bx_2 + cx_3}{b+c} \right)}{a + (b+c)}, \frac{ay_1 + (b+c) \left(\frac{by_2 + cy_3}{b+c} \right)}{a + (b+c)} \right)$$

$$\equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$

ILLUSTRATION 1.36

If a vertex of a triangle is $(1, 1)$, and the middle points of two sides passing through it are $(-2, 3)$ and $(5, 2)$, then find the centroid and the incentre of the triangle.

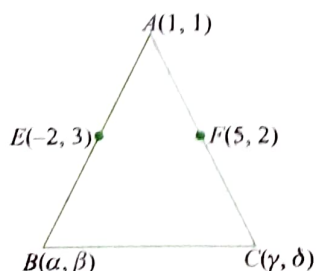
Sol. Let E and F be the midpoints of AB and AC .

Let the coordinates of B and C be (α, β) and (γ, δ) , respectively. Then

$$-2 = \frac{1+\alpha}{2}, 3 = \frac{1+\beta}{2},$$

$$5 = \frac{1+\gamma}{2}, 2 = \frac{1+\delta}{2}$$

$$\therefore \alpha = -5, \beta = 5, \gamma = 9, \delta = 3$$



Therefore, the coordinates of B and C are $(-5, 5)$ and $(9, 3)$, respectively.

Then, the centroid is

$$\left(\frac{1-5+9}{3}, \frac{1+5+3}{3} \right) \equiv \left(\frac{5}{3}, 3 \right)$$

$$\text{and } a = BC = \sqrt{(-5-9)^2 + (5-3)^2} = 10\sqrt{2}$$

$$b = CA = \sqrt{(9-1)^2 + (3-1)^2} = 2\sqrt{17}$$

$$\text{and } c = AB = \sqrt{(1+5)^2 + (1-5)^2} = 2\sqrt{13}$$

Then, the incenter is

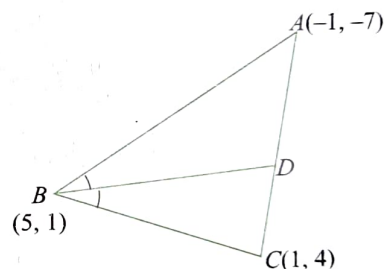
$$\left(\frac{10\sqrt{2}(1) + 2\sqrt{17}(-5) + 2\sqrt{13}(9)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}}, \frac{10\sqrt{2}(1) + 2\sqrt{17}(5) + 2\sqrt{13}(3)}{10\sqrt{2} + 2\sqrt{17} + 2\sqrt{13}} \right)$$

$$\text{i.e., } \left(\frac{5\sqrt{2} - 5\sqrt{17} + 9\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}}, \frac{5\sqrt{2} + 5\sqrt{17} + 3\sqrt{13}}{5\sqrt{2} + \sqrt{17} + \sqrt{13}} \right)$$

ILLUSTRATION 1.37

The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$. If the internal angle bisector of $\angle B$ meets the side AC in D , then find the length of AD .

Sol. The angle bisector of $\angle B$ meets the opposite side in D .



Using angle bisector theorem, we have

$$AD : DC = AB : BC$$

$$\text{Now, } AB = \sqrt{(5+1)^2 + (1+7)^2} = 10$$

$$\text{and } BC = \sqrt{(5-1)^2 + (1-4)^2} = 5$$

$$\therefore D \equiv \left(\frac{2(1) + 1(-1)}{2+1}, \frac{2(4) + 1(-7)}{2+1} \right)$$

$$\equiv \left(\frac{1}{3}, \frac{1}{3} \right)$$

$$\therefore BD = \sqrt{\left(5 - \frac{1}{3} \right)^2 + \left(1 - \frac{1}{3} \right)^2} = \frac{10\sqrt{2}}{3}$$

EXCENTRES OF TRIANGLE

An excircle or escribed circle of a triangle is a circle which lies outside the triangle such that it is tangent to one of triangle's sides and tangent to the extensions of the other two. Every triangle has three distinct excircles, one on each side.

The centre of an excircle is the intersection of the internal bisector of one angle of triangle and the external bisectors of the other two angles. The centre of excircle is called the excentre relative to the vertex opposite the side tangent to circle.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle ABC such that $BC = a$, $CA = b$ and $AB = c$.

The circle which touches the side BC , AB produced and AC produced is called the escribed circle opposite the angle A .

The bisectors of the external angle B and C meet at a point I_1 which is the centre of the escribed circle opposite the angle A .

L divides BC internally in the ratio $b : c$ and I_1 divides AL externally in the ratio $\frac{b+c}{a}$.

$$\text{Thus, point } L \equiv \left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c} \right)$$

$$\text{and } I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

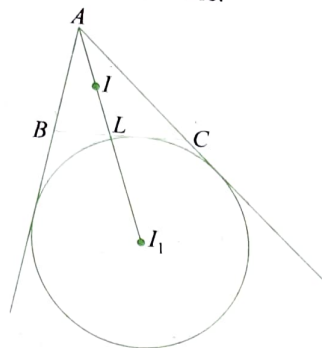
Similarly, we can find the coordinates of the other two excentres I_2 and I_3 opposite the vertices B and C , respectively.

$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right)$$

$$\text{and } I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$

CONCEPT APPLICATION EXERCISE 1.3

- If point $P(3, 2)$ divides the line segment AB internally in the ratio of $3 : 2$ and point $Q(-2, 3)$ divides AB externally in the ratio $4 : 3$, then find the coordinates of points A and B .
- If the points $(x, -1)$, $(3, y)$, $(-2, 3)$, and $(-3, -2)$ taken in order are the vertices of a parallelogram, then find the values of x and y .
- If the midpoints of the sides of a triangle are $(2, 1)$, $(-1, -3)$, and $(4, 5)$, then find the coordinates of its vertices.
- The line joining $A(b \cos \alpha, b \sin \alpha)$ and $B(a \cos \beta, a \sin \beta)$ is produced to the point $M(x, y)$ so that AM and BM are in the ratio $b : a$. Then prove that $x + y \tan[(\alpha + \beta)/2] = 0$.
- If the middle points of the sides of a triangle are $(-2, 3)$, $(4, -3)$, and $(4, 5)$, then find the centroid of the triangle.
- Find the incentre of the triangle with vertices $A(1, \sqrt{3})$, $B(0, 0)$ and $C(2, 0)$.
- If $(1, 4)$ is the centroid of a triangle and the coordinates of its any two vertices are $(4, -8)$ and $(-9, 7)$, find the area of the triangle.



- The vertices of a triangle are $A(x_1, x_1 \tan \theta_1)$, $B(x_2, x_2 \tan \theta_2)$, and $C(x_3, x_3 \tan \theta_3)$. If the circumcenter of $\triangle ABC$ coincides with the origin and $H(a, b)$ is the orthocenter, show that

$$\frac{a}{b} = \frac{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}$$

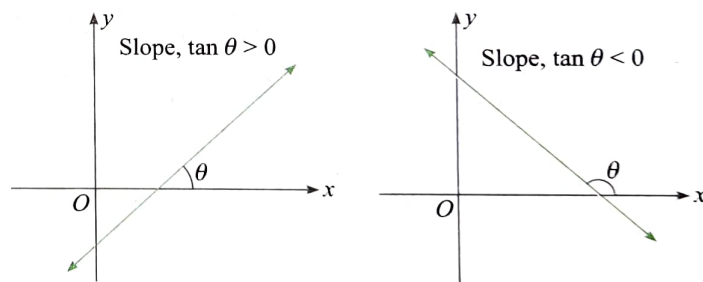
- If (x_i, y_i) , $i = 1, 2, 3$, are the vertices of an equilateral triangle such that $(x_1 + 2)^2 + (y_1 - 3)^2 = (x_2 + 2)^2 + (y_2 - 3)^2 = (x_3 + 2)^2 + (y_3 - 3)^2$, then find the value of $\frac{x_1 + x_2 + x_3}{y_1 + y_2 + y_3}$.
- The vertices of a triangle are $(at_1t_2, a(t_1 + t_2))$, $(at_2t_3, a(t_2 + t_3))$, and $(at_3t_1, a(t_3 + t_1))$. Find the orthocenter of the triangle.

ANSWERS

- $A \equiv \left(\frac{66}{17}, \frac{31}{17} \right)$ and $B \equiv \left(\frac{41}{17}, \frac{36}{17} \right)$
- $x = 2, y = 4$
- $(7, 9), (-3, -7), (1, 1)$
- $(2, 5/3)$
- $\left(1, \frac{1}{\sqrt{3}} \right)$
- 166.5 sq. unit
- $-2/3$
- $(-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$

SLOPE OF A LINE

Slope is the measure of direction in which a line is drawn. A line in a coordinate plane makes two angles with x -axis which are supplementary. The angle made by the line with positive direction of x -axis and measured anticlockwise is called inclination of the line. The trigonometric tangent of this angle is called the slope or gradient of the line.



If θ is the inclination of the line, then its slope is $\tan \theta$, generally denoted by m .

Thus, $m = \tan \theta$.

If θ is acute, then slope, $\tan \theta > 0$.

If θ is obtuse, then slope, $\tan \theta < 0$.

Note:

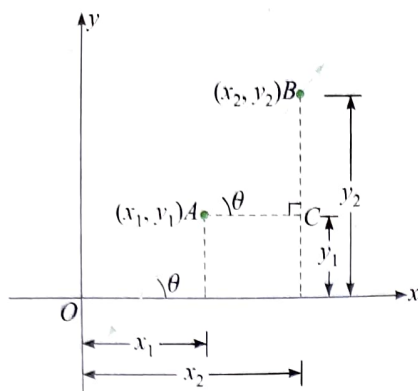
- The inclination of line parallel to x -axis is 0° . Thus, slope of horizontal line is $\tan 0^\circ = 0$.
- The inclination of line parallel to y -axis is 90° . Thus, slope of vertical line is $\lim_{\theta \rightarrow 90^\circ} \tan \theta = \infty$.

Here, we are considering limiting value of $\tan \theta$ as tangent function is not defined for angle 90° .

- The inclination of line which is equally inclined to coordinate axes is 45° or 135° . Therefore, slope of line is either -1 or $+1$.

SLOPE OF LINE JOINING GIVEN TWO POINTS

Consider a line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$.



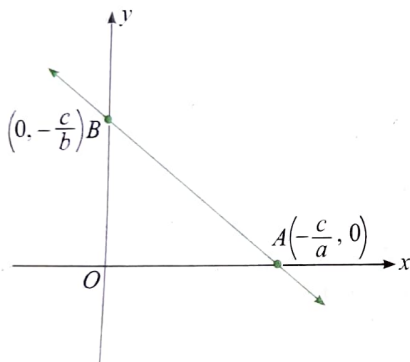
From the figure, in triangle ABC , $\tan \theta = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}$

SLOPE OF LINE HAVING GIVEN EQUATION

The general equation of a straight line is $ax + by + c = 0$.

To get the slope of the line, we require two points on the line.

This line meets axes at points $A(-c/a, 0)$ and $B(0, -c/b)$.



$$\begin{aligned} \therefore \text{Slope of line} &= \frac{-\frac{c}{b} - 0}{0 - \left(-\frac{c}{a}\right)} = -\frac{a}{b} \\ &= -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} \end{aligned}$$

Note:

- If three points A , B and C are collinear, then slope of AB = slope of BC = slope of AC
- If two lines are parallel, then their inclinations and, hence, slopes are same.

ILLUSTRATION 1.38

Determine the value of x so that the line passing through $(3, 4)$ and $(x, 5)$ makes an angle of 135° with the positive direction of x -axis.

Sol. Slope of line passing through $(3, 4)$ and $(x, 5)$ is $\frac{5-4}{x-3}$. Also, inclination of line is 135° .

$$\therefore \tan 135^\circ = \frac{5-4}{x-3}$$

$$\begin{aligned} \Rightarrow -1 &= \frac{5-4}{x-3} \\ \Rightarrow -x+3 &= 1 \\ \Rightarrow x &= 2 \end{aligned}$$

ILLUSTRATION 1.39

Which line is having the greatest inclination with the positive direction of the x -axis?

- Line joining the points $(1, 3)$ and $(4, 7)$
- Line $3x - 4y + 3 = 0$

Sol.

- Slope of the line joining the points $A(1, 3)$ and $B(4, 7)$ is

$$\frac{7-3}{4-1} = \frac{4}{3} = \tan \alpha$$

- Slope of the line $3x - 4y + 3 = 0$ is

$$-\frac{3}{-4} = \frac{3}{4} = \tan \beta$$

$$\begin{aligned} \text{Now, } \tan \alpha &> \tan \beta \\ \text{or } \alpha &> \beta \end{aligned}$$

Thus, (i) is having the greatest inclination with the positive direction of the x -axis.

ILLUSTRATION 1.40

If the points $(2, 3)$, $(1, 1)$, and $(x, 3x)$ are collinear, then find the value of x , using slope method.

Sol. Given points $A(2, 3)$, $B(1, 1)$, and $C(x, 3x)$ are collinear. Then

$$\text{Slope of } AB = \text{Slope of } BC$$

$$\text{or } \frac{1-3}{1-2} = \frac{3x-1}{x-1}$$

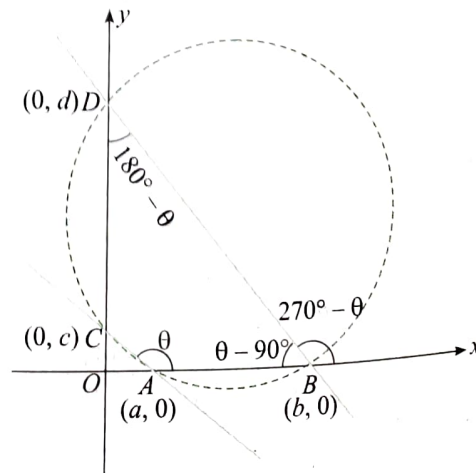
$$\text{or } 3x-1 = 2x-2$$

$$\text{or } x = -1$$

ILLUSTRATION 1.41

If the points $(a, 0)$, $(b, 0)$, $(0, c)$ and $(0, d)$ are concyclic ($a, b, c, d > 0$), then prove that $ab = cd$.

Sol. Without loss of generality, assume that $b > a$ and $d > c$. Since $ABCD$ is cyclic quadrilateral, sum of opposite angles is 180° .



i.e., $\angle BAC + \angle CDB = 180^\circ$

Let $\angle BAC = \theta$

$\therefore \angle CDB = 180^\circ - \theta$ (= inclination of line AC)

Therefore, inclination of line BD is $270^\circ - \theta$.

Slope of line AC,

$$\tan \theta = -\frac{c}{a} \quad \dots(1)$$

Slope of BD, $\tan (270^\circ - \theta) = -\frac{d}{b}$

or $\cot \theta = -\frac{d}{b} \quad \dots(2)$

From (1) and (2), we get

$$(\tan \theta)(\cot \theta) = \left(-\frac{c}{a}\right)\left(-\frac{d}{b}\right)$$

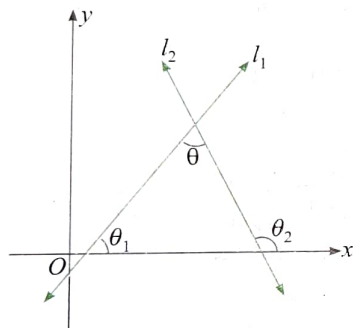
or $ab = cd$

ANGLE BETWEEN TWO LINES

Let us get the formula for the angle between two lines having slopes m_1 and m_2 .

Let inclinations of lines l_1 and l_2 be θ_1 and θ_2 , respectively.

So, $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$



As shown in the figure, one of the angles between lines l_1 and l_2 is θ .

Clearly, $\theta_2 = \theta + \theta_1$

or $\theta = \theta_2 - \theta_1$

$\therefore \tan \theta = \tan (\theta_2 - \theta_1)$

$$\Rightarrow \tan \theta = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1}$$

$$\Rightarrow \tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Another angle between lines is $\theta' = (180^\circ - \theta)$.

$\therefore \tan \theta' = \tan (180^\circ - \theta)$

$$= -\tan \theta = -\frac{m_2 - m_1}{1 + m_1 m_2}$$

Thus, if angle between lines is θ , then

$$\tan \theta = \pm \frac{m_2 - m_1}{1 + m_1 m_2}$$

If θ is acute angle between lines, then $\theta = \tan^{-1} \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$.

If θ is obtuse angle between lines,

then $\theta = 180^\circ - \tan^{-1} \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$.

Special cases:

(i) *Parallel lines:* If l_1 and l_2 are parallel, then $m_1 = m_2$. So, $\tan \theta = 0$ or $\theta = 0^\circ$.

(ii) *Perpendicular lines:* If l_1 and l_2 are perpendicular, then

$$\tan 90^\circ = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|. \text{ So, } 1 + m_1 m_2 = 0 \text{ or } m_1 m_2 = -1.$$

Thus, product of slopes of two perpendicular lines is -1 .

If m is the slope of one line then slope of the perpendicular line is $-\frac{1}{m}$.

ILLUSTRATION 1.42

If $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$ are three points, find the angle between BA and BC .

Sol. Let m_1 and m_2 be the slopes of BA and BC , respectively.

$$\text{Then, } m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}$$

$$m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

Acute angle between lines is

$$\begin{aligned} \theta &= \tan^{-1} \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \\ &= \tan^{-1} \left| \frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}} \right| \\ &= \tan^{-1} \left| \frac{\frac{10}{8}}{\frac{15}{8}} \right| = \tan^{-1} \frac{2}{3} \end{aligned}$$

Therefore, obtuse angle between lines is $180^\circ - \tan^{-1} \frac{2}{3}$.

ILLUSTRATION 1.43

Angle of a line with the positive direction of the x -axis is θ (θ is a cute). The line is rotated about some point on it in anticlockwise direction by angle 45° and its slope becomes 3. Find the angle θ .

Sol. Originally, the slope of the line is $\tan \theta = m$.

Now, the slope of the line after rotation is 3.

Angle between the old position and the new position of lines is 45° . Therefore, we have

$$\tan 45^\circ = \frac{3 - m}{1 + 3m}$$

or $1 + 3m = 3 - m$

or $4m = 2$

or $m = \frac{1}{2} = \tan \theta$

or $\theta = \tan^{-1} \left(\frac{1}{2} \right)$

ILLUSTRATION 1.44

Let $A(6, 4)$ and $B(2, 12)$ be two given points. Find the slope of a line perpendicular to AB .

Sol. Let m be the slope of AB . Then,

$$m = \frac{12-4}{2-6} = \frac{8}{-4} = -2$$

So, the slope of a line perpendicular to AB is

$$-\frac{1}{m} = \frac{1}{2}$$

ILLUSTRATION 1.45

If line $3x - ay - 1 = 0$ is parallel to the line $(a+2)x - y + 3 = 0$ then find the values of a .

Sol. The slope of the line $3x - ay - 1 = 0$ is $3/a$.

The slope of the line $(a+2)x - y + 3 = 0$ is $a+2$.

Since the lines are parallel, we have

$$a+2 = \frac{3}{a}$$

$$\text{or } a^2 + 2a - 3 = 0$$

$$\text{or } (a-1)(a+3) = 0$$

$$\text{or } a = 1 \text{ or } a = -3$$

ILLUSTRATION 1.46

If $A(2, -1)$ and $B(6, 5)$ are two points, then find the ratio in which the foot of the perpendicular from $(4, 1)$ to AB divides it.

Sol. Let $\frac{AM}{MB} = \frac{\lambda}{1}$

Therefore, the coordinates of M are

$$\left(\frac{6\lambda+2}{\lambda+1}, \frac{5\lambda-1}{\lambda+1} \right)$$

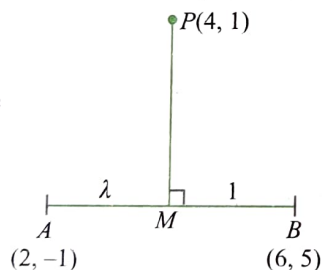
$$\therefore PM \perp AB$$

$$\text{or } \left(\frac{5\lambda-1}{\lambda+1} - 1 \right) \left\{ \frac{5-(-1)}{6-2} \right\} = -1$$

$$\text{or } \left(\frac{2\lambda-1}{\lambda-1} \right) \left(\frac{3}{2} \right) = -1$$

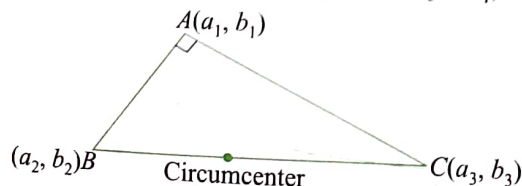
$$\text{or } 6\lambda - 3 = 2 - 2\lambda$$

$$\text{or } \lambda = \frac{5}{8}$$

**ILLUSTRATION 1.47**

If $(b_2 - b_1)(b_3 - b_1) + (a_2 - a_1)(a_3 - a_1) = 0$, then prove that the circumcenter of the triangle having vertices (a_1, b_1) , (a_2, b_2) , and (a_3, b_3) is $\left(\frac{a_2 + a_3}{2}, \frac{b_2 + b_3}{2} \right)$.

Sol. Given $(b_2 - b_1)(b_3 - b_1) + (a_2 - a_1)(a_3 - a_1) = 0$



$$\text{or } (b_2 - b_1)(b_3 - b_1) = -(a_2 - a_1)(a_3 - a_1)$$

$$\text{or } \left(\frac{b_2 - b_1}{a_2 - a_1} \right) \left(\frac{b_3 - b_1}{a_3 - a_1} \right) = -1$$

$$\text{or } (\text{slope of } AB) \times (\text{slope of } AC) = -1$$

Therefore, the triangle is right-angled at point (a_1, b_1) .

Hence, the circumcenter is the midpoint of (a_2, b_2) and (a_3, b_3)

$$\text{which is } \left(\frac{a_2 + a_3}{2}, \frac{b_2 + b_3}{2} \right).$$

ILLUSTRATION 1.48

Find the orthocenter of $\triangle ABC$ with vertices $A(1, 0)$, $B(-2, 1)$, and $C(5, 2)$.

Sol. Let the orthocenter be $H(h, k)$.

$$\text{Slope of } BC = \frac{2-1}{5-(-2)} = \frac{1}{7}$$

$$\text{Slope of } AC = \frac{2-0}{5-1} = \frac{1}{2}$$

$$AH \perp BC$$

$$\therefore \text{Slope of } AH = -7$$

$$\text{or } \frac{k-0}{h-1} = -7$$

$$\therefore -7h + 7 = k \text{ or } 7h + k = 7 \quad (1)$$

Also, $BH \perp AC$

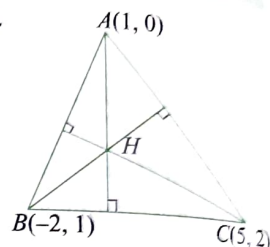
$$\therefore \text{Slope of } BH = -2$$

$$\text{or } \frac{k-1}{h+2} = -2$$

$$\therefore k - 1 = -2h - 4 \text{ or } 2h + k = -3 \quad (2)$$

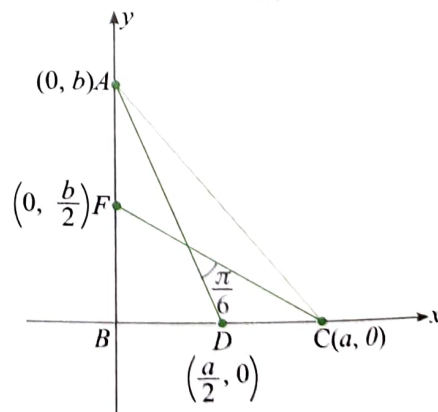
Solving (1) and (2), we get $h = 2$ and $k = -7$.

Hence, the orthocenter is $(2, -7)$.

**ILLUSTRATION 1.49**

Two medians drawn from acute angles of a right angled triangles intersect at an angle of $\pi/6$. If the length of the hypotenuse of the triangle is 3 units, then find the area of the triangle.

Sol.



In the figure, ABC is right angled triangle.

Given that $AC = 3$.

$$\text{So, } a^2 + b^2 = 9$$

AD and CF are two medians.

$$\text{So, } D \equiv \left(\frac{a}{2}, 0 \right) \text{ and } F \equiv \left(0, \frac{b}{2} \right)$$

$$\text{Slope of } AD = \frac{-2b}{a}$$

$$\text{Slope of } CF = \frac{-b}{2a}$$

$$\therefore \tan \frac{\pi}{6} = \frac{\left| \frac{-2b}{a} + \frac{b}{2a} \right|}{\left| 1 + \frac{b^2}{a^2} \right|} = \frac{\left| \frac{-3ab}{2(a^2 + b^2)} \right|}{\frac{9 \times 2}{9 \times 2}} = \frac{3ab}{9 \times 2}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{ab}{6}$$

$$\Rightarrow ab = 2\sqrt{3}$$

$$\therefore \text{Area of triangle } ABC = \frac{1}{2} ab = \sqrt{3}$$

CONCEPT APPLICATION EXERCISE 1.4

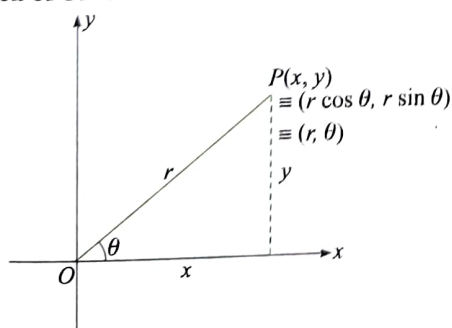
- The line joining the points $(x, 2x)$ and $(3, 5)$ makes an obtuse angle with the positive direction of the x -axis. Then find the values of x .
- If the line passing through $(4, 3)$ and $(2, k)$ is parallel to the line $y = 2x + 3$, then find the value of k .
- Triangle ABC lies in the cartesian plane and has an area of 70 sq. units. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively. The line containing the median to the side BC has slope -5 . Find the possible coordinates of point A .
- $ABCD$ is a rhombus of side 10 units where slope of AB is $4/3$ and slope of AD is $3/4$. If coordinates of A are $(0, 0)$, then find the coordinates of B, C and D .
- The line joining the points $A(2, 1)$ and $B(3, 2)$ is perpendicular to the line $(a^2)x + (a+2)y + 2 = 0$. Find the values of a .
- Find the angle between the line joining the points $(1, -2)$, $(3, 2)$ and the line $x + 2y - 7 = 0$.
- The orthocenter of $\triangle ABC$ with vertices $B(1, -2)$ and $C(-2, 0)$ is $H(3, -1)$. Find the vertex A .
- The medians AD and BE of the triangle with vertices $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are mutually perpendicular. Prove that $a^2 = 2b^2$.

ANSWERS

- $x \in (5/2, 3)$
- -1
- $(15, 32)$ or $(20, 7)$
- $B(6, 8), C(14, 14), D(8, 6)$
- $a = 2, -1$
- $\pi/2$
- $(3/7, -34/7)$

POLAR COORDINATES

Consider point $P(x, y)$ in the first quadrant. Let distance OP be r and inclination of OP be θ .

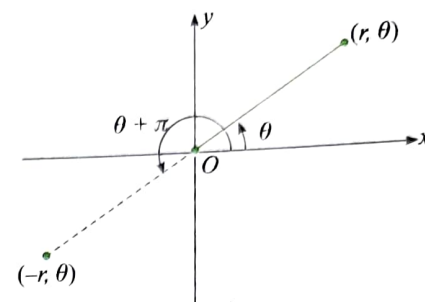


$$\therefore x = r \cos \theta \text{ and } y = r \sin \theta$$

$$\text{Thus, } (x, y) \equiv (r \cos \theta, r \sin \theta)$$

The polar coordinates of point P are defined as (r, θ) , where r is the distance of P from the origin (pole) and θ is the angle of OP with positive direction of x -axis. For point P , $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$.

We extend the definition of polar coordinates (r, θ) to the case in which r is negative by agreeing that the points $(-r, \theta)$ and (r, θ) lie on the same line through origin O and at the same distance $|r|$ from O , but on opposite sides of O .



If $r > 0$, the point (r, θ) lies in the same quadrant as θ and if $r < 0$, then it lies in the quadrant on the opposite side of the pole.

From the figure, $(-r, \theta)$ is the same as $(r, \theta + \pi)$.

In the Cartesian coordinate system, every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ can be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$, etc.

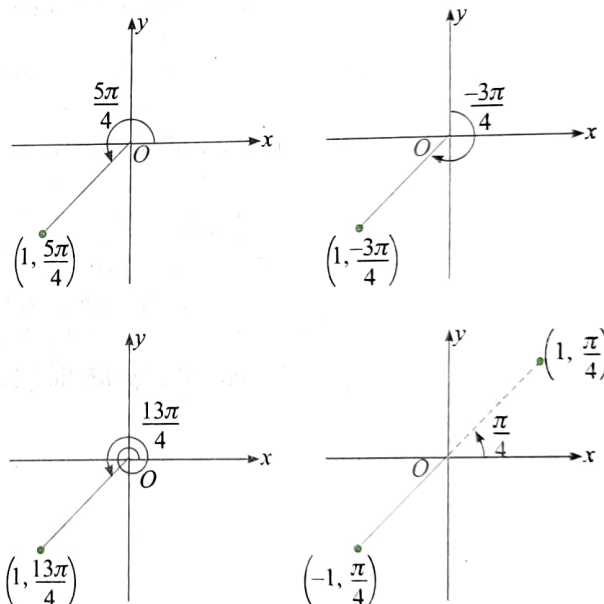
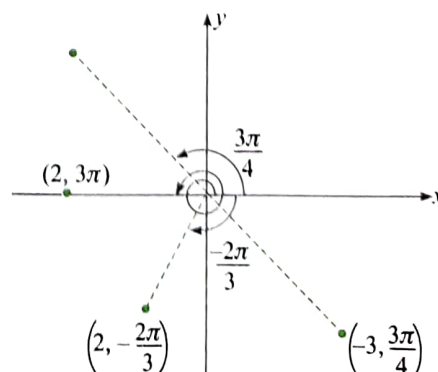


ILLUSTRATION 1.50

Plot the points whose polar coordinates are given below.

- $(2, 3\pi)$
- $(2, -2\pi/3)$
- $(-3, 3\pi/4)$

Sol.



COORDINATE CONVERSION

The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows:

Polar to rectangular	Rectangular to Polar
$x = r \cos \theta$	$\tan \theta = \frac{y}{x}$
$y = r \sin \theta$	$r^2 = x^2 + y^2$

ILLUSTRATION 1.51

Convert the following points from polar coordinates to the corresponding Cartesian coordinates.

- (i) $(2, \pi/3)$ (ii) $(0, \pi/2)$ (iii) $(-\sqrt{2}, \pi/4)$

Sol.

- (i) $(2, \pi/3) \equiv (r, \theta)$

$$\text{We have } x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \times \frac{1}{2} = 1$$

$$\text{and } y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in the Cartesian coordinates.

- (ii) Here, $x = 0 \cdot \cos(\pi/2) = 0$

$$\text{and } y = 0 \cdot \sin(\pi/2) = 0$$

So, $(0, \pi/2)$ is equivalent to $(0, 0)$ in the Cartesian coordinates.

$$(iii) \quad x = -\sqrt{2} \cos \frac{\pi}{4} = -\sqrt{2} \left(\frac{1}{\sqrt{2}} \right) = -1$$

$$y = -\sqrt{2} \sin \frac{\pi}{4} = -\sqrt{2} \left(\frac{1}{\sqrt{2}} \right) = -1$$

So, the equivalent Cartesian coordinates for the given polar coordinates are $(-1, -1)$.

ILLUSTRATION 1.52

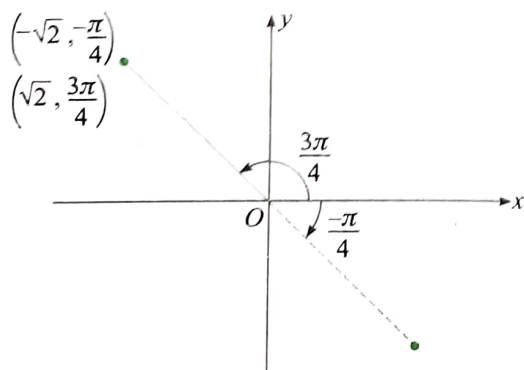
Convert the following cartesian coordinates to corresponding polar coordinates using positive r and negative r .

- (i) $(-1, 1)$ (ii) $(2, -3)$

Sol.

- (i) For the second quadrant point $(x, y) \equiv (-1, 1)$, we have

$$\tan \theta = \frac{y}{x} = -1 \Rightarrow \theta = \frac{3\pi}{4}$$



Using positive r , we have

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

So, one set of polar coordinates is $(r, \theta) \equiv (\sqrt{2}, 3\pi/4)$.

Using negative r , we have coordinates $\equiv (-\sqrt{2}, -\pi/4)$.

- (ii) For the fourth quadrant point $(x, y) \equiv (2, -3)$, we have

$$\tan \theta = \frac{-3}{2}$$

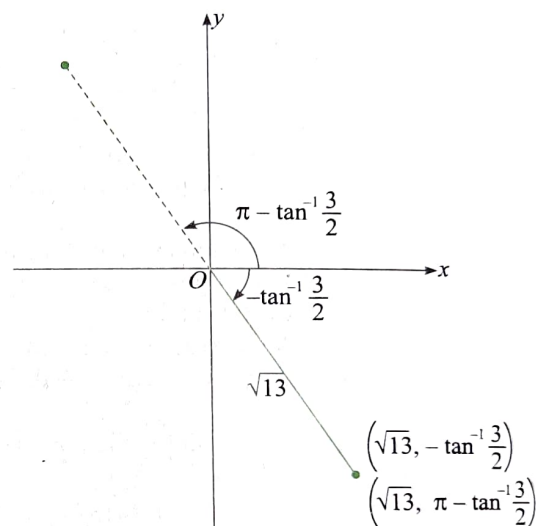
$$\Rightarrow \theta = \tan^{-1} \left(\frac{-3}{2} \right) = -\tan^{-1} \frac{3}{2}$$

Using positive r , we have

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$$

So, one set of polar coordinates is

$$(r, \theta) \equiv \left(\sqrt{13}, -\tan^{-1} \frac{3}{2} \right)$$



Using negative r , we have coordinates

$$= \left(-\sqrt{13}, \pi - \tan^{-1} \frac{3}{2} \right)$$

ILLUSTRATION 1.53

Convert Cartesian equation $y = 10$ into a polar equation.

$$\text{Sol. } y = 10$$

$$\text{or } r \sin \theta = 10$$

$$\text{or } r = \frac{10}{\sin \theta}$$

$$\text{or } r = 10 \operatorname{cosec} \theta$$

ILLUSTRATION 1.54

Express the polar equation $r = 2 \cos \theta$ in rectangular coordinates.

Sol. We use the formulas $r^2 = x^2 + y^2$ and $x = r \cos \theta$.

Given $r = 2 \cos \theta$. Therefore,

$$r^2 = 2r \cos \theta$$

$$\text{or } x^2 + y^2 = 2x$$

$$\text{or } x^2 - 2x + y^2 = 0$$

$$\text{or } x^2 - 2x + 1 + y^2 = 1$$

$$\text{or } (x-1)^2 + y^2 = 1$$

ILLUSTRATION 1.55

Convert $x^2 - y^2 = 4$ into a polar equation.

Sol. $x^2 - y^2 = 4$

$$\text{or } (r \cos \theta)^2 - (r \sin \theta)^2 = 4$$

$$\text{or } r^2 \cos^2 \theta - r^2 \sin^2 \theta = 4$$

$$\text{or } r^2 (\cos^2 \theta - \sin^2 \theta) = 4$$

$$\text{or } r^2 \cos 2\theta = 4$$

ILLUSTRATION 1.56

Convert $r \sin \theta = r \cos \theta + 4$ into its equivalent Cartesian equation.

Sol. Given $r \sin \theta = r \cos \theta + 4$. Therefore,
 $y = x + 4$

ILLUSTRATION 1.57

Convert $r = \operatorname{cosec} \theta e^{r \cos \theta}$ into its equivalent Cartesian equation.

Sol. $r = \operatorname{cosec} \theta e^{r \cos \theta} = \frac{1}{\sin \theta} e^{r \cos \theta}$

$$\text{or } r \sin \theta = e^{r \cos \theta}$$

$$\text{or } y = e^x$$

ILLUSTRATION 1.58

Find the maximum distance of any point on the curve $x^2 + 2y^2 + 2xy = 1$ from the origin.

Sol. Let the polar coordinates of any point on the curve be $P(r, \theta)$.

Then its Cartesian coordinates are $P(r \cos \theta, r \sin \theta)$.

Point P lies on the curve. Then

$$r^2 \cos^2 \theta + 2r^2 \sin^2 \theta + 2r^2 \sin \theta \cos \theta = 1$$

$$\text{or } r^2 = \frac{1}{\cos^2 \theta + 2 \sin^2 \theta + \sin 2\theta}$$

$$= \frac{1}{\sin^2 \theta + 1 + \sin 2\theta}$$

$$= \frac{2}{1 - \cos 2\theta + 2 + 2 \sin 2\theta}$$

$$= \frac{2}{3 - \cos 2\theta + 2 \sin 2\theta}$$

Now, $-\sqrt{5} \leq -\cos 2\theta + 2 \sin 2\theta \leq \sqrt{5}$

or $3 - \sqrt{5} \leq -\cos 2\theta + 2 \sin 2\theta \leq 3 + \sqrt{5}$

or $r_{\max}^2 = \frac{2}{3 - \sqrt{5}}$

or $r_{\max} = \frac{\sqrt{2}}{\sqrt{3 - \sqrt{5}}}$

CONCEPT APPLICATION EXERCISE 1.5

- Convert the following polar coordinates to its equivalent Cartesian coordinates.
 (i) $(2, \pi)$ (ii) $(\sqrt{3}, \pi/6)$
- Convert the following Cartesian coordinates to the corresponding polar coordinates using positive r .
 (i) $(1, -1)$ (ii) $(-3, -4)$
- Convert $2x^2 + 3y^2 = 6$ into a polar equation.
- Convert $r = 4 \tan \theta \sec \theta$ into its equivalent Cartesian equation.
- Find the minimum distance of any point on the line $3x + 4y - 10 = 0$ from the origin using polar coordinates.

ANSWERS

- (i) $(-2, 0)$ (ii) $(3/2, \sqrt{3}/2)$
- (i) $(\sqrt{2}, -\pi/4)$ or $(\sqrt{2}, 7\pi/4)$
 (ii) $(5, \pi + \tan^{-1} 4/3)$ or $(5, -\pi + \tan^{-1} 4/3)$
- $r^2(2 + \sin^2 \theta) = 6$
- $x^2 = 4y$ 5. 2 units

LOCUS AND ITS EQUATION

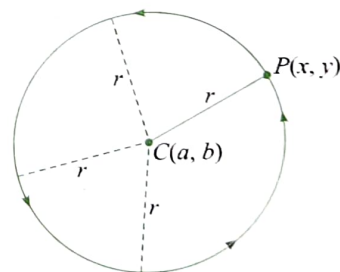
The curve described by a variable point on the plane which moves under given condition or a set of conditions is called its locus.

For example, let $C(a, b)$ is a fixed point and $P(x, y)$ is a variable point in the same plane. If P moves in such a way that the distance CP is constant r , then point P traces out a circle whose centre is C and radius r .

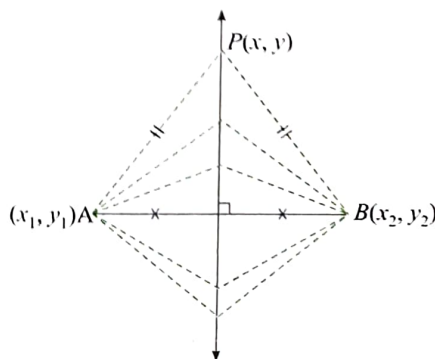
Since $CP = r$, we have

$$(x - a)^2 + (y - b)^2 = r^2,$$

which is the equation of locus of point P .



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two fixed points in a plane. If $P(x, y)$ is a variable point in the same plane such that $AP = BP$, then P lies on the perpendicular bisector of AB or locus of point P is line which is the perpendicular bisector of AB .



To get the equation of locus, we have

$$AP^2 = BP^2$$

$$\therefore (x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$$

$$\text{or } 2(x_2 - x_1)x + 2(y_2 - y_1)y = x_2^2 + y_2^2 - x_1^2 - y_1^2$$

To find the equation of locus of variable point we often let the variable point as (h, k) and find the relation between h and k using given condition(s). Replacing h by x and k by y , we get the required equation of locus.

ILLUSTRATION 1.59

The sum of the squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.

Sol. Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$, $B(-a, 0)$ be the given points. Then, we have

$$PA^2 + PB^2 = 2c^2 \quad (\text{Given})$$

$$\text{or } (h - a)^2 + (k - 0)^2 + (h + a)^2 + (k - 0)^2 = 2c^2$$

$$\text{or } 2h^2 + 2k^2 + 2a^2 = 2c^2$$

$$\text{or } h^2 + k^2 = c^2 - a^2$$

Hence, the equation to locus (h, k) is $x^2 + y^2 = c^2 - a^2$.

ILLUSTRATION 1.60

Find the locus of a point, so that the join of $(-5, 1)$ and $(3, 2)$ subtends a right angle at the moving point.

Sol. Let $P(h, k)$ be a moving point and let $A(-5, 1)$ and $B(3, 2)$ be the given points.

From the given condition, we have

$$\angle APB = 90^\circ$$

Therefore, $\triangle APB$ is a right-angled triangle. Hence,

$$AB^2 = AP^2 + PB^2$$

$$\text{or } (3 + 5)^2 + (2 - 1)^2 = (h + 5)^2 + (k - 1)^2 + (h - 3)^2 + (k - 2)^2$$

$$\text{or } 65 = 2(h^2 + k^2 + 2h - 3k) + 39$$

$$\text{or } h^2 + k^2 + 2h - 3k - 13 = 0$$

Hence, the locus of (h, k) is $x^2 + y^2 + 2x - 3y - 13 = 0$.

ILLUSTRATION 1.61

Find the locus of a point such that the sum of its distance from the points $(0, 2)$ and $(0, -2)$ is 6.

Sol. Let $P(h, k)$ be any point on the locus and let $A(0, 2)$ and $B(0, -2)$ be the given points.

By the given condition, we get

$$PA + PB = 6$$

$$\text{or } \sqrt{(h - 0)^2 + (k - 2)^2} + \sqrt{(h - 0)^2 + (k + 2)^2} = 6$$

$$\text{or } \sqrt{h^2 + (k - 2)^2} = 6 - \sqrt{h^2 + (k + 2)^2}$$

$$\text{or } h^2 + (k - 2)^2 = 36 - 12\sqrt{h^2 + (k + 2)^2} + h^2 + (k + 2)^2$$

$$\text{or } -8k - 36 = -12\sqrt{h^2 + (k + 2)^2}$$

$$\text{or } (2k + 9) = 3\sqrt{h^2 + (k + 2)^2}$$

$$\text{or } (2k + 9)^2 = 9\{h^2 + (k + 2)^2\}$$

$$\text{or } 4k^2 + 36k + 81 = 9h^2 + 9k^2 + 36k + 36$$

$$\text{or } 9h^2 + 5k^2 = 45$$

Hence, the locus of (h, k) is $9x^2 + 5y^2 = 45$.

ILLUSTRATION 1.62

AB is a variable line sliding between the coordinate axes in such a way that A lies on the x -axis and B lies on the y -axis. If P is a variable point on AB such that $PA = b$, $PB = a$, and $AB = a + b$, find the equation of the locus of P .

Sol. Let $P(h, k)$ be a variable point on AB such that $\angle OAB = \theta$.

Here, θ is a variable.

From triangles ALP and PMB , we have

$$\sin \theta = \frac{k}{b} \quad (1)$$

$$\cos \theta = \frac{h}{a} \quad (2)$$

Here, θ is a variable. So, we have to eliminate θ . Squaring (1) and (2) and adding, we get

$$\frac{k^2}{b^2} + \frac{h^2}{a^2} = 1$$

Hence, the locus of (h, k) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

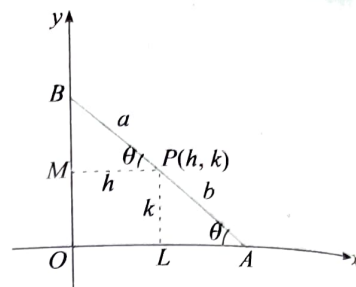
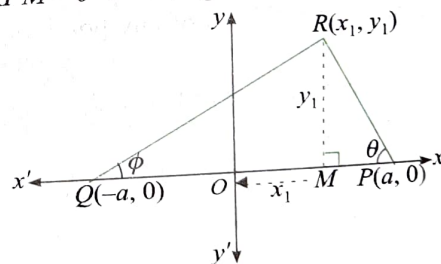


ILLUSTRATION 1.63

Two points $P(a, 0)$ and $Q(-a, 0)$ are given. R is a variable point above x -axis such that $\angle RPQ - \angle RQP$ is a positive constant 2α . Find the locus of the point R .

Sol. Let $R \equiv (x_1, y_1)$.

Also, let, $\angle RPM = \theta$ and $\angle RQM = \phi$.



$$\text{In } \triangle RMP, \tan \theta = \frac{RM}{MP} = \frac{y_1}{a - x_1} \quad (1)$$

$$\text{In } \triangle RQM, \tan \phi = \frac{RM}{QM} = \frac{y_1}{a + x_1} \quad (2)$$

But given $\angle RPQ - \angle RQP = 2\alpha$ (constant). Therefore,

$$\theta - \phi = 2\alpha$$

$$\text{or } \tan(\theta - \phi) = \tan 2\alpha$$

$$\text{or } \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} = \tan 2\alpha$$

$$\text{or } \frac{\frac{y_1}{a - x_1} - \frac{y_1}{a + x_1}}{1 + \frac{y_1}{a - x_1} \cdot \frac{y_1}{a + x_1}} = \tan 2\alpha$$

$$\text{or } \frac{2x_1 y_1}{a^2 - x_1^2 + y_1^2} = \tan 2\alpha$$

$$\text{or } a^2 - x_1^2 + y_1^2 = 2x_1y_1 \cot 2\alpha$$

$$\text{or } x_1^2 - y_1^2 + 2x_1y_1 \cot 2\alpha = a^2$$

Hence, the locus of the point $R(x_1, y_1)$ is

$$x^2 - y^2 + 2xy \cot 2\alpha = a^2$$

FINDING EQUATION OF LOCUS BY ELIMINATION OF GIVEN VARIABLE

In some cases, we are given coordinates of variable point in terms of some parameter (variable). To get the locus of such point we eliminate the parameter. For example, if we need to find the locus of variable point $(\cos \theta, \sin \theta)$, we let $x = \cos \theta$ and $y = \sin \theta$. To establish the relation between x and y , we eliminate parameter θ , which we can do here by squaring and adding.

So, required equation of locus is $x^2 + y^2 = 1$.

ILLUSTRATION 1.64

If the coordinates of a variable point P are $(a \cos \theta, b \sin \theta)$, where θ is a variable quantity, then find the locus of P .

Sol. Let $P \equiv (x, y)$. According to the question,

$$x = a \cos \theta \quad (1)$$

$$y = b \sin \theta \quad (2)$$

Squaring and adding (1) and (2), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \theta + \sin^2 \theta$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ILLUSTRATION 1.65

Find the locus of the point $(t^2 - t + 1, t^2 + t + 1)$, $t \in R$.

Sol.

$$\text{Let } (h, k) \equiv (t^2 - t + 1, t^2 + t + 1)$$

$$\text{or } h = t^2 - t + 1 \text{ and } k = t^2 + t + 1$$

$$\text{or } k - h = 2t$$

$$\text{or } t = \frac{k - h}{2}$$

$$\text{or } h = \left(\frac{k - h}{2} \right)^2 - \left(\frac{k - h}{2} \right) + 1$$

The required locus is

$$x = \left(\frac{y - x}{2} \right)^2 - \left(\frac{y - x}{2} \right) + 1$$

ILLUSTRATION 1.66

The line segment joining $A(5, 0)$ and $B(10 \cos \theta, 10 \sin \theta)$ is divided internally in the ratio $2 : 3$ at P . If θ varies, then find the locus of P .

Sol. Let point P be (h, k) .



$$\text{Here, } h = \frac{2(10 \cos \theta) + 3(5)}{2 + 3} = 4 \cos \theta + 3$$

$$\text{and } k = \frac{2(10 \sin \theta) + 3(0)}{2 + 3} = 4 \sin \theta$$

$$\therefore (h - 3)^2 + k^2 = 16$$

Therefore, locus of $P(h, k)$ is $(x - 3)^2 + y^2 = 16$.

ILLUSTRATION 1.67

If $A(\cos \alpha, \sin \alpha)$, $B(\sin \alpha, -\cos \alpha)$, $C(1, 2)$ are the vertices of $\triangle ABC$, then as α varies, find the locus of its centroid.

Sol. Let (h, k) be the centroid of the triangle. Then,

$$h = \frac{\cos \alpha + \sin \alpha + 1}{3}$$

$$\text{and } k = \frac{\sin \alpha - \cos \alpha + 2}{3}$$

$$\text{or } 3h - 1 = \cos \alpha + \sin \alpha$$

$$\text{and } 3k - 2 = \sin \alpha - \cos \alpha$$

Squaring and adding, we get

$$(3h - 1)^2 + (3k - 2)^2 = 2$$

$$\text{or } 9(h^2 + k^2) - 6h - 12k + 3 = 0$$

$$\text{or } 3(h^2 + k^2) - 2h - 4k + 1 = 0$$

Therefore, the locus of the centroid is $3(x^2 + y^2) - 2x - 4y + 1 = 0$.

CONCEPT APPLICATION EXERCISE 1.6

- Find the locus of a point whose distance from $(a, 0)$ is equal to its distance from the y -axis.
- The coordinates of the points A and B are $(a, 0)$ and $(-a, 0)$, respectively. If a point P moves so that $PA^2 - PB^2 = 2k^2$, when k is constant, then find the equation to the locus of the point P .
- Let $A(2, -3)$ and $B(-2, 1)$ be the vertices of $\triangle ABC$. If the centroid of the triangle moves on the line $2x + 3y = 1$, then find the locus of the vertex C .
- Q is a variable point whose locus is $2x + 3y + 4 = 0$; corresponding to a particular position of Q , P is the point of section of OQ , O being the origin, such that $OP : PQ = 3 : 1$. Find the locus of P .
- Find the locus of the middle point of the portion of the line $x \cos \alpha + y \sin \alpha = p$ which is intercepted between the axes, given that p remains constant.
- Find the locus of the point of intersection of lines $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = b$ (α is a variable).
- A point moves such that the area of the triangle formed by it with the points $(1, 5)$ and $(3, -7)$ is 21 sq. units. Then, find the locus of the point.
- A variable line through point $P(2, 1)$ meets the axes at A and B . Find the locus of the circumcenter of triangle OAB (where O is the origin).
- A straight line is drawn through $P(3, 4)$ to meet the axis of x and y at A and B , respectively. If the rectangle $OACB$ is completed, then find the locus of C .

ANSWERS

1. $y^2 - 2ax + a^2 = 0$
2. $2ax + k^2 = 0$
3. $2x + 3y = 9$
4. $2x + 3y + 3 = 0$
5. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$
6. $x^2 + y^2 = a^2 + b^2$
7. $6x + y = 32$ or $6x + y = -10$
8. $x + 2y - 2xy = 0$
9. $\frac{3}{x} + \frac{4}{y} = 1$

Solved Examples

EXAMPLE 1.1

If a, b, c are the p th, q th, r th terms, respectively, of an HP, show that the points (bc, p) , (ca, q) , and (ab, r) are collinear.

Sol. Let the first term be A and the common difference be D of the corresponding AP.

Given p th term of HP = a

$$\therefore p\text{th term of AP} = \frac{1}{a}$$

$$\therefore A + (p-1)D = \frac{1}{a}$$

$$\text{Similarly, } A + (q-1)D = \frac{1}{b}$$

$$A + (r-1)D = \frac{1}{c}$$

Subtracting, we get

$$(p-q)D = \frac{1}{a} - \frac{1}{b}$$

$$\text{and } (q-r)D = \frac{1}{b} - \frac{1}{c}$$

Dividing, we get

$$\frac{p-q}{q-r} = \frac{bc-ac}{ac-ab} \quad (1)$$

The given points are $A(bc, p)$, $B(ca, q)$, and $C(ab, r)$.

$$\text{Slope of } AB = \frac{q-p}{ca-bc}$$

$$\text{and Slope of } BC = \frac{r-q}{ab-ca} = \frac{q-p}{ca-bc} \quad [\text{Using (1)}]$$

$$\therefore \text{Slope of } BC = \text{Slope of } AB$$

Therefore, A, B , and C are collinear.

EXAMPLE 1.2

Prove that the circumcenter, orthocenter, incenter, and centroid of the triangle formed by the points $A(-1, 11)$, $B(-9, -8)$, and $C(15, -2)$ are collinear, without actually finding any of them.

Sol. The given points are $A(-1, 11)$, $B(-9, -8)$, and $C(15, -2)$.

$$AB = \sqrt{64+361} = \sqrt{425}$$

$$BC = \sqrt{576+36} = \sqrt{612}$$

$$AC = \sqrt{256+169} = \sqrt{425}$$

Thus, $AB = AC \neq BC$

Therefore, triangle is isosceles, and in an isosceles triangle, all centers are collinear.

EXAMPLE 1.3

A rod of length k slides in a vertical plane, its ends touching the coordinate axes. Prove that the locus of the foot of the perpendicular from the origin to the rod is $(x^2 + y^2)^3 = k^2 x^2 y^2$.

Sol. (Slope of OP) · (Slope of BP) = -1

$$\text{or } \frac{\beta}{\alpha} \cdot \frac{\beta-b}{\alpha-0} = -1$$

$$\text{or } b = \frac{\alpha^2 + \beta^2}{\beta}$$

and (Slope of OP) · (Slope of AP) = -1

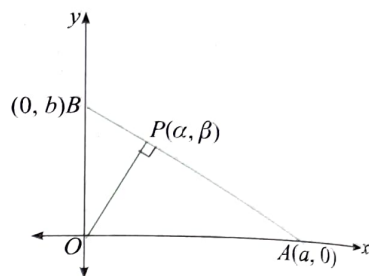
$$\text{or } \frac{\beta}{\alpha} \cdot \frac{\beta}{\alpha-a} = -1$$

$$\therefore a = \frac{\alpha^2 + \beta^2}{\alpha}$$

$$\text{Now, } a^2 + b^2 = k^2$$

$$\therefore (\alpha^2 + \beta^2)^3 = k^2 \alpha^2 \beta^2$$

$$\therefore (x^2 + y^2)^3 = k^2 x^2 y^2$$



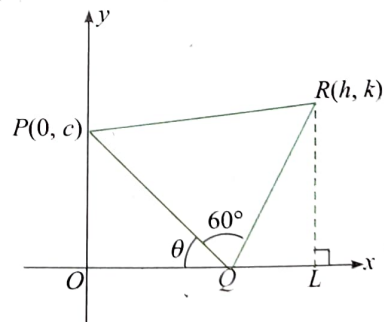
EXAMPLE 1.4

OX and OY are two coordinate axes. On OY is taken a fixed point $P(0, c)$ and on OX any point Q . On PQ , an equilateral triangle is described, its vertex R being on the right side of PQ .

Prove that the locus of R is $y = \sqrt{3}x - c$.

Sol. This given fixed point is $P(0, c)$.

From given Fig.,



$$h = OL = OQ + QL$$

$$= c \cot \theta + QR \cos(180^\circ - 60^\circ - \theta)$$

$$= c \cot \theta + PQ \{\cos 120^\circ \cdot \cos \theta + \sin 120^\circ \cdot \sin \theta\}$$

$$= c \cot \theta - \frac{PQ}{2} \cos \theta + \frac{\sqrt{3}}{2} PQ \sin \theta$$

$$= c \cot \theta - \frac{1}{2} c \operatorname{cosec} \theta \cos \theta + \frac{\sqrt{3}}{2} c \operatorname{cosec} \theta \sin \theta$$

$$\left[\because \sin \theta = \frac{c}{PQ} \right]$$

$$\therefore h = \frac{c}{2} (\cot \theta + \sqrt{3}) \quad (1)$$

$$\text{Now, } k = RL = RQ \sin(180^\circ - 60^\circ - \theta)$$

$$= PQ \left\{ \sin \theta \times \frac{1}{2} + \cos \theta \times \frac{\sqrt{3}}{2} \right\}$$

$$= c \operatorname{cosec} \theta \left\{ \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta \right\}$$

$$\therefore k = \frac{c}{2} (1 + \sqrt{3} \cot \theta) \quad (2)$$

Eliminating $\cot \theta$ from (1) and (2), we get $k = \sqrt{3}h - c$.

Therefore, the required locus is $y = \sqrt{3}x - c$, a straight line.

EXAMPLE 1.5

If (x, y) and (X, Y) are the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ux + vy$, where u and v are independent of x and y , becomes $UX + UY$, show that $u^2 + v^2 = U^2 + V^2$.

Sol. Let the axes rotate at angle θ . If (x, y) is the point with respect to the old axes and (X, Y) are the coordinates with respect to the new axes, then

$$\begin{cases} x = X \cos \theta - Y \sin \theta \\ y = X \sin \theta + Y \cos \theta \end{cases}$$

$$\text{Then } ux + vy = u(X \cos \theta - Y \sin \theta) + v(X \sin \theta + Y \cos \theta)$$

$$= (u \cos \theta + v \sin \theta) X + (-u \sin \theta + v \cos \theta) Y$$

But given new expression is $UX + UY$. Then,

$$UX + UY = (u \cos \theta + v \sin \theta) X + (-u \sin \theta + v \cos \theta) Y$$

On comparing the coefficients of X and Y , we get

$$u \cos \theta + v \sin \theta = U \quad \dots(1)$$

$$\text{and } -u \sin \theta + v \cos \theta = U \quad \dots(2)$$

Squaring and adding (1) and (2), we get

$$u^2 + v^2 = U^2 + V^2$$

EXAMPLE 1.6

What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to the rectangular axes through the point $(-2, -3)$, the new axes being inclined at an angle of 45° with the old axes?

Sol. Let O' be $(-2, -3)$. Since the axes are rotated about O' by an angle 45° in the anticlockwise direction, let (x', y') be the new coordinates with respect to new axes and (x, y) be the coordinates with respect to the old axes. Then, we have

$$x = -2 + x' \cos 45^\circ - y' \sin 45^\circ = -2 + \left(\frac{x' - y'}{\sqrt{2}} \right)$$

$$y = -3 + x' \sin 45^\circ + y' \cos 45^\circ = -3 + \left(\frac{x' + y'}{\sqrt{2}} \right)$$

The new equation will be

$$2 \left\{ -2 + \left(\frac{x' - y'}{\sqrt{2}} \right) \right\}^2 + 4 \left\{ -2 + \left(\frac{x' - y'}{\sqrt{2}} \right) \right\} \left\{ -3 + \left(\frac{x' + y'}{\sqrt{2}} \right) \right\}$$

$$- 5 \left\{ -3 + \left(\frac{x' + y'}{\sqrt{2}} \right) \right\}^2 + 20 \left\{ -2 + \left(\frac{x' - y'}{\sqrt{2}} \right) \right\}$$

$$- 22 \left\{ -3 + \left(\frac{x' + y'}{\sqrt{2}} \right) \right\} - 14 = 0$$

$$\text{or } x'^2 - 14x'y' - 7y'^2 - 2 = 0$$

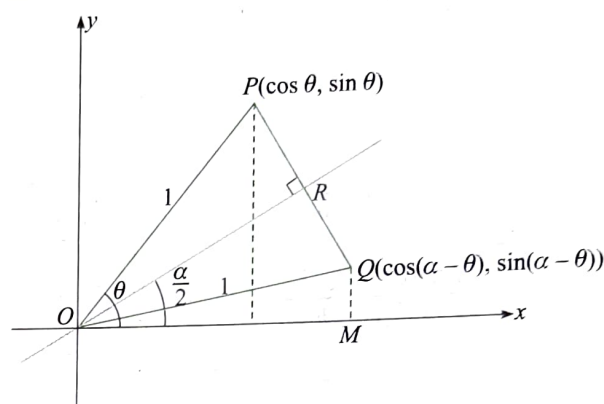
Hence, the new equation of the curve is

$$x^2 - 14xy - 7y^2 - 2 = 0$$

EXAMPLE 1.7

Prove that the image of point $P(\cos \theta, \sin \theta)$ in the line having slope $\tan(\alpha/2)$ and passing through origin is $Q(\cos(\alpha - \theta), \sin(\alpha - \theta))$.

Sol.



Clearly, $OP = OQ = 1$.

Now, we have to prove that Q is the image of P in the line OR which has slope $\tan(\alpha/2)$.

Triangle POQ is isosceles triangle.

If Q is the image of P in line OR , then OR is the perpendicular bisector of PQ .

We have to prove that $\angle QOM = \alpha - \theta$.

$$\angle ROQ = \angle POR = \theta - (\alpha/2)$$

$$\therefore \angle QOM = \angle ROM - \angle ROQ$$

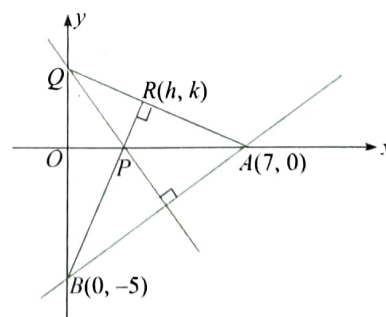
$$= (\alpha/2) - (\theta - (\alpha/2)) = \alpha - \theta$$

EXAMPLE 1.8

A line cuts the x -axis at $A(7, 0)$ and the y -axis at $B(0, -5)$. A variable line PQ is drawn perpendicular to AB cutting the x -axis at P and the y -axis at Q . If AQ and BP intersect at R , then find the locus of R .

Sol. From the figure in $\triangle ABQ$,

$$AP \perp BQ, PQ \perp AB$$



Then, we have $BP \perp AQ$ (as the altitudes of triangle are concurrent). Hence,

$$BR \perp AR$$

or $\frac{k-0}{h-7} \times \frac{k+5}{h-0} = -1$

Therefore, equation of the locus of R is $x^2 + y^2 - 7x + 5y = 0$.

EXAMPLE 1.9

Two straight lines rotate about two fixed points $(-a, 0)$ and $(a, 0)$ in anticlockwise sense. If they start from their position of coincidence such that one rotates at a rate double the other, then find the locus of curve.

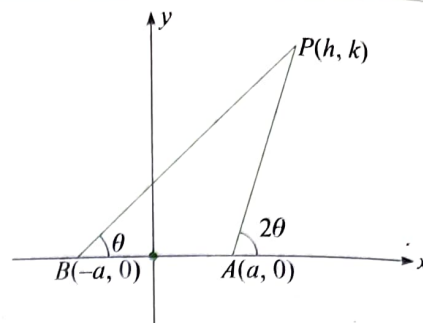
Sol. Let PB make angle θ and PA make angle 2θ with x -axis. Let P be (h, k) .

$$\tan \theta = \frac{k}{h+a} \quad (1)$$

$$\tan 2\theta = \frac{k}{h-a} \quad (2)$$

From (2), we get

$$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{k}{h-a}$$



Putting value of $\tan \theta$ from (1), we get

$$\frac{2 \left(\frac{k}{h+a} \right)}{1 - \left(\frac{k}{h+a} \right)^2} = \frac{k}{h-a}$$

$$\Rightarrow h^2 + k^2 - 2ah - 3a^2 = 0$$

$$\Rightarrow x^2 + y^2 - 2ax - 3a^2 = 0$$

Exercises

Single Correct Answer Type

- ABC is an isosceles triangle. If the coordinates of the base are $B(1, 3)$ and $C(-2, 7)$, the coordinates of vertex A can be
 - $(1, 6)$
 - $(-1/2, 5)$
 - $(5/6, 6)$
 - none of these
- If two vertices of a triangle are $(1, 3)$ and $(4, -1)$ and the area of triangle is 5 sq. units, then the angle at the third vertex lies in
 - $\left(0, 2 \tan^{-1} \frac{5}{4}\right]$
 - $\left[0, \tan^{-1} \frac{5}{4}\right)$
 - $\left[2 \tan^{-1} \frac{5}{4}, 2\right)$
 - none of these
- Which of the following sets of points form an equilateral triangle?
 - $(1, 0), (4, 0), (7, -1)$
 - $(0, 0), (3/2, 4/3), (4/3, 3/2)$
 - $(2/3, 0), (0, 2/3), (1, 1)$
 - None of these
- A particle p moves from the point $A(0, 4)$ to the point $(10, -4)$. The particle P can travel the upper-half plane $\{(x, y) | y \geq 0\}$ at the speed of 1 m/s and the lower-half plane $\{(x, y) | y \leq 0\}$ at the speed of 2 m/s. The coordinates of a point on the x -axis, if the sum of the squares of the travel times of the upper- and lower-half planes is minimum, are
 - $(1, 0)$
 - $(2, 0)$
 - $(4, 0)$
 - $(5, 0)$
- If

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$
 then the two triangles with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are
 - equal in area
 - similar
 - congruent
 - none of these
- $OPQR$ is a square and M, N are the middle points of the sides PQ and QR , respectively. Then the ratio of the area of the square to that of triangle OMN is
 - $4 : 1$
 - $2 : 1$
 - $8 : 3$
 - $7 : 3$
- A straight line passing through $P(3, 1)$ meets the coordinate axes at A and B . It is given that the distance of this straight line from the origin O is maximum. The area of triangle OAB is equal to
 - $50/3$ sq. units
 - $25/3$ sq. units
 - $20/3$ sq. units
 - $100/3$ sq. units
- Let $A \equiv (3, -4)$, $B \equiv (1, 2)$. Let $P \equiv (2k - 1, 2k + 1)$ be a variable point such that $PA + PB$ is the minimum. Then k is
 - $7/9$
 - 0
 - $7/8$
 - none of these
- The polar coordinates equivalent to $(-3, \sqrt{3})$ are
 - $\left(2\sqrt{3}, \frac{\pi}{6}\right)$
 - $\left(-2\sqrt{3}, \frac{5\pi}{6}\right)$
 - $\left(2\sqrt{3}, \frac{7\pi}{6}\right)$
 - $\left(2\sqrt{3}, \frac{5\pi}{6}\right)$
- If the point $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of (x_1, y_1) and (x_2, y_2) internally, then
 - $t < 0$
 - $0 < t < 1$
 - $t > 1$
 - $t = 1$
- P and Q are points on the line joining $A(-2, 5)$ and $B(3, 1)$ such that $AP = PQ = QB$. Then, the distance of the midpoint of PQ from the origin is
 - 3
 - $\sqrt{37}/2$
 - 4
 - 3.5
- In triangle ABC , angle B is right angle and $AC = 2$. If coordinates of A and B are $(2, 2)$ and $(1, 3)$, respectively, then the length of median AD is
 - $\frac{1}{2}$
 - $\sqrt{\frac{5}{2}}$
 - $\frac{5}{\sqrt{2}}$
 - $\frac{1}{\sqrt{2}}$
- One vertex of an equilateral triangle is $(2, 2)$ and its centroid is $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. The length of its side is
 - $4\sqrt{2}$
 - $4\sqrt{3}$
 - $3\sqrt{2}$
 - $5\sqrt{2}$
- $ABCD$ is a rectangle with $A(-1, 2)$, $B(3, 7)$ and $AB : BC = 4 : 3$. If P is the centre of the rectangle, then the distance of P from each corner is equal to
 - $\frac{\sqrt{14}}{2}$
 - $3\frac{\sqrt{41}}{4}$
 - $2\frac{\sqrt{41}}{3}$
 - $5\frac{\sqrt{41}}{8}$
- If $(2, -3)$, $(6, 5)$ and $(-2, 1)$ are three consecutive vertices of a rhombus, then its area is
 - 24
 - 36
 - 18
 - 48
- If points $A(3, 5)$ and B are equidistant from $H(\sqrt{2}, \sqrt{5})$ and B has rational coordinates, then $AB =$
 - $\sqrt{7}$
 - $\sqrt{(3 - \sqrt{2})^2 + (5 - \sqrt{5})^2}$
 - $\sqrt{34}$
 - None of these
- Let n be the number of points having rational coordinates at a fixed distance from the point $(0, \sqrt{3})$. Then
 - $n > 2$
 - $n \leq 1$
 - $n \leq 2$
 - $n = 1$
- In $\triangle ABC$, the sides $BC = 5$, $CA = 4$ and $AB = 3$. If $A \equiv (0, 0)$ and the internal bisector of angle A meets BC in $D\left(\frac{12}{7}, \frac{12}{7}\right)$, then incentre of $\triangle ABC$ is
 - $(2, 2)$
 - $(3, 2)$
 - $(2, 3)$
 - $(1, 1)$

19. If $A(0, 0)$, $B(1, 0)$ and $C\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ are vertices of a triangle, then the centre of the circle for which the lines AB , BC and CA are tangents is
- (1) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (2) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$
 (3) $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$ (4) $\left(\frac{1}{2}, -\frac{1}{\sqrt{3}}\right)$
20. **Statement 1:** If in a triangle, orthocentre, circumcentre and centroid are rational points, then its vertices must also be rational points.
Statement 2: If the vertices of a triangle are rational points, then the centroid, circumcentre and orthocentre are also rational points.
- (1) Statement 1 is true, statement 2 is true and statement 2 is correct explanation for statement 1.
 (2) Statement 1 is true, statement 2 is true and statement 2 is NOT the correct explanation for statement 1.
 (3) Statement 1 is true, statement 2 is false.
 (4) Statement 1 is false, statement 2 is true.
21. Consider three points $P \equiv (-\sin(\beta - \alpha), -\cos \beta)$, $Q \equiv (\cos(\beta - \alpha), \sin \beta)$, and $R \equiv (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \alpha, \beta, \theta < \pi/4$. Then
- (1) P lies on the line segment RQ
 (2) Q lies on the line segment PR
 (3) R lies on the line segment QP
 (4) P, Q, R are non-collinear
22. If two vertices of a triangle are $(-2, 3)$ and $(5, -1)$, the orthocenter lies at the origin, and the centroid on the line $x + y = 7$, then the third vertex lies at
- (1) $(7, 4)$ (2) $(8, 14)$
 (3) $(12, 21)$ (4) none of these
23. The vertices of a triangle are $(pq, 1/(pq))$, $(qr, 1/(qr))$, and $(rq, 1/(rq))$, where p, q , and r are the roots of the equation $y^3 - 3y^2 + 6y + 1 = 0$. The coordinates of its centroid are
- (1) $(1, 2)$ (2) $(2, -1)$
 (3) $(1, -1)$ (4) $(2, 3)$
24. If the vertices of a triangle are $(\sqrt{5}, 0)$, $(\sqrt{3}, \sqrt{2})$, and $(2, 1)$, then the orthocenter of the triangle is
- (1) $(\sqrt{5}, 0)$ (2) $(0, 0)$
 (3) $(\sqrt{5} + \sqrt{3} + 2, \sqrt{2} + 1)$ (4) none of these
25. Two vertices of a triangle are $(4, -3)$ and $(-2, 5)$. If the orthocenter of the triangle is at $(1, 2)$, then the third vertex is
- (1) $(-33, -26)$ (2) $(33, 26)$
 (3) $(26, 33)$ (4) none of these
26. In $\triangle ABC$, if the orthocenter is $(1, 2)$ and the circumcenter is $(0, 0)$, then centroid of $\triangle ABC$ is
- (1) $(1/2, 2/3)$ (2) $(1/3, 2/3)$
 (3) $(2/3, 1)$ (4) none of these
27. A triangle ABC with vertices $A(-1, 0)$, $B(-2, 3/4)$, and $C(-3, -7/6)$ has its orthocenter at H . Then, the orthocenter of triangle BCH will be
- (1) $(-3, -2)$ (2) $(1, 3)$
 (3) $(-1, 2)$ (4) none of these
28. If in triangle ABC , $A \equiv (1, 10)$, circumcenter $\equiv (-1/3, 2/3)$, and orthocenter $\equiv (11/3, 4/3)$, then the coordinates of the midpoint of the side opposite to A are
- (1) $(1, -11/3)$ (2) $(1, 5)$
 (3) $(1, -3)$ (4) $(1, 6)$
29. In $\triangle ABC$, the coordinates of B are $(0, 0)$, $AB = 2$, $\angle ABC = \pi/3$, and the middle point of BC has coordinates $(2, 0)$. The centroid of the triangle is
- (1) $(1/2, \sqrt{3}/2)$ (2) $(5/3, 1/\sqrt{3})$
 (3) $(4 + \sqrt{3}/3, 1/3)$ (4) none of these
30. If the origin is shifted to the point $(ab/(a - b), 0)$ without rotation, then the equation $(a - b)(x^2 + y^2) - 2abx = 0$ becomes
- (1) $(a - b)(x^2 + y^2) - (a + b)xy + abx = a^2$
 (2) $(a + b)(x^2 + y^2) = 2ab$
 (3) $(x^2 + y^2) = (a^2 + b^2)$
 (4) $(a - b)^2(x^2 + y^2) = a^2b^2$
31. A light ray emerging from the point source placed at $P(2, 3)$ is reflected at a point Q on the y -axis. It then passes through the point $R(5, 10)$. The coordinates of Q are
- (1) $(0, 3)$ (2) $(0, 2)$
 (3) $(0, 5)$ (4) none of these
32. Points $P(p, 0)$, $Q(q, 0)$, $R(0, p)$, $S(0, q)$ form
- (1) parallelogram (2) rhombus
 (3) cyclic quadrilateral (4) none of these
33. A rectangular billiard table has vertices at $P(0, 0)$, $Q(0, 7)$, $R(10, 7)$, and $S(10, 0)$. A small billiard ball starts at $M(3, 4)$, moves in a straight line to the top of the table, bounces to the right side of the table, and then comes to rest at $N(7, 1)$. The y -coordinate of the point where it hits the right side is
- (1) 3.7 (2) 3.8 (3) 3.9 (4) 4
34. $ABCD$ is a square. Points $E(4, 3)$ and $F(2, 5)$ lie on AB and CD , respectively, such that EF divides the square in two equal parts. If the coordinates of A are $(7, 3)$, then the coordinates of other vertices can be
- (1) $(7, 2)$ (2) $(7, 5)$
 (3) $(-1, 3)$ (4) $(-1, 5)$
35. If one side of a rhombus has endpoints $(4, 5)$ and $(1, 1)$, then the maximum area of the rhombus is
- (1) 50 sq. units (2) 25 sq. units
 (3) 30 sq. units (4) 20 sq. units
36. A rectangle $ABCD$, where $A \equiv (0, 0)$, $B \equiv (4, 0)$, $C \equiv (4, 2)$, $D \equiv (0, 2)$, undergoes the following transformations successively:
- i. $f_1(x, y) \rightarrow (y, x)$
 ii. $f_2(x, y) \rightarrow (x + 3y, y)$
 iii. $f_3(x, y) \rightarrow ((x - y)/2, (x + y)/2)$

The final figure will be

- (1) a square (2) a rhombus
(3) a rectangle (4) a parallelogram

37. If a straight line through the origin bisects the line passing through the given points $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$, then the lines

- (1) are perpendicular
(2) are parallel
(3) have an angle between them of $\pi/4$
(4) none of these

38. Let $A_r, r = 1, 2, 3, \dots$, be the points on the number line such that $OA_1, OA_2, OA_3, \dots$ are in GP, where O is the origin, and the common ratio of the GP be a positive proper fraction. Let M_r be the middle point of the line segment $A_r A_{r+1}$. Then the value of $\sum_{r=1}^{\infty} OM_r$ is equal to

- (1) $\frac{OA_1(OA_1 - OA_2)}{2(OA_1 + OA_2)}$ (2) $\frac{OA_1(OA_1 + OA_2)}{2(OA_1 - OA_2)}$
(3) $\frac{OA_1}{2(OA_1 - OA_2)}$ (4) ∞

39. The vertices of a parallelogram $ABCD$ are $A(3, 1), B(13, 6), C(13, 21)$, and $D(3, 16)$. If a line passing through the origin divides the parallelogram into two congruent parts, then the slope of the line is

- (1) $11/12$ (2) $11/8$ (3) $25/8$ (4) $13/8$

40. Points A and B are in the first quadrant; point O is the origin. If the slope of OA is 1, the slope of OB is 7, and $OA = OB$, then the slope of AB is

- (1) $-1/5$ (2) $-1/4$ (3) $-1/3$ (4) $-1/2$

41. Let a, b, c be in A.P. and x, y, z be in G.P.. Then the points $(a, x), (b, y)$ and (c, z) will be collinear if

- (1) $x^2 = y$ (2) $x = y = z$
(3) $y^2 = z$ (4) $x = z^2$

42. If $\sum_{i=1}^4 (x_i^2 + y_i^2) \leq 2x_1x_3 + 2x_2x_4 + 2y_2y_3 + 2y_1y_4$, the points

$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are

- (1) the vertices of a rectangle
(2) collinear
(3) the vertices of a trapezium
(4) none of these

43. The vertices A and D of square $ABCD$ lie on the positive sides of x - and y -axis, respectively. If the vertex C is the point $(12, 17)$, then the coordinates of vertex B are

- (1) $(14, 16)$ (2) $(15, 3)$
(3) $(17, 5)$ (4) $(17, 12)$

44. Through the point $P(\alpha, \beta)$, where $\alpha\beta > 0$, the straight line

$\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form a triangle of area S with the axes. If $ab > 0$, then the least value of S is

- (1) $\alpha\beta$ (2) $2\alpha\beta$ (3) $3\alpha\beta$ (4) none

45. The locus of the moving point whose coordinates are given by $(e^t + e^{-t}, e^t - e^{-t})$ where t is a parameter, is

- (1) $xy = 1$ (2) $x + y = 2$
(3) $x^2 - y^2 = 4$ (4) $x^2 - y^2 = 2$

46. The locus of a point represented by

$$x = \frac{a(t+1)}{2}, y = \frac{a(t-1)}{2}, \text{ where } t \in \mathbb{R} - \{0\}, \text{ is}$$

- (1) $x^2 + y^2 = a^2$ (2) $x^2 - y^2 = a^2$
(3) $x + y = a$ (4) $x - y = a$

47. The maximum area of the triangle whose sides a, b , and c satisfy $0 \leq a \leq 1, 1 \leq b \leq 2$, and $2 \leq c \leq 3$ is

- (1) 1 (2) $1/2$ (3) 2 (4) $3/2$

48. The vertices of a variable triangle are $(3, 4), (5 \cos \theta, 5 \sin \theta)$, and $(5 \sin \theta, -5 \cos \theta)$, where $\theta \in \mathbb{R}$. The locus of its orthocenter is

- (1) $(x + y - 1)^2 + (x - y - 7)^2 = 100$
(2) $(x + y - 7)^2 + (x - y - 1)^2 = 100$
(3) $(x + y - 7)^2 + (x + y - 1)^2 = 100$
(4) $(x + y - 7)^2 + (x - y + 1)^2 = 100$

49. From a point P , perpendiculars PM and PN are drawn to x and y axes, respectively. If MN passes through fixed point (a, b) , then locus of P is

- (1) $xy = ax + by$ (2) $xy = ab$
(3) $xy = bx + ay$ (4) $x + y = xy$

50. The locus of point of intersection of the lines $y + mx = \sqrt{a^2 m^2 + b^2}$ and $my - x = \sqrt{a^2 + b^2 m^2}$ is

- (1) $x^2 + y^2 = \frac{1}{a^2} + \frac{1}{b^2}$ (2) $x^2 + y^2 = a^2 + b^2$
(3) $x^2 - y^2 = a^2 - b^2$ (4) $\frac{1}{x^2} + \frac{1}{y^2} = a^2 - b^2$

51. If the roots of the equation

$$(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0 \quad (a > b)$$

are the slopes of two perpendicular lines intersecting at $P(x_1, y_1)$, then the locus of P is

- (1) $x^2 + y^2 = a^2 + b^2$ (2) $x^2 + y^2 = a^2 - b^2$
(3) $x^2 - y^2 = a^2 + b^2$ (4) $x^2 - y^2 = a^2 - b^2$

52. Through point $P(-1, 4)$, two perpendicular lines are drawn which intersect x -axis at Q and R . Find the locus of incentre of ΔPQR .

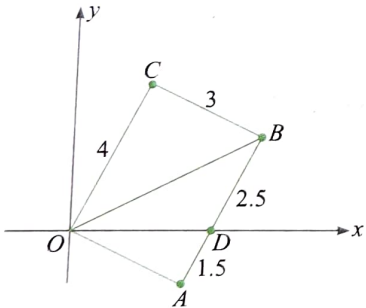
- (1) $x^2 + y^2 + 2x - 8y - 17 = 0$
(2) $x^2 - y^2 + 2x - 8y + 17 = 0$
(3) $x^2 + y^2 - 2x - 8y - 17 = 0$
(4) $x^2 - y^2 + 8x - 2y - 17 = 0$

53. The number of integral points (x, y) (i.e., x and y both are integers) which lie in the first quadrant but not on the coordinate axes and also on the straight line $3x + 5y = 2007$ is equal to

- (1) 133 (2) 135 (3) 138 (4) 140

54. The foot of the perpendicular on the line $3x + y = \lambda$ drawn from the origin is C . If the line cuts the x - and the y -axis at A and B , respectively, then $BC : CA$ is

- (1) $1 : 3$ (2) $3 : 1$ (3) $1 : 9$ (4) $9 : 1$

55. The image of $P(a, b)$ on the line $y = -x$ is Q and the image of Q on the line $y = x$ is R . Then the midpoint of PR is
 (1) $(a + b, b + a)$ (2) $((a + b)/2, (b + 2)/2)$
 (3) $(a - b, b - a)$ (4) $(0, 0)$
56. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is
 (1) $a_1^2 - a_2^2 + b_1^2 - b_2^2$ (2) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$
 (3) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ (4) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$
57. Consider three lines as follows.
 $L_1: 5x - y + 4 = 0$
 $L_2: 3x - y + 5 = 0$
 $L_3: x + y + 8 = 0$
 If these lines enclose a triangle ABC and the sum of the squares of the tangent to the interior angles can be expressed in the form p/q , where p and q are relatively prime numbers, then the value of $p + q$ is
 (1) 500 (2) 450 (3) 230 (4) 465
58. Consider a point $A(m, n)$, where m and n are positive integers. B is the reflection of A in the line $y = x$, C is the reflection of B in the y axis, D is the reflection of C in the x axis and E is the reflection of D in the y axis. The area of the pentagon $ABCDE$ is
 (1) $2m(m + n)$ (2) $m(m + 3n)$
 (3) $m(2m + 3n)$ (4) $2m(m + 3n)$
59. In the given figure, $OABC$ is a rectangle. Slope of OB is
- 
- (1) $1/4$ (2) $1/3$
 (3) $1/2$ (4) Cannot be determined
3. If $(-4, 0)$ and $(1, -1)$ are two vertices of a triangle of area 4 sq. units, then its third vertex lies on
 (1) $y = x$ (2) $5x + y + 12 = 0$
 (3) $x + 5y - 4 = 0$ (4) $x + 5y + 12 = 0$
4. The area of triangle ABC is 20 cm^2 . The coordinates of vertex A are $(-5, 0)$ and those of B are $(3, 0)$. The vertex C lies on the line $x - y = 2$. The coordinates of C are
 (1) $(5, 3)$ (2) $(-3, -5)$ (3) $(-5, -7)$ (4) $(7, 5)$
5. If $(a \cos \theta_1, a \sin \theta_1)$, $(a \cos \theta_2, a \sin \theta_2)$, and $(a \cos \theta_3, a \sin \theta_3)$ represent the vertices of an equilateral triangle inscribed in a circle, then
 (1) $\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 0$
 (2) $\sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 0$
 (3) $\tan \theta_1 + \tan \theta_2 + \tan \theta_3 = 0$
 (4) $\cot \theta_1 + \cot \theta_2 + \cot \theta_3 = 0$
6. If the points $A(0, 0)$, $B(\cos \alpha, \sin \alpha)$, and $C(\cos \beta, \sin \beta)$ are the vertices of a right-angled triangle, then
 (1) $\sin \frac{\alpha - \beta}{2} = \frac{1}{\sqrt{2}}$ (2) $\cos \frac{\alpha - \beta}{2} = -\frac{1}{\sqrt{2}}$
 (3) $\cos \frac{\alpha - \beta}{2} = \frac{1}{\sqrt{2}}$ (4) $\sin \frac{\alpha - \beta}{2} = -\frac{1}{\sqrt{2}}$
7. The ends of a diagonal of a square are $(2, -3)$ and $(-1, 1)$. Another vertex of the square can be
 (1) $(-3/2, -5/2)$ (2) $(5/2, 1/2)$
 (3) $(1/2, 5/2)$ (4) none of these
8. If all the vertices of a triangle have integral coordinates, then the triangle may be
 (1) right-angled (2) equilateral
 (3) isosceles (4) none of these
9. In a ΔABC , $A \equiv (\alpha, \beta)$, $B \equiv (1, 2)$, $C \equiv (2, 3)$, point A lies on the line $y = 2x + 3$, where α, β are integers, and the area of the triangle is S such that $[S] = 2$ where $[\cdot]$ denotes the greatest integer function. Then the possible coordinates of A can be
 (1) $(-7, -11)$ (2) $(-6, -9)$
 (3) $(2, 7)$ (4) $(3, 9)$
10. In an acute triangle ABC , if the coordinates of orthocenter H are $(4, b)$, of centroid G are $(b, 2b - 8)$, and of circumcenter C are $(-4, 8)$, then b cannot be
 (1) 4 (2) 8 (3) 12 (4) -12
11. Consider the points $O(0, 0)$, $A(0, 1)$, and $B(1, 1)$ in the $x-y$ plane. Suppose that points $C(x, 1)$ and $D(1, y)$ are chosen such that $0 < x < 1$ and such that O, C , and D are collinear. Let the sum of the area of triangles OAC and BCD be denoted by S . Then which of the following is/are correct?
 (1) Minimum value of S is irrational lying in $(1/3, 1/2)$.
 (2) Minimum value of S is irrational in $(2/3, 1)$.
 (3) The value of x for the minimum value of S lies in $(2/3, 1)$.
 (4) The value of x for the minimum values of S lies in $(1/3, 1/2)$.

Multiple Correct Answers Type

1. If $(-6, -4)$, $(3, 5)$, $(-2, 1)$ are the vertices of a parallelogram, then the remaining vertex can be
 (1) $(0, -1)$ (2) $(7, 10)$ (3) $(-1, 0)$ (4) $(-11, -8)$
2. Let $O \equiv (0, 0)$, $A \equiv (0, 4)$, $B \equiv (6, 0)$. Let P be a moving point such that the area of triangle POA is two times the area of triangle POB . The locus of P will be a straight line whose equation can be
 (1) $x + 3y = 0$ (2) $x + 2y = 0$
 (3) $2x - 3y = 0$ (4) $3y - x = 0$

12. Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point $(1, 2)$ and the vertex A is on the y -axis, then vertex A can be
- (1) $(0, 3)$ (2) $(0, 5/2)$
 (3) $(0, 0)$ (4) $(0, 5)$
13. A triangle ABC right angled at C having $CA = a$ and $CB = b$ moves in a way such that the angular points A and B slide along x -axis and y -axis, respectively. Then C lies on
- (1) $ax + by + 1 = 0$ (2) $ax + by = 0$
 (3) $ax^2 \pm 2by + y^2 = 0$ (4) $ax - by = 0$

Linked Comprehension Type

For Problems 1–3

For points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ of the coordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Let $O \equiv (0, 0)$ and $A \equiv (3, 2)$. Consider the set of points P in the first quadrant which are equidistant (with respect to the new distance) from O and A .

- The set of points P consists of
 - one straight line only
 - union of two line segments
 - union of two infinite rays
 - union of a line segment of finite length and an infinite ray
- The area of the region bounded by the locus of P and the line $y = 4$ in the first quadrant is
 - 2 sq. units
 - 4 sq. units
 - 6 sq. units
 - none of these
- The locus of point P is
 - one-one and onto function
 - many-one and onto function
 - one-one and into function
 - relation but not function

For Problems 4 and 5

Consider the triangle having vertices $O(0, 0)$, $A(2, 0)$, and $B(1, \sqrt{3})$. Also, $b \leq \min\{a_1, a_2, a_3, \dots, a_n\}$ means $b \leq a_1$ when a_1 is least; $b \leq a_2$ when a_2 is least, and so on. From this, we can say $b \leq a_1, b \leq a_2, \dots, b \leq a_n$.

- Let R be the region consisting of all those points P inside $\triangle OAB$ which satisfy $d(P, OA) \leq \min[d(P, OB), d(P, AB)]$, where d denotes the distance from the point to the corresponding line. Then the area of the region R is
 - $\sqrt{3}$ sq. units
 - $(2 + \sqrt{3})$ sq. units
 - $\sqrt{3}/2$ sq. units
 - $1/\sqrt{3}$ sq. units
- Let R be the region consisting of all those points P inside $\triangle OAB$ which satisfy $OP \leq \min[BP, AP]$. Then the area of the region R is
 - $\sqrt{3}$ sq. units
 - $1/\sqrt{3}$ sq. units
 - $\sqrt{3}/2$ sq. units
 - none of these

For Problems 6–8

Let $ABCD$ be a square with sides of unit length. Points E and F are taken on sides AB and AD , respectively, so that $AE = AF$. Let P be a point inside the square $ABCD$.

- The maximum possible area of quadrilateral $CDFE$ is
 - $1/8$
 - $1/4$
 - $5/8$
 - $3/8$
- The value of $(PA)^2 - (PB)^2 + (PC)^2 - (PD)^2$ is equal to
 - 3
 - 2
 - 1
 - 0
- Let a line passing through point A divides the square $ABCD$ into two parts so that the area of one portion is double the other. Then the length of the portion of line inside the square is
 - $\sqrt{10}/3$
 - $\sqrt{13}/3$
 - $\sqrt{11}/3$
 - $2/\sqrt{3}$

For Problems 9 and 10

Let ABC be an acute-angled triangle and AD , BE , and CF be its medians, where E and F are at $(3, 4)$ and $(1, 2)$, respectively. The centroid of $\triangle ABC$ is $G(3, 2)$.

- The coordinates of D are
 - $(7, -4)$
 - $(5, 0)$
 - $(7, 4)$
 - $(-3, 0)$
- The height of the altitude drawn from point A is (in units)
 - $4\sqrt{2}$
 - $3\sqrt{2}$
 - $6\sqrt{2}$
 - $2\sqrt{3}$

Matrix Match Type

- O is the origin and B is a point on the x -axis at a distance of 2 units from the origin. Match the following lists:

List I	List II
a. If $\triangle AOB$ is an equilateral triangle, then the coordinates of A can be	p. $(-1, \sqrt{3})$
b. If $\triangle AOB$ is isosceles such that $\angle OAB$ is 30° , then the coordinates of A can be	q. $(-1, 2 + \sqrt{3})$
c. If OB is one side of a rhombus of area $\sqrt{3}$ units, then the other vertices of the rhombus can be	r. $(-3, -\sqrt{3})$
d. If OB is a chord of circle with radius equal to OB , then the coordinates of point A on the circumference of the circle such that $\triangle OAB$ is isosceles can be	s. $(1, 2 + \sqrt{3})$

- Consider the triangle whose vertices are $(0, 0)$, $(5, 12)$ and $(16, 12)$. Match the following lists:

List I	List II
a. Centroid of the triangle	p. $\left(\frac{21}{2}, \frac{8}{3}\right)$
b. Circumcenter of the triangle	q. $(7, 9)$
c. Incenter of the triangle	r. $(27, -21)$
d. Excenter opposite to vertex $(5, 12)$	s. $(7, 8)$

3. Match the following lists:

List I	List II
a. The locus of $P(x, y)$ such that $\sqrt{x^2 + y^2 + 8y + 16}$ $-\sqrt{x^2 + y^2 - 6x + 9} = 5$	p. No such point P exists
b. The locus of $P(x, y)$ such that $\sqrt{x^2 + y^2 + 8y + 16}$ $-\sqrt{x^2 + y^2 - 6x + 9} = \pm 5$	q. Line segment
c. The locus of $P(x, y)$ such that $\sqrt{x^2 + y^2 + 8y + 16}$ $+\sqrt{x^2 + y^2 - 6x + 9} = 5$	r. A ray
d. The locus of $P(x, y)$ such that $\sqrt{x^2 + y^2 + 8y + 16}$ $-\sqrt{x^2 + y^2 - 6x + 9} = 7$	s. Two rays

4. Match the following lists and then choose the correct code.

List I	List II
a. Distance between feet of perpendicular drawn from a point $(-3, 4)$ on both axes is	p. 0
b. An equilateral triangle whose circumcentre is $(3, -2)$ and one side is on Y -axis has side length L , then the value of $L/\sqrt{3}$ is	q. 1
c. Area of the triangle formed by (x_1, y_1) , (x_2, y_2) and $(3x_2 - 2x_1, 3y_2 - 2y_1)$ (in sq. units) is	r. 5
d. If the points $(-2, 0)$, $\left(-1, \frac{1}{\sqrt{3}}\right)$, $(\cos \theta, \sin \theta)$ are collinear, then the number of values of $\theta \in [0, 2\pi]$ is	s. 4

Codes:

- a b c d
- (1) q r s p
- (2) r s p q
- (3) r p q s
- (4) s r p q

Numerical Value Type

- Line AB passes through point $(2, 3)$ and intersects the positive x and y -axes at $A(a, 0)$ and $B(0, b)$, respectively. If the area of ΔAOB is 11, then the value of $4b^2 + 9a^2$ is _____.
- A point A divides the join of $P(-5, 1)$ and $Q(3, 5)$ in the ratio $k : 1$. Then the integral value of k for which the area of ΔABC , where B is $(1, 5)$ and C is $(7, -2)$, is equal to 2 units in magnitude is _____.
- The distance between the circumcenter and the orthocenter of the triangle whose vertices are $(0, 0)$, $(6, 8)$, and $(-4, 3)$ is L . Then the value of $\frac{2}{\sqrt{5}}L$ is _____.
- A man starts from the point $P(-3, 4)$ and reaches the point $Q(0, 1)$ touching the x -axis at $R(\alpha, 0)$ such that $PR + RQ$ is minimum. Then $|\alpha| =$ _____.
- Let $A(0, 1)$, $B(1, 1)$, $C(1, -1)$, $D(-1, 0)$ be four points. If P is any other point, then $PA + PB + PC + PD \geq d$, when $[d]$ is _____, where $[.]$ represents greatest integer.
- A triangle ABC has vertices $A(5, 1)$, $B(-1, -7)$, and $C(1, 4)$, respectively. L be the line mirror passing through C and parallel to AB . A light ray emanating from point A goes along the direction of the internal bisector of angle A , which meets the mirror and BC at E and D , respectively. Then the sum of the areas of ΔACE and ΔABC is _____.
- If the area of the triangle formed by the points $(2a, b)$, $(a + b, 2b + a)$, and $(2b, 2a)$ is 2 sq. units, then the area of the triangle whose vertices are $(a + b, a - b)$, $(3b - a, b + 3a)$, and $(3a - b, 3b - a)$ will be _____.
- Lines L_1 and L_2 have slopes m and n , respectively. Suppose L_1 makes twice as large angle with the horizontal (measured counter clockwise from the positive x -axis) as does L_2 and L_1 has 4 times the slope of L_2 . If L_1 is not horizontal, then the value of the product mn equals _____.
- If lines $2x - 3y + 6 = 0$ and $kx + 2y + 12 = 0$ cut the coordinate axes in concyclic points, then the value of $|k|$ is _____.
- Perpendiculars from the point $P(4, 4)$ to the straight lines $3x + 4y + 5 = 0$ and $y = mx + 7$ meet at Q and R , respectively. If the area of triangle PQR is maximum, then the value of $3m$ is _____.
- The value of a for which the image of the point $(a, a - 1)$ w.r.t. the line mirror $3x + y = 6a$ is the point $(a^2 + 1, a)$ is _____.
- The maximum area of the convex polygon formed by joining the points $A(0, 0)$, $B(2t^2, 0)$, $C(18, 2)$, $D\left(\frac{8}{t^2}, 4\right)$ and $E(0, 2)$, where $t \in \mathbb{R} - \{0\}$ and interior angle at vertex B is greater than or equal to 90° is _____.

Archives

JEE MAIN

Single Correct Answer Type

- The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for
 - no value of p .
 - exactly one value of p .
 - exactly two values of p .
 - more than two values of p .
 (AIEEE 2009)
- If the line $2x + y = k$ passes through the point which divides the line segment joining the points $(1, 1)$ and $(2, 4)$ in the ratio $3 : 2$, then k equals.
 - $\frac{29}{5}$
 - 5
 - 6
 - $\frac{11}{5}$
 (AIEEE 2012)
- The number of points having both coordinates as integers, that lie in the interior of the triangle with vertices $(0, 0)$, $(0, 41)$ and $(41, 0)$ is
 - 901
 - 861
 - 820
 - 780
 (JEE Main 2015)
- Let k be an integer such that triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point

- $\left(2, \frac{1}{2}\right)$
- $\left(2, -\frac{1}{2}\right)$
- $\left(1, \frac{3}{4}\right)$
- $\left(1, -\frac{3}{4}\right)$

(JEE Main 2017)

- Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$, respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is

- $\frac{3\sqrt{5}}{2}$
- $\sqrt{10}$
- $2\sqrt{10}$
- $3\sqrt{\frac{5}{2}}$

(JEE Main 2018)

- A straight line through a fixed point $(2, 3)$ intersects the coordinate axes at distinct points P and Q . If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is

- $3x + 2y = 6xy$
- $3x + 2y = 6$
- $2x + 3y = xy$
- $3x + 2y = xy$

(JEE Main 2018)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (3) | 2. (1) | 3. (4) | 4. (2) | 5. (1) |
| 6. (3) | 7. (1) | 8. (3) | 9. (4) | 10. (2) |
| 11. (2) | 12. (2) | 13. (1) | 14. (4) | 15. (4) |
| 16. (4) | 17. (3) | 18. (4) | 19. (3) | 20. (4) |
| 21. (4) | 22. (4) | 23. (2) | 24. (3) | 25. (2) |
| 26. (2) | 27. (4) | 28. (1) | 29. (2) | 30. (4) |
| 31. (3) | 32. (3) | 33. (1) | 34. (4) | 35. (2) |
| 36. (4) | 37. (1) | 38. (2) | 39. (2) | 40. (4) |
| 41. (2) | 42. (1) | 43. (3) | 44. (2) | 45. (3) |
| 46. (3) | 47. (1) | 48. (4) | 49. (3) | 50. (2) |
| 51. (2) | 52. (2) | 53. (1) | 54. (4) | 55. (4) |
| 56. (4) | 57. (4) | 58. (2) | 59. (3) | |

Multiple Correct Answers Type

- | | |
|-----------------------|------------------------|
| 1. (2), (3), (4) | 2. (1), (4) |
| 3. (3), (4) | 4. (2), (4) |
| 5. (1), (2) | 6. (1), (3), (4) |
| 7. (1), (2) | 8. (1), (3) |
| 9. (1), (2), (3), (4) | 10. (1), (2), (3), (4) |

11. (1), (3)

12. (2), (3)

13. (2), (4)

Linked Comprehension Type

- | | | | | |
|--------|--------|--------|--------|---------|
| 1. (4) | 2. (2) | 3. (4) | 4. (4) | 5. (2) |
| 6. (3) | 7. (4) | 8. (2) | 9. (2) | 10. (3) |

Matrix Match Type

- $a \rightarrow p$; $b \rightarrow p, s$; $c \rightarrow p, r$; $d \rightarrow q, s$
- $a \rightarrow s$; $b \rightarrow p$; $c \rightarrow q$; $d \rightarrow r$
- $a \rightarrow r$; $b \rightarrow s$; $c \rightarrow q$; $d \rightarrow p$
- (2)

Numerical Value Type

- | | | | | |
|-----------|----------|--------|----------|---------|
| 1. (220) | 2. (7) | 3. (5) | 4. (0.6) | 5. (4) |
| 6. (37.5) | 7. (8) | 8. (2) | 9. (3) | 10. (4) |
| 11. (2) | 12. (54) | | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (2) | 2. (3) | 3. (4) | 4. (1) | 5. (4) |
| 6. (4) | | | | |

2

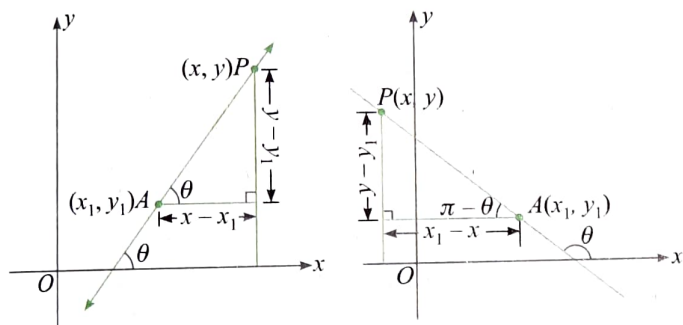
Straight Lines

EQUATION OF LINE IN DIFFERENT FORMS

We know that equation of straight line is linear equation in variables x and y . The general form of equation of straight line is $ax + by + c = 0$, where $a, b, c \in \mathbb{R}$ and at least one of a and b is non-zero. We have different forms of equation of straight line depending upon through which point/points it is passing and in which direction it is drawn. Here, direction of line means slope of line.

EQUATION OF LINE IN POINT-SLOPE FORM

Let the line having slope m pass through the point $A(x_1, y_1)$.



For any point $P(x, y)$ on the line, we have

$$\frac{y - y_1}{x - x_1} = m$$

or $y - y_1 = m(x - x_1)$

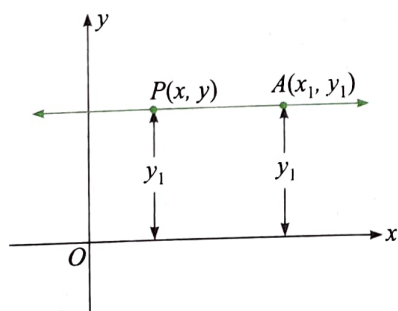
This is the required equation of straight line. This is the most commonly used form of straight line.

If the line is passing through the point $A(x_1, y_1)$ and is inclined at an angle θ with positive x -axis, then its equation is

$$y - y_1 = (\tan \theta)(x - x_1)$$

Equation of Line Parallel to an Axis

Let the line pass through the point $A(x_1, y_1)$ and be parallel to x -axis, i.e., horizontal.



Here, slope of line is zero or we can say that $m = 0$.

So, equation of line is $y - y_1 = 0$ or $y = y_1$.

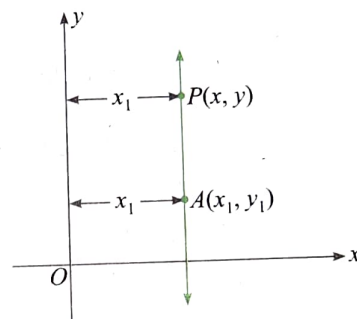
For any point on the line, the value of ordinate is y_1 , i.e., each point on the line is at distance $|y_1|$ from x -axis. So, equation of locus is $y = y_1$.

Let the line pass through point $A(x_1, y_1)$ and be parallel to y -axis, i.e., vertical.

Here, slope of line is infinity, i.e., $m \rightarrow \infty$.

So, equation of line is

$$\frac{y - y_1}{\infty} = x - x_1 \quad \text{or} \quad x - x_1 = 0 \quad \text{or} \quad x = x_1$$



For any point on the line, the value of abscissa is x_1 , i.e., each point on the line is at distance $|x_1|$ from y -axis. So, equation of locus is $x = x_1$.

Equation of Line Parallel or Perpendicular to the Given Line

Let the given line be $ax + by + c = 0$.

Slope of line is $-\frac{a}{b}$.

Then slope of line parallel to given line is $-\frac{a}{b}$.

So, equation of line parallel to given line is $ax + by + \lambda = 0$.

The value of λ can be obtained using some condition given in question.

Now, slope of the line perpendicular to given line is $\frac{b}{a}$.

So, equation of line perpendicular to given line is $bx - ay + \lambda = 0$.

ILLUSTRATION 2.1

Find the equation of line passing through point $(2, 3)$ which is

- parallel to the x -axis
- parallel to the y -axis

Sol. Equation of the line passing through (2, 3) which is (i) parallel to x -axis is $y = 3$ and (ii) parallel to y -axis is $x = 2$.

ILLUSTRATION 2.2

Find the equation of line passing through point (2, -5) which is

- (i) parallel to the line $3x + 2y + 4 = 0$
 (ii) perpendicular to the line $3x + 2y - 4 = 0$

Sol.

- (i) Equation of line parallel to the line $3x + 2y - 4 = 0$ is $3x + 2y + \lambda = 0$.

If this line passes through (2, -5), then

$$3(2) + 2(-5) + \lambda = 0$$

$$\therefore \lambda = 4$$

So, equation of line is $3x + 2y + 4 = 0$.

Alternative method:

Slope of given line $3x + 2y - 4 = 0$ is $-\frac{3}{2}$.

Thus, equation of line through point (2, -5) having slope

$$-\frac{3}{2} \text{ is}$$

$$y + 5 = -\frac{3}{2}(x - 2)$$

$$\text{or } 3x + 2y + 4 = 0$$

- (ii) Equation of line perpendicular to the line $3x + 2y - 4 = 0$ is $2x - 3y + \lambda = 0$.

If this line passes through (2, -5), then

$$2(2) - 3(-5) + \lambda = 0$$

$$\therefore \lambda = -19$$

So, equation of line is $2x - 3y - 19 = 0$.

Alternative method:

Slope of line perpendicular to the line $3x + 2y - 4 = 0$ is $\frac{2}{3}$.

Thus, equation of line through point (2, -5) having slope $\frac{2}{3}$ is

$$y + 5 = \frac{2}{3}(x - 2)$$

$$\text{or } 2x - 3y - 19 = 0$$

ILLUSTRATION 2.3

Find the equation of the perpendicular bisector of the line segment joining the points $A(2, 3)$ and $B(6, -5)$.

Sol. The slope of AB is given by

$$m = \frac{-5 - 3}{6 - 2} = -2$$

Therefore, the slope of the line perpendicular to AB is

$$-\frac{1}{m} = \frac{1}{2}$$

Let P be the midpoint of AB . Then, the coordinates of P are

$$\left(\frac{2+6}{2}, \frac{3-5}{2}\right), \text{ i.e., } (4, -1)$$

Thus, the required line passes through $P(4, -1)$ and has slope $1/2$.

So, its equation is

$$y + 1 = \frac{1}{2}(x - 4)$$

$$\text{or } x - 2y - 6 = 0$$

$$[(\text{Using } y - y_1 = m(x - x_1))]$$

ILLUSTRATION 2.4

Find the locus of a point P which moves such that its distance from the line $y = \sqrt{3}x - 7$ is the same as its distance from $(2\sqrt{3}, -1)$.

Sol. The point $(2\sqrt{3}, -1)$ lies on the line $y = \sqrt{3}x - 7$.

Therefore, locus of the point is a straight line perpendicular to the given line passing through the given point, i.e.,

$$y + 1 = -\frac{1}{\sqrt{3}}(x - 2\sqrt{3})$$

$$\text{or } x + \sqrt{3}y = \sqrt{3}$$

ILLUSTRATION 2.5

Consider a triangle with vertices $A(1, 2)$, $B(3, 1)$, and $C(-3, 0)$. Find

- (i) the equation of altitude through vertex A .
 (ii) the equation of median through vertex A .
 (iii) the equation of internal angle bisector of $\angle A$.

Sol.

- (i) Let the altitude through A be AD .

Now, $AD \perp BC$

$$\text{Slope of } BC = \frac{1 - 0}{3 - (-3)} = \frac{1}{6}$$

$$\therefore \text{Slope of } AD = -6$$

Therefore, the equation of altitude AD is

$$y - 2 = -6(x - 1)$$

$$\text{or } 6x + y - 8 = 0$$

- (ii) The median through A passes through the midpoint of BC . Now, the midpoint of BC is $E((3 - 3)/2, (1 + 0)/2)$ or $E(0, 1/2)$.

Therefore, the equation of

$$\text{median } AE \text{ is } y - 2 = \frac{2 - (1/2)}{1 - 0}(x - 1)$$

$$\text{or } 3x - 2y + 1 = 0$$

- (iii) Let the internal bisector of A meet the side BC in F .

Now, the point F divides BC in the ratio

$$\frac{BF}{CF} = \frac{AB}{AC} = \frac{\sqrt{5}}{\sqrt{20}} = \frac{1}{2}$$

Therefore, the coordinates of F are

$$\left(\frac{2(3) + 1(-3)}{2 + 1}, \frac{2(1) + 1(0)}{2 + 1}\right) = \left(1, \frac{2}{3}\right)$$

Therefore, the slope of AF is ∞ .

Hence, the equation of AF is $x - 1 = 0$.

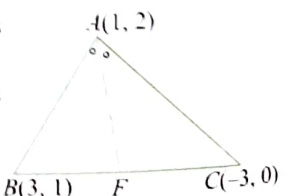
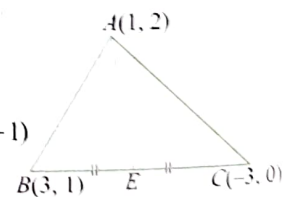
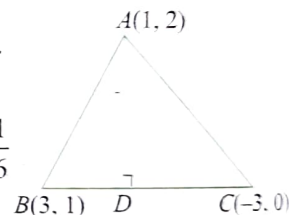
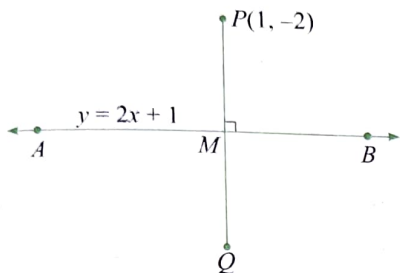


ILLUSTRATION 2.6

Find the coordinates of the foot of the perpendicular drawn from the point $P(1, -2)$ on the line $y = 2x + 1$. Also, find the image of P in the line.

Sol. Given line is $y = 2x + 1$.

Let M be the foot of the perpendicular drawn from $P(1, -2)$ on the line given line.



Slope of given line (AB) is 2.

$$\therefore \text{Slope of } PM = -\frac{1}{2}$$

Thus, equation of line PM is

$$y + 2 = -\frac{1}{2}(x - 1)$$

$$\text{or } x + 2y + 3 = 0 \quad (2)$$

Solving equations (1) and (2), we get $M(-1, -1)$, which is the required foot of the perpendicular on given line. Also, if Q is image of P in the line, then M is the midpoint of PQ .

Therefore, coordinates of point Q are $(-3, 0)$.

ILLUSTRATION 2.7

If the line $\frac{x}{a} + \frac{y}{b} = 1$ moves in such a way that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$,

where c is a constant, then prove that the foot of perpendicular from the origin upon the straight line describes the curve $x^2 + y^2 = c^2$.

Sol. Given variable line is $\frac{x}{a} + \frac{y}{b} = 1$ (1)

Line perpendicular to (1) and passing through the origin is

$$\frac{x}{b} - \frac{y}{a} = 0 \quad (2)$$

Now, foot of perpendicular $P(h, k)$ from the origin upon the line (1) is the point of intersection of lines (1) and (2).

$$\frac{h}{a} + \frac{k}{b} = 1 \quad (3)$$

$$\text{and } \frac{h}{b} - \frac{k}{a} = 0 \quad (4)$$

Squaring and adding (3) and (4), we get

$$h^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) + k^2 \left(\frac{1}{b^2} + \frac{1}{a^2} \right) = 1$$

But, it is given that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

$$\therefore (h^2 + k^2) \frac{1}{c^2} = 1.$$

Hence, equation of locus of P is $x^2 + y^2 = c^2$.

EQUATION OF LINE PASSING THROUGH TWO POINTS (TWO POINTS FORM)

Let a line passes through points $A(x_1, y_1)$ and $B(x_2, y_2)$. Now, consider a variable point $P(x, y)$ on this line. We have to find the equation of locus of point P .

Clearly,

Slope of AP = Slope of AB

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} = m \text{ (slope)}$$

$$\Rightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad (1)$$

This is the required equation of straight line.

Also, we have

Slope of BP = Slope of AB

$$\therefore \frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow y - y_2 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_2) \quad (2)$$

Equations (1) and (2) represent the same straight line.

Further, since points A, B and P are collinear, we have

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \quad (3)$$

This is the equation of line in determinant form.

We can get the equation of line using section formula also.

Since point P lies on the line through points A and B , it divides AB in some ratio say ' t '.

$$\therefore x = \frac{tx_2 + x_1}{t + 1}, y = \frac{ty_2 + y_1}{t + 1}, \text{ where } t \text{ is parameter.}$$

Eliminating t , we get the required equation of straight line.

ILLUSTRATION 2.8

In what ratio does the line joining the points $(2, 3)$ and $(4, 1)$ divide the segment joining the points $(1, 2)$ and $(4, 3)$?

Sol. The equation of the line joining the points $(2, 3)$ and $(4, 1)$ is

$$y - 3 = \frac{1 - 3}{4 - 2} (x - 2)$$

$$\text{or } y - 3 = -x + 2$$

$$\text{or } x + y - 5 = 0 \quad (1)$$

Suppose the line joining $(2, 3)$ and $(4, 1)$ divides the segment joining $(1, 2)$ and $(4, 3)$ at point P in the ratio $\lambda : 1$.

Then the coordinates of P are

$$\left(\frac{4\lambda + 1}{\lambda + 1}, \frac{3\lambda + 2}{\lambda + 1} \right)$$

Clearly, P lies on (1). Therefore,

$$\frac{4\lambda + 1}{\lambda + 1} + \frac{3\lambda + 2}{\lambda + 1} = 5$$

or $\lambda = 1$

Hence, the required ratio is $\lambda : 1$, i.e., $1 : 1$.

ILLUSTRATION 2.9

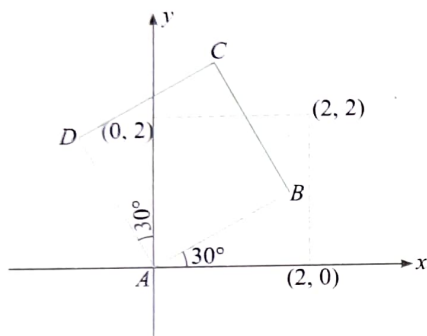
$ABCD$ is a square whose vertices are $A(0, 0)$, $B(2, 0)$, $C(2, 2)$, and $D(0, 2)$. The square is rotated in the XY -plane through an angle 30° in the anticlockwise sense about an axis passing through A perpendicular to the XY -plane. Find the equation of the diagonal BD of this rotated square.

Sol. We have

$$B \equiv (2 \cos 30^\circ, 2 \sin 30^\circ) = (\sqrt{3}, 1)$$

$$D \equiv (-2 \sin 30^\circ, 2 \cos 30^\circ) = (-1, \sqrt{3})$$

Hence, the equation of BD is



$$y - 1 = \frac{\sqrt{3} - 1}{-(\sqrt{3} + 1)}(x - \sqrt{3})$$

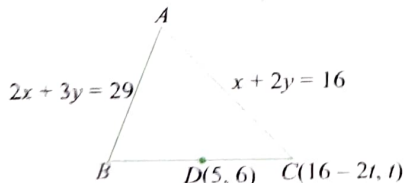
$$= (\sqrt{3} - 2)(x - \sqrt{3})$$

$$\text{i.e., } (2 - \sqrt{3})x + y = 2\sqrt{3} - 2$$

ILLUSTRATION 2.10

In a triangle ABC , side AB has the equation $2x + 3y = 29$ and side AC has the equation $x + 2y = 16$. If the midpoint of BC is $(5, 6)$, then find the equation of BC .

Sol.



Vertex C lies on the line $x + 2y = 16$.

So, let point C be $(16 - 2t, t)$.

$D(5, 6)$ is midpoint of BC .

So, point B is $(2t - 6, 12 - t)$

Point B lies on the line $2x + 3y - 29 = 0$

$$\therefore 2(2t - 6) + 3(12 - t) - 29 = 0$$

$$\Rightarrow t = 5$$

Hence, point B is $(4, 7)$

Therefore, equation of line BC is

$$y - 7 = \frac{6 - 7}{5 - 4}(x - 4)$$

$$\text{or } x + y = 11$$

ILLUSTRATION 2.11

Two consecutive sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation of one diagonal is $11x + 7y = 9$, find the equation of the other diagonal.

Sol. Let the equations of sides AB and AD of the parallelogram $ABCD$ be as given in (1) and (2), respectively, i.e.,

$$4x + 5y = 0 \quad (1)$$

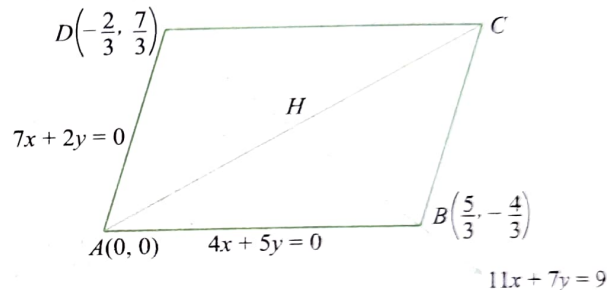
$$\text{and } 7x + 2y = 0 \quad (2)$$

Solving (1) and (2), we have

$$x = 0, y = 0$$

$$\therefore A \equiv (0, 0)$$

$$D \left(-\frac{2}{3}, \frac{7}{3} \right)$$



The equation of one diagonal of the parallelogram is

$$11x + 7y = 9 \quad (3)$$

Clearly, $A(0, 0)$ does not lie on the diagonal as shown in (3).

Therefore, (3) is the equation of diagonal BD .

Solving (1) and (3), we get $B \equiv (5/3, -4/3)$.

Solving (2) and (3), we get $D \equiv (-2/3, 7/3)$.

Since H is the middle point of BD , we have

$$H \equiv \left(\frac{1}{2}, \frac{1}{2} \right)$$

Now, the equation of diagonal AC which passes through $A(0, 0)$ and $H(1/2, 1/2)$ is

$$y - 0 = \frac{0 - (1/2)}{0 - (1/2)}(x - 0) \text{ or } y - x = 0$$

ILLUSTRATION 2.12

If one of the sides of a square is $3x - 4y - 12 = 0$ and the center is $(0, 0)$, then find the equations of the diagonals of the square.

Sol. The diagonals of square are inclined at an angle of 45° with the sides.

Slope of the line $3x - 4y - 12 = 0$ is $3/4$.

Let the slope of the diagonal be m . Then,

$$\tan 45^\circ = \left| \frac{m - (3/4)}{1 + (3/4)m} \right|$$

$$\therefore 4 + 3m = \pm(4m - 3)$$

$$\text{or } 4 + 3m = 4m - 3 \text{ or } 4 + 3m = 3 - 4m$$

or $m = 7$ or $m = -\frac{1}{7}$

Therefore, the equations of diagonals are $7x - y = 0$ and $x + 7y = 0$.

ILLUSTRATION 2.13

A vertex of an equilateral triangle is $(2, 3)$ and the equation of the opposite side is $x + y = 2$. Find the equation of the other sides of the triangle.

Sol. Given line is

$$x + y - 2 = 0 \quad (1)$$

Its slope is $m_1 = -1$.

Let the slope of the line be m which makes an angle of 60° with the line in (1). Then,

$$\tan 60^\circ = \left| \frac{m_1 - m}{1 + m_1 m} \right| \text{ or } \sqrt{3} = \left| \frac{-1 - m}{1 - m} \right|$$

$$\text{or } \sqrt{3} = \left| \frac{1 + m}{m - 1} \right| \text{ or } \frac{1 + m}{m - 1} = \pm \sqrt{3}$$

$$\text{or } 1 + m = \pm \sqrt{3}(m - 1)$$

$$\text{or } m = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}, \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= 2 + \sqrt{3}, 2 - \sqrt{3}$$

Therefore, the equations of other two sides of the triangle are

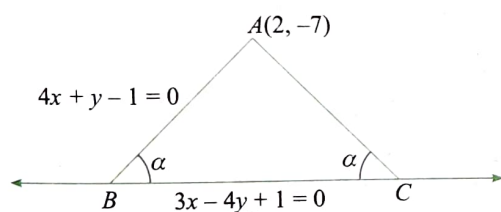
$$y - 3 = (2 + \sqrt{3})(x - 2)$$

$$\text{and } y - 3 = (2 - \sqrt{3})(x - 2)$$

ILLUSTRATION 2.14

A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B . Find the equation of the line AC , so that $AB = AC$.

Sol.



Let the equation of BC be

$$3x - 4y + 1 = 0 \quad (1)$$

and the equation of AB be

$$4x + y - 1 = 0 \quad (2)$$

Since $AB = AC$, we have

$$\angle ABC = \angle ACB = \alpha \text{ (say)}$$

$$\text{Slope of line } BC = \frac{3}{4}$$

$$\text{and Slope of } AB = -4$$

Let the slope of AC be m . Equating the two values of $\tan \alpha$, we get

$$\left| \frac{-4 - \frac{3}{4}}{1 - 4 \times \frac{3}{4}} \right| = \left| \frac{\frac{3}{4} - m}{1 + \frac{3}{4}m} \right|$$

$$\text{or } \pm \frac{19}{8} = \frac{3 - 4m}{4 + 3m}$$

$$\text{or } m = \frac{52}{89}, -4$$

Therefore, the equation of AC is

$$y + 7 = -\left(\frac{52}{89}\right)(x - 2) \text{ or } 52x + 89y + 519 = 0$$

ILLUSTRATION 2.15

A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Sol. Incident ray is along the line

$$x - 2y - 3 = 0 \quad (1)$$

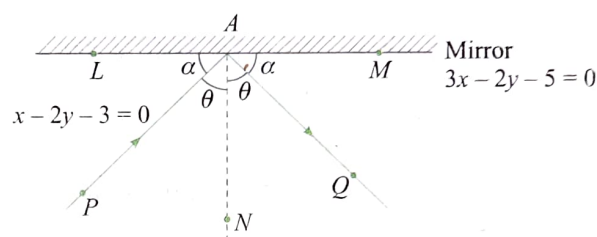
Mirror is along the line

$$3x - 2y - 5 = 0 \quad (2)$$

In the figure, incident ray PA strikes the mirror at point A .

Solving the incident ray and mirror line, we get $A \equiv (1, -1)$.

Let the slope of AQ be m .



Also, slope of LM is $\frac{3}{2}$ and slope of PA is $\frac{1}{2}$.

Now, incident ray and reflected ray make same angle α with the mirror line.

Let the slope of reflected ray be m .

Now, $\angle LAP = \angle QAM = \alpha$

$$\Rightarrow \tan \alpha = \left| \frac{\frac{3}{2} - \frac{1}{2}}{1 + \frac{3}{2} \cdot \frac{1}{2}} \right| = \left| \frac{m - \frac{3}{2}}{1 + m \cdot \frac{3}{2}} \right|$$

$$\Rightarrow \frac{4}{7} = \left| \frac{2m - 3}{2 + 3m} \right|$$

$$\Rightarrow \frac{2m - 3}{2 + 3m} = \pm \frac{4}{7}$$

$$\Rightarrow m = \frac{1}{2}, \frac{29}{2}$$

But slope of incident ray is $\frac{1}{2}$.

Thus, slope of reflected ray is $\frac{29}{2}$.

Hence, the equation of reflected rays is

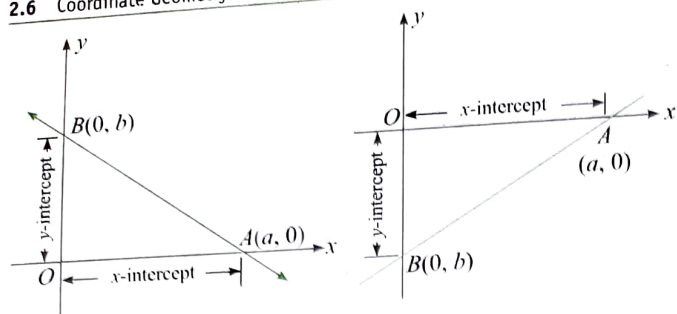
$$y + 1 = \frac{29}{2}(x - 1)$$

$$\text{or } 29x - 2y - 31 = 0$$

EQUATION OF LINE IN INTERCEPT FORM

One of the important aspects of straight line is its intercepts.

Line has two intercepts: x -intercept and y -intercept.



If the line intersects axes at $A(a, 0)$ and $B(0, b)$, then a and b are called its x -intercept and y -intercept, respectively. For example, line $2x - 3y + 6 = 0$ intersects axes at $P(-3, 0)$ and $Q(0, 2)$. So, x -intercept of line is -3 and y -intercept is 2 .

Conversely, if the line has intercepts a and b on axes, then it meets the axes at $A(a, 0)$ and $B(0, b)$ and slope of line is $-\frac{b}{a}$.

A line having equal intercepts, cuts the positive axes or negative axes. Slope of such line is -1 .

Slope-Intercept Form of Line

For a line, let the y -intercept be c and the slope be m .

Line meets y -axis at $(0, c)$.

So, using point-slope form, equation of line is

$$y - c = m(x - 0) \text{ or } y = mx + c$$

This form of line is called slope-intercept form.

Here, c is y -intercept of the line.

This form of straight line is useful in expressing linear function as $f(x) = mx + c$.

Intercept Form of Line

Let x -intercept and y -intercept of line be a and b , respectively.

Clearly, line meets the axes at $A(a, 0)$ and $B(0, b)$.

$$\text{Slope of line} = \frac{0 - b}{a - 0} = -\frac{b}{a}$$

So, equation of line is

$$y - 0 = -\frac{b}{a}(x - a)$$

$$\text{or } \frac{x}{a} + \frac{y}{b} = 1.$$

This is the intercept form of line.

ILLUSTRATION 2.16

Find the equation of the line which intersects the y -axis at a distance of 2 units above the origin and makes an angle of 30° with the positive direction of the x -axis.

Sol. Here, $c = 2$ and $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$.

Thus, the required equation of line is

$$y = \frac{1}{\sqrt{3}}x + 2$$

$$\text{or } x - \sqrt{3}y + 2\sqrt{3} = 0$$

ILLUSTRATION 2.17

Find the equation of a straight line cutting off an intercept -1 from the y -axis and being equally inclined to the axes.

Sol. Since the required line is equally inclined with the coordinate axes, it makes an angle of either 45° or 135° with the x -axis.

So, its slope is either $m = \tan 45^\circ$ or $m = \tan 135^\circ$, i.e., $m = 1$ or -1 .

It is given that $c = -1$. Hence, the equations of the lines are

$$y = x - 1 \text{ and } y = -x - 1$$

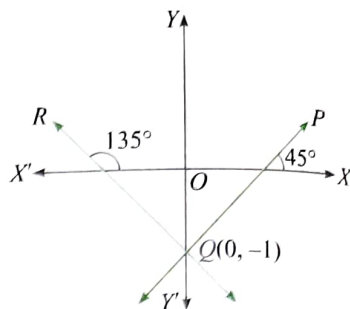


ILLUSTRATION 2.18

Find the equation of a line that has y -intercept 4 and is perpendicular to the line joining $(2, -3)$ and $(4, 2)$.

Sol. Let m be the slope of the required line.

Since the required line is perpendicular to the line joining $A(2, -3)$ and $B(4, 2)$, we have

$$m \times \text{Slope of } AB = -1$$

$$\text{or } m \times \frac{2 + 3}{4 - 2} = -1$$

$$\text{or } m = -\frac{2}{5}$$

The required line cuts-off an intercept of 4 on the y -axis. So, $c = 4$.

Hence, the equation of the required line is

$$y = -\frac{2}{5}x + 4$$

$$\text{or } 2x + 5y - 20 = 0$$

ILLUSTRATION 2.19

Find the equation of the line passing through the point $(2, 2)$ and cutting off intercepts on the axes whose sum is 9.

Sol. The equation of a line in the intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1$$

This line passes through $(2, 2)$. Therefore,

$$\frac{2}{a} + \frac{2}{b} = 1 \quad (1)$$

It is given that $a + b = 9$, i.e.,

$$b = 9 - a \quad (2)$$

From (1) and (2), we get

$$\frac{2}{a} + \frac{2}{9 - a} = 1$$

$$\text{or } a^2 - 9a + 18 = 0$$

$$\text{or } (a - 6)(a - 3) = 0$$

$$\text{i.e., } a = 6 \text{ or } a = 3$$

If $a = 6$ and $b = 9 - 6 = 3$, then the equation of the line is

$$\frac{x}{6} + \frac{y}{3} = 1 \text{ or } x + 2y - 6 = 0$$

If $a = 3$ and $b = 9 - 3 = 6$, then the equation of the line is

$$\frac{x}{3} + \frac{y}{6} = 1 \text{ or } 2x + y - 6 = 0$$

ILLUSTRATION 2.20

Find the equation of the straight line that

- makes equal intercepts on the axes and passes through the point $(2, 3)$.
- passes through the point $(-5, 4)$ and is such that the portion intercepted between the axes is divided by the point in the ratio $1 : 2$.

Sol.

- Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Since it makes equal intercepts on the coordinate axes, we have $a = b$.

So, the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \text{ or } x + y = a$$

The line passes through the point $(2, 3)$. Therefore,

$$2 + 3 = a$$

$$\text{or } a = 5$$

Thus, the equation of the required line is $x + y = 5$.

- Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

Clearly, this line meets the coordinate axes at $A(a, 0)$ and $B(0, b)$, respectively.

The coordinates of the point that divides the line joining $A(a, 0)$ and $B(0, b)$ in the ratio $1 : 2$ are

$$\left(\frac{1(0) + 2(a)}{1+2}, \frac{1(b) + 2(0)}{1+2} \right) \equiv \left(\frac{2a}{3}, \frac{b}{3} \right)$$

It is given that the point $(-5, 4)$ divides AB in the ratio $1 : 2$.

Therefore, $2a/3 = -5$ and $b/3 = 4$, i.e.,

$$a = -\frac{15}{2} \text{ and } b = 12$$

Hence, the equation of the required line is

$$-\frac{x}{15/2} + \frac{y}{12} = 1$$

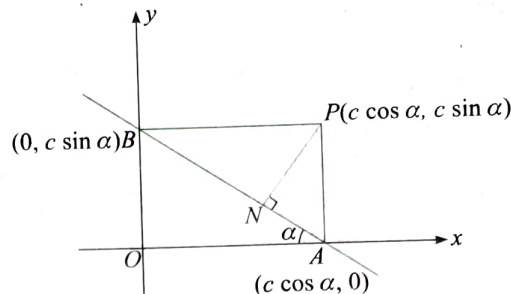
$$\text{or } 8x - 5y + 60 = 0$$

ILLUSTRATION 2.21

Line segment AB of fixed length c slides between coordinate axes such that its ends A and B lie on the axes. If O is origin and rectangle $OAPB$ is completed, then show that the locus of the foot of the perpendicular drawn from P to AB is

$$\frac{2}{x^3} + \frac{2}{y^3} = \frac{2}{c^3}$$

Sol.



As shown in the figure, A is on x -axis and B is on y -axis.

$$AB = c \text{ (given)}$$

Given that $OAPB$ is a rectangle.

Let $\angle BAO = \alpha$.

$$\therefore OA = c \cos \alpha \text{ and } OB = c \sin \alpha$$

$$\therefore P \equiv (c \cos \alpha, c \sin \alpha)$$

Equation of AB is

$$\frac{x}{c \cos \alpha} + \frac{y}{c \sin \alpha} = 1$$

$$\text{or } x \sin \alpha + y \cos \alpha = c \sin \alpha \cos \alpha \quad (1)$$

Equation of PN (perpendicular from P to AB) is

$$y - c \sin \alpha = \cot \alpha (x - c \cos \alpha)$$

$$\Rightarrow x \cos \alpha - y \sin \alpha = c(\cos^2 \alpha - \sin^2 \alpha) \quad (2)$$

N is point of intersection of (1) and (2).

Solving (1) and (2), we get

$$x = c \cos^3 \alpha, y = c \sin^3 \alpha$$

$$\therefore \cos \alpha = \left(\frac{x}{c} \right)^{1/3}, \sin \alpha = \left(\frac{y}{c} \right)^{1/3}$$

Squaring and adding, we get

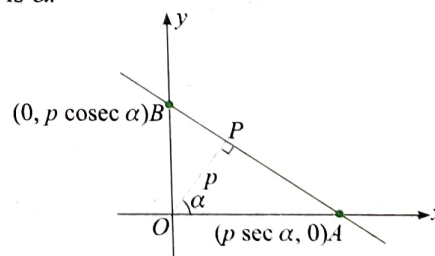
$$\left(\frac{x}{c} \right)^{2/3} + \left(\frac{y}{c} \right)^{2/3} = 1$$

or $x^{2/3} + y^{2/3} = c^{2/3}$, which is the required locus.

EQUATION OF LINE IN NORMAL FORM

Let the line L be such that

- length of the perpendicular (normal) from origin to the line is p , and
- angle which normal makes with the positive direction of x -axis is α .



In the above figure, line L meets x -axis at A and y -axis at B on positive sides.

Also, OP is perpendicular to the line L .

From the figure, x -intercept of line is $(p \sec \alpha)$ and y -intercept is $(p \csc \alpha)$.

So, intercept form of line is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1$$

or $x \cos \alpha + y \sin \alpha = p$

The angle α of normal with positive x -axis for different orientations of line is as shown in the following figures.

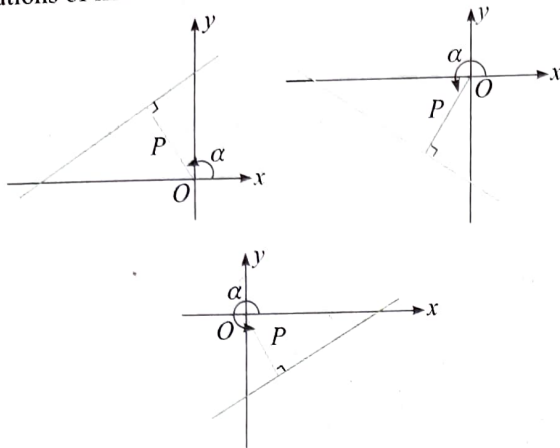


ILLUSTRATION 2.22

Reduce the line $2x - 3y + 5 = 0$, in slope-intercept, intercept and normal forms. Also, find the distance of the line from origin and inclination of normal to the line with x -axis.

Sol. Equation of given line is

$$2x - 3y + 5 = 0$$

or $y = \frac{2x}{3} + \frac{5}{3}$

This is of the form $y = mx + c$.

Slope of the line is $m = \tan \theta = \frac{2}{3}$ and y -intercept is $c = \frac{5}{3}$.

Here, θ is the angle of the line with positive x -axis.

Intercept form of the line is $\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{3}\right)} = 1$, where x -intercept

is $a = -\frac{5}{2}$ and y -intercept is $b = \frac{5}{3}$.

Further to get the equation in normal form, divide both sides by $\sqrt{2^2 + 3^2} = \sqrt{13}$.

So, equation of line becomes $-\frac{2x}{\sqrt{13}} + \frac{3y}{\sqrt{13}} = \frac{5}{\sqrt{13}}$.

Comparing with standard normal form $x \cos \alpha + y \sin \alpha = p$, we get

$$\cos \alpha = -\frac{2}{\sqrt{13}}, \sin \alpha = \frac{3}{\sqrt{13}} \text{ and}$$

$$p = \frac{5}{\sqrt{13}} = \text{Distance of line from origin}$$

Also, inclination of normal to the line with x -axis is

$$\alpha = 180^\circ - \sin^{-1} \frac{3}{\sqrt{13}}.$$

ILLUSTRATION 2.23

Find the equation of the line which is at a perpendicular distance of 5 units from the origin and the angle made by the perpendicular with the positive x -axis is 30° .

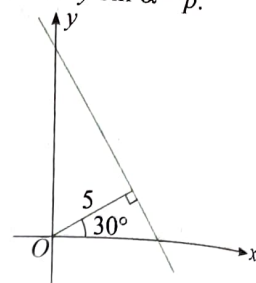
Sol. Data given in the question is for normal form of the line. Standard equation of normal form is $x \cos \alpha + y \sin \alpha = p$.

$$\therefore p = 5 \text{ units and } \alpha = 30^\circ.$$

Thus, the required equation of the given line is

$$x \cos 30^\circ + y \sin 30^\circ = 5$$

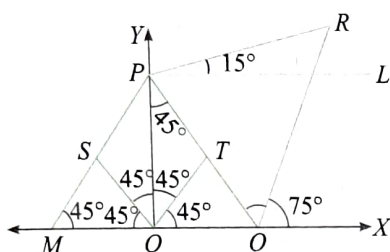
or $\sqrt{3}x + y = 10$



CONCEPT APPLICATION EXERCISE 2.1

- Find the equation of the right bisector of the line segment joining the points $(3, 4)$ and $(-1, 2)$.
- If the coordinates of the points A, B, C , and D are (a, b) , (a', b') , $(-a, b)$, and $(a', -b')$, respectively, then find the equation of the line bisecting the line segments AB and CD .
- If the coordinates of the vertices of triangle ABC are $(-1, 6)$, $(-3, -9)$, and $(5, -8)$, respectively, then find the equation of the median through C .
- Find the equation of the line perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$ and passing through a point at which it cuts the x -axis.
- If the middle points of the sides BC, CA , and AB of triangle ABC are $(1, 3)$, $(5, 7)$, and $(-5, 7)$, respectively, then find the equation of the side AB .
- Find the equations of the lines which pass through the origin and are inclined at an angle $\tan^{-1} m$ to the line $y = mx + c$.
- If $(-2, 6)$ is the image of the point $(4, 2)$ with respect to line $L = 0$, then find the equation of line L .
- Find the area bounded by the curves $x + 2|y| = 1$ and $x = 0$.
- Find the equation of the straight line passing through the intersection of the lines $x - 2y = 1$ and $x + 3y = 2$ and parallel to $3x + 4y = 0$.
- If the foot of the perpendicular from the origin to a straight line is at $(3, -4)$, then find the equation of the line.
- A straight line through the point $(2, 2)$ intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B , respectively. Then find the equation of the line AB so that triangle OAB is equilateral.
- Find the equation of the straight line passing through the point $(4, 3)$ and making intercepts on the coordinate axes whose sum is -1 .
- A straight line through the point $A(3, 4)$ is such that its intercept between the axis is bisected at A . Find its equation.
- A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by line L and the coordinate axes is 5. Find the equation of line L .
- One side of a rectangle lies along the line $4x + 7y + 5 = 0$. Two of its vertices are $(-3, 1)$ and $(1, 1)$. Find the equations of the other three sides.

16. A line $L_1 \equiv 3y - 2x - 6 = 0$ is rotated about its point of intersection with the y -axis in the clockwise direction to make it L_2 such that the area formed by L_1, L_2 , the x -axis, and line $x = 5$ is $49/3$ sq. units if its point of intersection with $x = 5$ lies below the x -axis. Find the equation of L_2 .
17. The diagonals AC and BD of a rhombus intersect at $(5, 6)$. If $A \equiv (3, 2)$, then find the equation of diagonal BD .
18. Find the equation of the straight line which passes through the origin and makes angle 60° with the line $x + \sqrt{3}y + 3\sqrt{3} = 0$.
19. A line intersects the straight lines $5x - y - 4 = 0$ and $3x - 4y - 4 = 0$ at A and B , respectively. If a point $P(1, 5)$ on the line AB is such that $AP : PB = 2 : 1$ (internally), find point A .
20. In the given figure, PQR is an equilateral triangle and $OSPT$ is a square. If $OT = 2\sqrt{2}$ units, find the equation of lines OT, OS, SP, QR, PR , and PQ .



21. Two fixed points A and B are taken on the coordinates axes such that $OA = a$ and $OB = b$. Two variable points A' and B' are taken on the same axes such that $OA' + OB' = OA + OB$. Find the locus of the point of intersection of AB' and $A'B$.
22. A regular polygon has two of its consecutive diagonals as the lines $\sqrt{3}x + y = \sqrt{3}$ and $2y = \sqrt{3}$. Point $(1, c)$ is one of its vertices. Find the equation of the sides of the polygon and also find the coordinates of the vertices.
23. Find the direction in which a straight line must be drawn through the point $(-1, 2)$ so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

ANSWERS

- | | |
|---|---|
| 1. $2x + y = 5$ | 2. $2ay - 2b'x = ab - a'b'$ |
| 3. $13x + 14y + 47 = 0$ | 4. $\frac{x}{b} + \frac{y}{a} = \frac{a}{b}$ |
| 5. $x - y + 12 = 0$ | 6. $y = 0, y = \frac{2mx}{1 - m^2}$ |
| 7. $3x - 2y + 5 = 0$ | 8. $1/2$ |
| 9. $3x + 4y - 5 = 0$ | 10. $3x - 4y = 25$ |
| 11. $y - 2 = 0$ | 12. $\frac{x}{2} + \frac{y}{-3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$ |
| 13. $4x + 3y = 24$ | 14. $x + 5y \pm 5\sqrt{2} = 0$ |
| 16. $x + y = 2$ | 17. $2y + x - 17 = 0$ |
| 18. $x = 0, x - \sqrt{3}y = 0$ | 19. $(75/17, 307/17)$ |
| 21. $\frac{x}{aa'} + \frac{y}{bb'} = 0$ | 23. Parallel to the x -axis |

EQUATION OF STRAIGHT LINE IN SYMMETRIC FORM (PARAMETRIC OR DISTANCE FORM)

Consider a line passing through the point $P(x_1, y_1)$ and making an angle θ with positive x -axis.

Equation of line is

$$y - y_1 = \tan \theta (x - x_1)$$

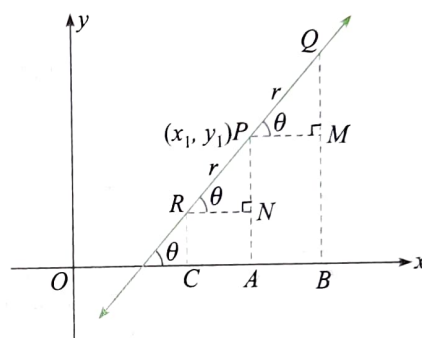
or $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$ (say)

$\therefore x = x_1 + r \cos \theta$ and $y = y_1 + r \sin \theta$

Here, $(x_1 + r \cos \theta, y_1 + r \sin \theta)$ are coordinates of point which is lying at distance r from the point $P(x_1, y_1)$ on the above line.

In fact, there must be two points at distance r from point P on the line.

The coordinates of other point are $(x_1 - r \cos \theta, y_1 - r \sin \theta)$.



In the above figure, $OA = x_1, AP = y_1$ and $PQ = PR = r$.

In triangle PMQ , $QM = r \sin \theta$ and $PM = r \cos \theta$.

$\therefore OB = OA + AB = OA + PM = x_1 + r \cos \theta$

and $BQ = BM + QM = AP + QM = y_1 + r \sin \theta$

$\therefore Q \equiv (x_1 + r \cos \theta, y_1 + r \sin \theta)$

Similarly, in triangle PNR , $PN = r \sin \theta$ and $RN = r \cos \theta$.

$\therefore OC = OA - AC = OA - RN = x_1 - r \cos \theta$

and $CR = AN = AP - PN = y_1 - r \sin \theta$

$\therefore R \equiv (x_1 - r \cos \theta, y_1 - r \sin \theta)$.

This form of straight line is effective tool in solving geometry related problems.

ILLUSTRATION 2.24

A straight line is drawn through the point $P(2, 3)$ and is inclined at an angle of 30° with the x -axis. Find the coordinates of two points on it at a distance of 4 units from P .

Sol. Here, $P(x_1, y_1) \equiv P(2, 3)$, $\theta = 30^\circ$.

So, the required coordinates of points at distance 4 units from point P are

$$\begin{aligned} & (x_1 \pm r \cos \theta, y_1 \pm r \sin \theta) \\ & \equiv (2 \pm 4 \cos 30^\circ, 3 \pm 4 \sin 30^\circ) \\ & \equiv \left(2 \pm 4 \frac{\sqrt{3}}{2}, 3 \pm 4 \frac{1}{2} \right) \\ & \equiv (2 + 2\sqrt{3}, 5), (2 - 2\sqrt{3}, 1) \end{aligned}$$

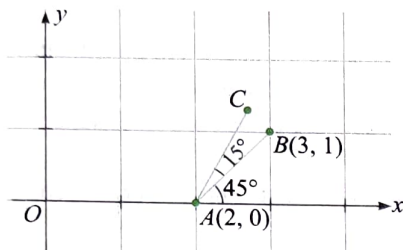
ILLUSTRATION 2.25

The line joining two points $A(2, 0)$ and $B(3, 1)$ is rotated about A in anticlockwise direction through an angle of 15° . Find the equation of the line in the new position. If B goes to C in the new position, then what will be the coordinates of C ?

Sol. The slope of the line AB is

$$m = \frac{1-0}{3-2} = 1 = \tan 45^\circ$$

So, inclination of line AB with x -axis is 45° .



Now, AB is rotated through an angle 15° in anticlockwise direction about A and point B goes to C .

So, inclination of line AC is 60° .

Hence, equation of line AC is

$$\frac{x-2}{\cos 60^\circ} = \frac{y-0}{\sin 60^\circ}$$

$$\text{or } \frac{x-2}{1/2} = \frac{y-0}{\sqrt{3}/2} = r \text{ (say)}$$

$$AC = AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}$$

So, for point C , $r = \sqrt{2}$.

$$\therefore \frac{x-2}{1/2} = \frac{y-0}{\sqrt{3}/2} = \sqrt{2}$$

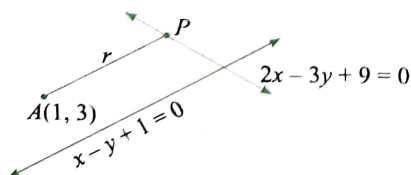
$$\Rightarrow x = 2 + \frac{1}{2}\sqrt{2} = 2 + \frac{\sqrt{2}}{2} \text{ and } y = \frac{\sqrt{3}}{2} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}$$

$$\text{Therefore, } C \equiv \left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2} \right)$$

ILLUSTRATION 2.26

A line through point $A(1, 3)$ and parallel to the line $x - y + 1 = 0$ meets the line $2x - 3y + 9 = 0$ at point P . Find distance AP without finding point P .

Sol.



Slope of line $x - y + 1 = 0$ is 1, i.e., $\tan 45^\circ$.

So, the equation of line through $A(1, 3)$ and having slope 1 is

$$\frac{x-1}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ}$$

$$\text{or } \frac{x-1}{1/\sqrt{2}} = \frac{y-3}{1/\sqrt{2}} = r \text{ (say)}$$

This line meets the given line $2x - 3y + 9 = 0$ at point P .

$$\text{Let } AP = r. \text{ Then } P \equiv \left(1 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}} \right).$$

But point P lies on the line $2x - 3y + 9 = 0$.

$$\therefore 2\left(1 + \frac{r}{\sqrt{2}}\right) - 3\left(3 + \frac{r}{\sqrt{2}}\right) + 9 = 0$$

$$\Rightarrow \frac{r}{\sqrt{2}} = 2$$

$$\Rightarrow r = 2\sqrt{2}$$

ILLUSTRATION 2.27

Two adjacent vertices of a square are $(1, 2)$ and $(-2, 6)$. Find the other vertices.

Sol. Given two adjacent vertices are $A(1, 2)$ and $B(-2, 6)$. Then,

$$AB = 5$$

Also, the slope of AB is

$$m_{AB} = \frac{6-2}{-2-1} = -\frac{4}{3}$$

$$\text{Slope of } AD = \frac{3}{4} = \tan \theta$$

The coordinates of D and D' are

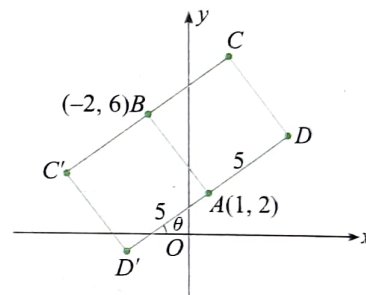
$$(1 \pm 5 \cos \theta, 2 \pm 5 \sin \theta) \equiv \left(1 \pm 5 \frac{4}{5}, 2 \pm 5 \frac{3}{5} \right)$$

$$\text{or } D(5, 5) \text{ and } D'(-3, -1)$$

Similarly, the coordinates of C and C' are

$$(-2 \pm 5 \cos \theta, 6 \pm 5 \sin \theta) \equiv \left(-2 \pm 5 \frac{4}{5}, 6 \pm 5 \frac{3}{5} \right)$$

$$\text{or } C(2, 9) \text{ and } C'(-6, 3)$$

**ILLUSTRATION 2.28**

A line through the variable point $A(k+1, 2k)$ meets the lines $7x + y - 16 = 0$, $5x - y - 8 = 0$ and $x - 5y + 8 = 0$, at B , C and D , respectively. Prove that AC , AB and AD are in H.P.

Sol. Let the line passing through point $A(k+1, 2k)$ make an angle θ with positive x -axis direction.

Then equation of line is

$$\frac{x - (k+1)}{\cos \theta} = \frac{y - 2k}{\sin \theta} = r_1, r_2, r_3;$$

where $AB = r_1$, $AC = r_2$, $AD = r_3$

$$\therefore B \equiv ((k+1) + r_1 \cos \theta, 2k + r_1 \sin \theta)$$

$$C \equiv ((k+1) + r_2 \cos \theta, 2k + r_2 \sin \theta)$$

$$D \equiv ((k+1) + r_3 \cos \theta, 2k + r_3 \sin \theta)$$

Points B , C and D satisfy lines $7x + y - 16 = 0$, $5x - y - 8 = 0$ and $x - 5y + 8 = 0$, respectively.

So, putting these points in the corresponding lines, we get

$$\therefore r_1 = \frac{9(1-k)}{7 \cos \theta + \sin \theta}, r_2 = \frac{3(1-k)}{5 \cos \theta - \sin \theta}$$

$$\text{and } r_3 = \frac{9(1-k)}{5 \sin \theta - \cos \theta}$$

$$\begin{aligned} \therefore \frac{1}{r_2} + \frac{1}{r_3} &= \frac{5 \cos \theta - \sin \theta}{3(1-k)} + \frac{(5 \sin \theta - \cos \theta)}{9(1-k)} \\ &= \frac{15 \cos \theta - 3 \sin \theta + 5 \sin \theta - \cos \theta}{9(1-k)} \\ &= \frac{14 \cos \theta + 2 \sin \theta}{9(1-k)} = \frac{2}{r_1} \end{aligned}$$

Clearly, AC , AB and AD are in H.P.

CONCEPT APPLICATION EXERCISE 2.2

- Two particles start from the point $(2, -1)$; one moves 2 units along the line $x + y = 1$ and the other moves 5 units along the line $x - 2y = 4$. If the particles move upward w.r.t. coordinate axes, then find their new positions.
- The centre of a square is at the origin and one vertex is $A(2, 1)$. Find the coordinates of other vertices of the square.
- The straight line passing through $P(x_1, y_1)$ and making an angle α with x -axis intersects $Ax + By + C = 0$ in Q . Find the length PQ .
- The centroid of an equilateral triangle is $(0, 0)$. If two vertices of the triangle lie on $x + y = 2\sqrt{2}$, then find all the possible vertices of triangle.

ANSWERS

- $(2 - \sqrt{2}, -1 + \sqrt{2})$ and $(2 + 2\sqrt{5}, -1 + \sqrt{5})$
- $(-2, -1)$, $(-1, 2)$ and $(1, -2)$
- $\frac{|Ax_1 + By_1 + C|}{|A \cos \alpha + B \sin \alpha|}$
- $(-2\sqrt{2}, -2\sqrt{2})$, $(\sqrt{2} - \sqrt{6}, \sqrt{2} + \sqrt{6})$,
 $(\sqrt{2} + \sqrt{6}, \sqrt{2} - \sqrt{6})$

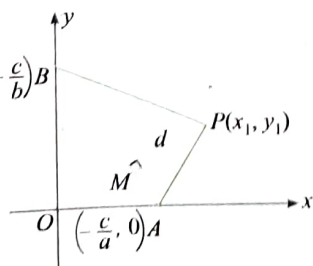
DISTANCE OF A POINT FROM A LINE

The distance of a point from a line is the length of the perpendicular drawn from the point to the line.

Let $L \equiv ax + by + c = 0$ be a line, whose distance from the point $P(x_1, y_1)$ is d .

Line L meets the axes at $A(-c/a, 0)$ and $B(0, -c/b)$.

Draw a perpendicular PM from the point P to the line L .



Area of triangle PAB

$$\begin{aligned} &= \frac{1}{2} \left| x_1 \left(0 + \frac{c}{b} \right) - \frac{c}{a} \left(-\frac{c}{b} - y_1 \right) + 0(y_1 - 0) \right| \\ &= \frac{1}{2} \left| \frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab} \right| \\ &= \left| \frac{ax_1 + by_1 + c}{2ab} \right| \end{aligned} \quad (1)$$

Also, area of $\triangle PAB = \frac{1}{2} AB \times PM$

$$\begin{aligned} &= \frac{1}{2} \sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \cdot d \\ &= \left| \frac{c}{2ab} \right| \sqrt{a^2 + b^2} \cdot d \end{aligned} \quad (2)$$

From (1) and (2), we get

$$\begin{aligned} \left| \frac{ax_1 + by_1 + c}{2ab} \right| &= \left| \frac{c}{2ab} \right| \sqrt{a^2 + b^2} \cdot d \\ d &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \end{aligned}$$

ILLUSTRATION 2.29

If p is the length of the perpendicular from the origin to the line $\frac{x}{a} + \frac{y}{b} = 1$, then prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Sol. The given line is

$$bx + ay - ab = 0 \quad (1)$$

It is given that p is the length of the perpendicular from the origin to (1), that is,

$$\begin{aligned} p &= \frac{|b(0) + a(0) - ab|}{\sqrt{b^2 + a^2}} \\ &= \frac{ab}{\sqrt{a^2 + b^2}} \end{aligned}$$

$$\text{or } p^2 = \frac{a^2 b^2}{a^2 + b^2}$$

$$\text{or } \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

ILLUSTRATION 2.30

Find the coordinates of a point on $x + y + 3 = 0$, whose distance from $x + 2y + 2 = 0$ is $\sqrt{5}$.

Sol. Putting $x = t$ in $x + y + 3 = 0$, we get $y = -3 - t$. So, let the required point be $(t, -3 - t)$. This point is at a distance of $\sqrt{5}$ units from $x + 2y + 2 = 0$. Therefore,

$$\left| \frac{t - 6 - 2t + 2}{\sqrt{1^2 + 2^2}} \right| = \sqrt{5}$$

$$\text{or } \left| \frac{-t - 4}{\sqrt{5}} \right| = \sqrt{5}$$

$$\text{or } t + 4 = \pm 5$$

$$\text{or } t = 1, -9$$

Hence, the required points are $(1, -4)$ and $(-9, 6)$.

ILLUSTRATION 2.31

Find the least and the greatest values of distance of the point $(\cos \theta, \sin \theta)$, $\theta \in R$, from the line $3x - 4y + 10 = 0$.

Sol. Distance of point $(\cos \theta, \sin \theta)$ from the line $3x - 4y + 10 = 0$ is

$$d = \frac{|3 \cos \theta - 4 \sin \theta + 10|}{\sqrt{3^2 + (-4)^2}} = \frac{|3 \cos \theta - 4 \sin \theta + 10|}{5}$$

Now, $-5 \leq 3 \cos \theta - 4 \sin \theta \leq 5$

$$5 \leq 3 \cos \theta - 4 \sin \theta + 10 \leq 15$$

$$\Rightarrow 1 \leq \frac{|3 \cos \theta - 4 \sin \theta + 10|}{5} \leq 3$$

Hence, the least distance is 1 and the greatest distance is 3.

ILLUSTRATION 2.32

Prove that the product of the lengths of the perpendiculars drawn from the points $(\sqrt{a^2 - b^2}, 0)$ and $(-\sqrt{a^2 - b^2}, 0)$

to the line $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$ is b^2 .

Sol. The equation of the given line is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

$$\text{or } bx \cos \theta + ay \sin \theta - ab = 0 \quad (1)$$

The length of the perpendicular from the point $(\sqrt{a^2 - b^2}, 0)$ to line (1) is

$$\begin{aligned} p_1 &= \frac{|b \cos \theta \sqrt{a^2 - b^2} + a \sin \theta (0) - ab|}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \\ &= \frac{|b \cos \theta \sqrt{a^2 - b^2} - ab|}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \quad (2) \end{aligned}$$

The length of the perpendicular from the point $(-\sqrt{a^2 - b^2}, 0)$ to line (1) is

$$\begin{aligned} p_2 &= \frac{|b \cos \theta (-\sqrt{a^2 - b^2}) + a \sin \theta (0) - ab|}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \\ &= \frac{|b \cos \theta \sqrt{a^2 - b^2} + ab|}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \quad (3) \end{aligned}$$

$$\begin{aligned} \therefore p_1 p_2 &= \frac{|b \cos \theta \sqrt{a^2 - b^2} - ab| |b \cos \theta \sqrt{a^2 - b^2} + ab|}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} \\ &= \frac{|b^2 \cos^2 \theta (a^2 - b^2) - a^2 b^2|}{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)} \\ &= \frac{|a^2 b^2 \cos^2 \theta - b^4 \cos^2 \theta - a^2 b^2|}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} \\ &= \frac{b^2 |a^2 \cos^2 \theta - b^2 \cos^2 \theta - a^2|}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} \end{aligned}$$

$$\begin{aligned} &= \frac{b^2 |-b^2 \cos^2 \theta - a^2 (1 - \cos^2 \theta)|}{b^2 \cos^2 \theta + a^2 \sin^2 \theta} \\ &= \frac{b^2 (b^2 \cos^2 \theta + a^2 \sin^2 \theta)}{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)} = b^2 \end{aligned}$$

Hence, proved.

ILLUSTRATION 2.33

Find the least value of $(x-1)^2 + (y-2)^2$ under the condition $3x + 4y - 2 = 0$.

Sol. $(x-1)^2 + (y-2)^2$ is square of distances between the points $A(1, 2)$ and $P(x, y)$, where P lies on the line $3x + 4y - 2 = 0$.

Least value of $(x-1)^2 + (y-2)^2$

= Square of shortest distance of point $(1, 2)$ from the line $3x + 4y - 2 = 0$

$$= \left(\frac{|3(1) + 4(2) - 2|}{\sqrt{3^2 + 4^2}} \right)^2 = \frac{9}{25}$$

ILLUSTRATION 2.34

ABC is an equilateral triangle with $A(0, 0)$ and $B(a, 0)$, $(a > 0)$. L, M , and N are the foot of the perpendiculars drawn from a point P to the sides AB, BC , and CA , respectively. If P lies inside the triangle and satisfies the condition $PL^2 = PM \cdot PN$, then find the locus of P .

Sol. The equations of lines AC and BC are, respectively, given by

$$y - \sqrt{3}x = 0$$

$$y + \sqrt{3}(x - a) = 0$$

Let $P(h, k)$ be the coordinates of the point whose locus is to be found. Now, according to the given condition, we have

$$PL^2 = PM \cdot PN$$

$$\text{i.e., } k^2 = \frac{|k - \sqrt{3}h|}{2} \cdot \frac{|k + \sqrt{3}(h - a)|}{2}$$

$$\text{i.e., } 4k^2 = (k - \sqrt{3}h) [k + \sqrt{3}(h - a)]$$

[$\because P$ lies below both the lines]

$$\text{i.e., } 3(h^2 + k^2) - 3ah + \sqrt{3}ak = 0$$

$$\text{i.e., } h^2 + k^2 - ah + \frac{a}{\sqrt{3}}k = 0$$

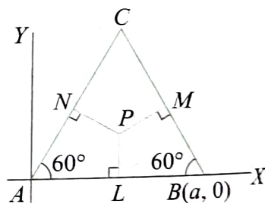
Putting (x, y) in place of (h, k) gives the equation of the required locus as

$$x^2 + y^2 - ax + \frac{a}{\sqrt{3}}y = 0$$

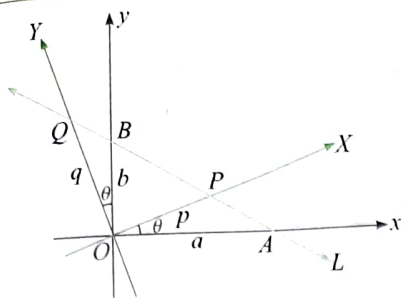
ILLUSTRATION 2.35

Line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle keeping the origin fixed, the same line L has intercepts p and q . Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$$



Sol.



As shown in the figure, axes are rotated by an angle θ about origin in anticlockwise direction.

Line L has intercepts a and b on original axes and intercepts p and q on rotated axes.

Equation of line L w.r.t. original axes is

$$\frac{x}{a} + \frac{y}{b} - 1 = 0 \quad (1)$$

Equation of line L w.r.t. rotated axes is

$$\frac{x}{p} + \frac{y}{q} - 1 = 0 \quad (2)$$

Since origin and line L are not changing their positions, distance of line from origin remains the same even though the axes are rotated.

So, comparing distances of lines given by equations (1) and (2) from origin, we get

$$\begin{aligned} \left| \frac{0}{a} + \frac{0}{b} - 1 \right| &= \left| \frac{0}{p} + \frac{0}{q} - 1 \right| \\ \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} &= \frac{1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}} \\ \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} &= \frac{1}{p^2} + \frac{1}{q^2} \end{aligned}$$

Distance between Two Parallel Lines

Let the two parallel lines be

$$ax + by + c = 0 \quad (1)$$

$$\text{and } ax + by + c' = 0 \quad (2)$$

Distance between parallel lines is equal to the length of the perpendicular drawn from any point on line (1) to line (2).

Consider point $P(x_1, y_1)$ on line (1).

$$\therefore ax_1 + by_1 + c = 0 \quad (3)$$

Perpendicular distance d of point P from the line (2) is given by

$$d = \frac{|ax_1 + by_1 + c'|}{\sqrt{a^2 + b^2}}$$

$$\text{or } d = \frac{|c' - c|}{\sqrt{a^2 + b^2}} \quad [\text{From (3), } ax_1 + by_1 = -c]$$

ILLUSTRATION 2.36

Two sides of a square lie on the lines $x + y = 1$ and $x + y + 2 = 0$. What is its area?

Sol. Clearly, the length of the side of the square is equal to the distance between the parallel lines.

$$x + y - 1 = 0 \quad (1)$$

$$\text{and } x + y + 2 = 0 \quad (2)$$

$$\text{Hence, side length} = \frac{|2 - (-1)|}{\sqrt{(1+1)}} = \frac{3}{\sqrt{2}}$$

$$\text{Therefore, area of square} = \frac{9}{2} \text{ sq. units.}$$

ILLUSTRATION 2.37

Find the equation of the line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

Sol. The equations of the given lines are

$$9x + 6y - 7 = 0 \quad (1)$$

$$\text{and } 3x + 2y + 6 = 0 \quad (2)$$

$$\text{or } 9x + 6y + 18 = 0 \quad (2)$$

The line parallel to the above lines is

$$9x + 6y + c = 0 \quad (3)$$

The distance of line (3) from lines (1) and (2) is the same. Therefore,

$$\frac{|c - 18|}{\sqrt{9^2 + 6^2}} = \frac{|c + 7|}{\sqrt{9^2 + 6^2}}$$

$$\text{or } c - 18 = -c - 7$$

$$c = \frac{11}{2}$$

Therefore, the equation of the line is $18x + 12y + 11 = 0$.

ILLUSTRATION 2.38

If one side of the square is $2x - y + 6 = 0$ and one of the vertices is $(2, 1)$, then find the other sides of the square.

Sol. Vertex $A(2, 1)$ does not lie on the line $BC \equiv 2x - y + 6 = 0$ (1)

Hence, one of the side of the square passing through A is parallel to BC .

Therefore, the equation of AD is

$$y - 1 = 2(x - 2)$$

$$\text{or } 2x - y - 3 = 0 \quad (2)$$

Another side AB of the square is passing through $(2, 1)$ and is perpendicular to BC .

Therefore, the equation of AB is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$\text{or } x + 2y - 4 = 0$$

The third side is parallel to AB , which has equation

$$x + 2y + c = 0$$

Now, the distance between the lines BC and AD is $9/\sqrt{5}$.

Therefore, the distance between the lines (3) and (4) is

$$\frac{|c + 4|}{\sqrt{5}} = \frac{9}{\sqrt{5}}$$

$$\therefore c = 5 \text{ or } -13$$

Therefore, the equation of the third side is

$$x + 2y + 5 = 0 (\equiv C'D')$$

$$\text{or } x + 2y - 13 = 0 (\equiv CD)$$

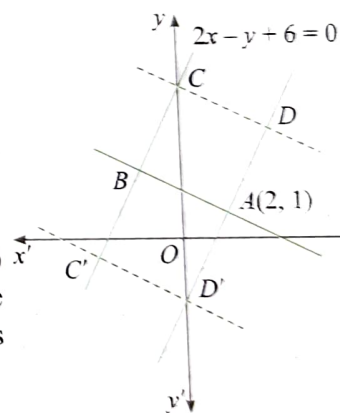
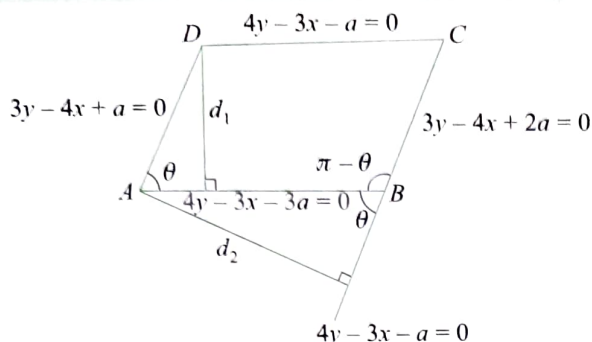


ILLUSTRATION 2.39

Prove that the area of the parallelogram contained by the lines $4y - 3x - a = 0$, $3y - 4x + a = 0$, $4y - 3x - 3a = 0$, and $3y - 4x + 2a = 0$ is $(2/7)a^2$.

Sol.

$$\tan \theta = \frac{(4/3) - (3/4)}{1 + (4/3) \times (3/4)} = \frac{7}{24}$$

or $\operatorname{cosec} \theta = \frac{25}{7}$

Now $d_1 = \frac{|-a + 3a|}{5} = \frac{2a}{5}$

and $d_2 = \frac{|2a - a|}{5} = \frac{a}{5}$

$$\therefore \text{Area} = 2(\text{Area of } \triangle ABD) = 2 \left(\frac{1}{2} d_1 \cdot AB \right) = d_1 d_2 \operatorname{cosec} \theta = \left(\frac{a}{5} \right) \left(\frac{2a}{5} \right) \frac{25}{7} = \frac{2a^2}{7}$$

ILLUSTRATION 2.40

Find the equations of straight lines passing through $(-2, -7)$ and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.

Sol. Given lines are

$$4x + 3y - 12 = 0 \quad (1)$$

$$\text{and } 4x + 3y - 3 = 0 \quad (2)$$

Given $AB = 3$.

The distance between parallel lines (1) and (2),

$$AL = \frac{|-12 + 3|}{\sqrt{4^2 + 3^2}} = \frac{9}{5}$$

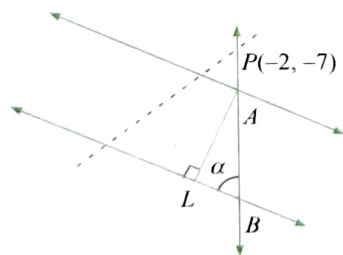
From $\triangle ALB$, we get

$$\begin{aligned} LB^2 &= AB^2 - AL^2 \\ &= 3^2 - \frac{81}{25} = 9 - \frac{81}{25} = \frac{16}{25} \end{aligned}$$

$$\therefore LB = \frac{4}{5}$$

$$\tan \alpha = \frac{AL}{LB} = \frac{3}{4}$$

Also, $\tan \theta = \text{Slope of line (1)} = -\frac{4}{3}$

Let the slope of PB be m . Now,

$$\tan \alpha = \frac{3}{4} = \frac{m - (-4/3)}{1 + m(-4/3)} = \frac{3m + 4}{3 - 4m}$$

or $m = \infty$ or $m = -\frac{7}{24}$

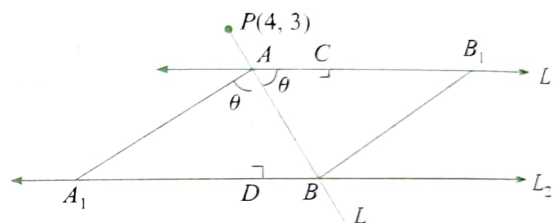
Therefore, the equation of the line is

$$x + 2 = 0 \text{ and } y + 7 = -\frac{7}{24}(x + 2)$$

or $x + 2 = 0$ and $7x + 24y + 182 = 0$

ILLUSTRATION 2.41

A line L is drawn from $P(4, 3)$ to meet the lines L_1 and L_2 given by $3x + 4y + 5 = 0$ and $3x + 4y + 15 = 0$ at points A and B , respectively. From A , a line perpendicular to L is drawn meeting the line L_2 at A_1 . Similarly, from point B , a line perpendicular to L is drawn meeting the line L_1 at B_1 . Thus, a parallelogram AA_1BB_1 is formed. Then find the equation of L so that the area of the parallelogram AA_1BB_1 is the least.

Sol.

The given lines (L_1 and L_2) are parallel and the distance between them (BC or AD) is $(15 - 5)/5 = 2$ units. Let $\angle BAC = \theta$. So, $AB = BC \operatorname{cosec} \theta = 2 \operatorname{cosec} \theta$ and $AA_1 = AD \sec \theta = 2 \sec \theta$. Now, the area of parallelogram AA_1BB_1 is

$$\begin{aligned} \Delta &= AB \times AA_1 = 4 \sec \theta \operatorname{cosec} \theta \\ &= \frac{8}{\sin 2\theta} \end{aligned}$$

Clearly, Δ is the least for $\theta = \pi/4$. Let the slope of AB be m . Then,

$$\tan 45^\circ = \frac{m + 3/4}{1 - (3m/4)}$$

or $4m + 3 = \pm(4 - 3m)$ or $m = \frac{1}{7}$ or -7

Hence, the equation of L is

$$x - 7y + 17 = 0$$

or $7x + y - 31 = 0$

CONCEPT APPLICATION EXERCISE 2.3

- Find the points on the y -axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.
- If p and p' are the distances of the origin from the lines $x \sec \alpha + y \operatorname{cosec} \alpha = k$ and $x \cos \alpha - y \sin \alpha = k \cos 2\alpha$, then prove that $4p^2 + p'^2 = k^2$.
- Prove that the lengths of the perpendiculars from the points $(m^2, 2m)$, $(mm', m + m')$, and $(m'^2, 2m')$ to the line $x + y + 1 = 0$ are in GP.
- Find the ratio in which the line $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$.

5. Find the incentre of a triangle formed by the lines $x \cos \frac{\pi}{9} + y \sin \frac{\pi}{9} = \pi$, $x \cos \frac{8\pi}{9} + y \sin \frac{8\pi}{9} = \pi$ and $x \cos \frac{13\pi}{9} + y \sin \frac{13\pi}{9} = \pi$.
6. Find the equations of lines parallel to $3x - 4y - 5 = 0$ at a unit distance from it.
7. Find the equation of a straight line passing through the point $(-5, 4)$ and which cuts off an intercept of $\sqrt{2}$ units between the lines $x + y + 1 = 0$ and $x + y - 1 = 0$.

ANSWERS

1. $(0, 1)$ and $(0, -9)$ 4. $3 : 7$
 5. $(0, 0)$ 6. $3x - 4y - 10 = 0$ or $3x - 4y = 0$
 7. $x - y + 9 = 0$

POSITION OF A POINT W.R.T. A LINE

Let the equation of the given line be

$$ax + by + c = 0 \quad (1)$$

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

None of these points is lying on the line.

Let us find the condition for which P and Q lie either on the same side or on opposite sides of line (1).

Let line (1) divide the segment joining the points P and Q in the ratio $\lambda : 1$ at point R .

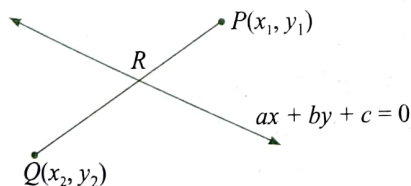
$$\therefore R \equiv \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1} \right)$$

Since point R lies on the line (1), we have

$$a \left(\frac{\lambda x_2 + x_1}{\lambda + 1} \right) + b \left(\frac{\lambda y_2 + y_1}{\lambda + 1} \right) + c = 0$$

$$\Rightarrow \lambda = - \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$$

If points P and Q lie on the opposite sides of line, R will divide PQ internally.



In that case,

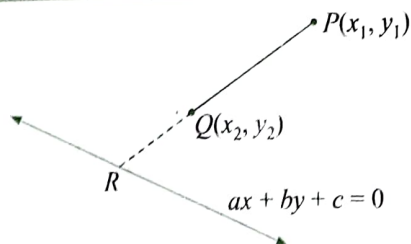
$$\lambda > 0$$

$$\therefore - \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

$$\therefore \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

Hence, $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ will have opposite signs.

If points P and Q lie on the same side of line, R will divide PQ externally.



In that case,

$$\lambda < 0$$

$$\therefore - \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

$$\therefore \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

Hence, $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ will have same sign.

ILLUSTRATION 2.42

Are the points $(3, 4)$ and $(2, -6)$ on the same or opposite sides of the line $3x - 4y = 8$?

Sol. We have line $3x - 4y - 8 = 0$.

$$\text{Let } L(x, y) = 3x - 4y - 8$$

$$\therefore L(3, 4) = 3(3) - 4(4) - 8 = -15$$

$$\text{and } L(2, -6) = 3(2) - 4(-6) - 8 = 22$$

Since $L(3, 4)$ and $L(2, -6)$ have opposite signs, point $(3, 4)$ and $(2, -6)$ lie on opposite sides of the line.

ILLUSTRATION 2.43

Find the set of positive values of b for which the origin and the point $(1, 1)$ lie on the same side of the straight line, $a^2x + a by + 1 = 0, \forall a \in \mathbb{R}$.

Sol. Points $(0, 0)$ and $(1, 1)$ lie on same side of the line $a^2x + aby + 1 = 0$.

For $(0, 0)$,

$$a^2(0) + ab(0) + 1 > 0$$

Therefore, for $(1, 1)$, we must have

$$a^2 + ab + 1 > 0 \quad \forall a \in \mathbb{R}$$

$$\therefore D < 0$$

$$\therefore b^2 - 4 < 0$$

$$\text{or } b \in (-2, 2)$$

$$\text{But } b > 0$$

$$\therefore b \in (0, 2)$$

ILLUSTRATION 2.44

If point $(a^2, a + 1)$ lies in the angle between the line $3x - y + 1 = 0$ and $x + 2y - 5 = 0$ containing the origin, then find the values of a .

Sol. Given lines are

$$L_1 = 3x - y + 1 = 0 \quad (1)$$

$$L_2 = x + 2y - 5 = 0 \quad (2)$$

$P(a^2, a + 1)$ lies in the angle formed by above two lines containing $O(0, 0)$. Then O and P must lie on the same side w.r.t. both the lines

Now, $L_1(0, 0) = 1 > 0$

So, we must have

$$L_1(a^2, a+1) > 0$$

$$\text{or } 3a^2 - (a+1) + 1 > 0$$

$$\text{or } a(3a-1) > 0$$

$$\Rightarrow a \in (-\infty, 0) \cup \left(\frac{1}{3}, \infty\right) \quad (3)$$

$$L_2(0, 0) = -5 < 0$$

So, we must have

$$L_2(a^2, a+1) < 0$$

$$\text{or } a^2 + 2(a+1) - 5 < 0$$

$$a^2 + 2a - 3 < 0$$

$$\Rightarrow (a-1)(a+3) < 0$$

$$a \in (-3, 1) \quad (4)$$

From (3) and (4),

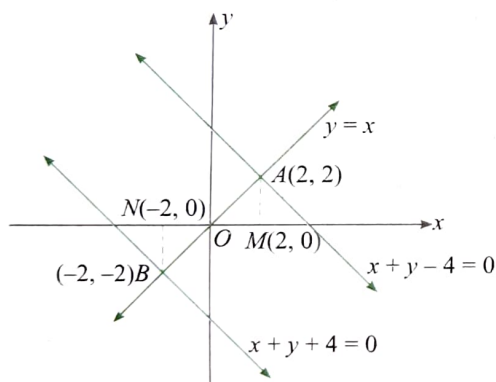
$$a \in (-3, 0) \cup \left(\frac{1}{3}, 1\right)$$

ILLUSTRATION 2.45

If the point (a, a) is placed in between the lines $|x+y|=4$, then find the values of a .

Sol. We have lines $x+y-4=0$ and $x+y+4=0$

Point (a, a) lies on the line $y=x$.



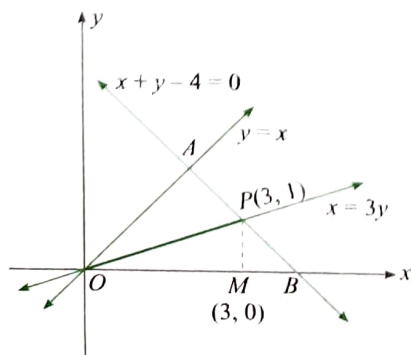
Line $y=x$ intersects the given lines at $A(2, 2)$ and $B(-2, -2)$.

So, if (a, a) lies on the line $y=x$ between points A and B , we must have $-2 < a < 2$.

ILLUSTRATION 2.46

Find the values of a for which point $(3a, a)$ lies inside the triangle formed by the lines $y=x$, the x -axis and line $x+y=4$.

Sol.



In the figure, triangle OAB is formed by lines $y=x$, $x+y=4$ and x -axis.

Variable point $(3a, a)$ lies on the line $x=3y$ for all real values of a .

Line $x=3y$ meets the line $x+y-4=0$ at $P(3, 1)$.

Hence, point $(3a, a)$ must lie between O and P on line $3x=y$.

$$\therefore 0 < a < 1$$

ILLUSTRATION 2.47

Determine all the values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x+3y-1=0$, $x+2y-3=0$ and $5x-6y-1=0$.

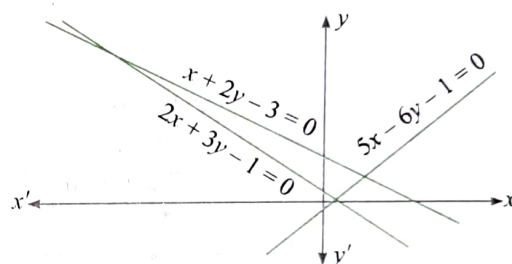
Sol. Given lines are

$$L_1(x, y) = 2x + 3y - 1 = 0$$

$$L_2(x, y) = 5x - 6y - 1 = 0$$

$$L_3(x, y) = x + 2y - 3 = 0$$

Draw these lines on coordinate axes.



Point $P(\alpha, \alpha^2)$ lies inside the triangle formed by these lines.

Clearly, from the figure, origin and point P lie on the opposite sides w.r.t. $2x+3y-1=0$.

$$L_1(0, 0) = -1 < 0$$

So, we must have

$$L_1(\alpha, \alpha^2) = 2\alpha + 3\alpha^2 - 1 > 0$$

$$\Rightarrow (3\alpha-1)(\alpha+1) > 0$$

$$\Rightarrow \alpha < -1 \text{ or } \alpha > 1/3 \quad (1)$$

Also, origin and point P lie on the same side w.r.t. line $x+2y-3=0$.

$$L_3(0, 0) = -3 < 0$$

So, we must have

$$L_3(\alpha, \alpha^2) = \alpha + 2\alpha^2 - 3 < 0$$

$$\Rightarrow (2\alpha+3)(\alpha-1) < 0$$

$$\Rightarrow -3/2 < \alpha < 1 \quad (2)$$

Further origin and point P lie on the same side w.r.t. line $5x-6y-1=0$.

$$\Rightarrow L_2(0, 0) = -1 < 0$$

So, we must have

$$L_2(\alpha, \alpha^2) = 5\alpha - 6\alpha^2 - 1 < 0$$

$$\Rightarrow 6\alpha^2 - 5\alpha + 1 > 0$$

$$\Rightarrow (3\alpha-1)(2\alpha-1) > 0$$

$$\Rightarrow \alpha < 1/3 \text{ or } \alpha > 1/2 \quad (3)$$

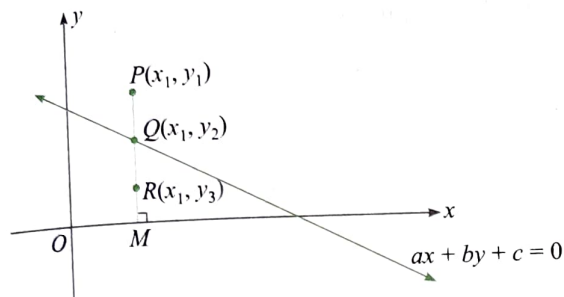
From (1), (2) and (3), common values of α are $(-3/2, -1) \cup (1/2, 1)$.

DIVISION OF PLANE BY LINE

A line drawn on Cartesian plane divides it into two regions (above and below the line). Now, we have three sets of points on the plane w.r.t. a line; set of points lying on the line, set of points lying above the line and region of points lying below the line. All the points in a particular set satisfy a certain relation.

Consider the straight line $ax + by + c = 0$. Let $L(x, y) = ax + by + c$.

All the points lying on the line, satisfy the equation $ax + by + c = 0$.



Consider the point $P(x_1, y_1)$ lying in the region above the line. Through this point, draw perpendicular to x -axis meeting line at $Q(x_1, y_2)$ and x -axis at M .

Clearly, $PM > QM$.

$$\therefore y_1 > y_2$$

$$\text{Let } b > 0.$$

$$\therefore by_1 > by_2$$

$$\Rightarrow ax_1 + by_1 + c > ax_1 + by_2 + c$$

$$\Rightarrow ax_1 + by_1 + c > 0 \quad [\text{As } (x_1, y_2) \text{ lies on the line}]$$

Thus, for all the points lying in the region above the line, we have $ax + by + c > 0$.

For point $R(x_1, y_3)$, we will find that $ax_1 + by_3 + c < 0$.

Thus, for all the points lying in the region below the line, we have $ax + by + c < 0$.

Also, we have considered $b > 0$. So, we must have $c < 0$ as the line is cutting positive x -axis.

$$\therefore L(0, 0) = c < 0$$

So, for all the points in the region of the plane w.r.t. line where origin lies, $ax + by + c < 0$ and for the region where origin does not lie, $ax + by + c > 0$.

Similarly, for $b < 0$, we will find that the line divides the plane into two halves. In one half for all the points, $L(x, y) > 0$ and in other half for all the points, $L(x, y) < 0$.

ILLUSTRATION 2.48

Sketch the region in which the points satisfying the following inequality lie.

(i) $2x - 3y - 5 > 0$

(ii) $-3x + 4y + 7 > 0$

(iii) $x > 2$

(iv) $y > -3$

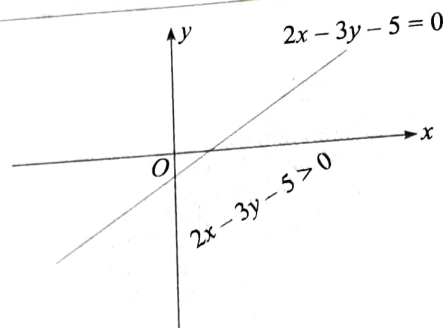
Sol.

(i) $2x - 3y - 5 > 0$

Let $L(x, y) = 2x - 3y - 5$

$L(0, 0) = -5 < 0$

So, $2x - 3y - 5 > 0$ represents the half region of the plane where origin does not lie, i.e., below line $2x - 3y - 5 = 0$.

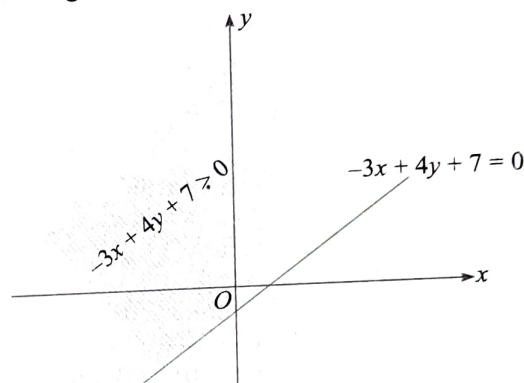


(ii) $-3x + 4y + 7 > 0$

Let $L(x, y) = -3x + 4y + 7$

$L(0, 0) = 7 > 0$

So, $-3x + 4y + 7 > 0$ represents the half region of the plane where origin lies, i.e., above line $-3x + 4y + 7 = 0$.

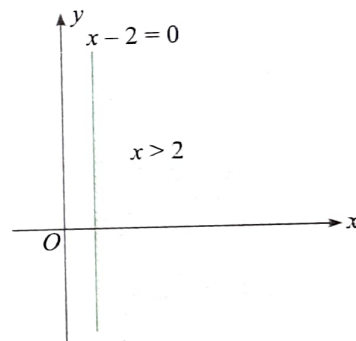


(iii) $x > 2$ or $x - 2 > 0$

Let $L(x, y) = x - 2$

$L(0, 0) = 0 - 2 < 0$

So, $x - 2 > 0$ represents the half region of the plane where origin does not lie, i.e., to the right of the line $x = 2$.



(iv) $y > -3$ or $y + 3 > 0$

Let $L(x, y) = y + 3$

$L(0, 0) = 0 + 3 > 0$

So, $y + 3 > 0$ represents the half region of the plane where origin lies, i.e., above the line $y + 3 = 0$.

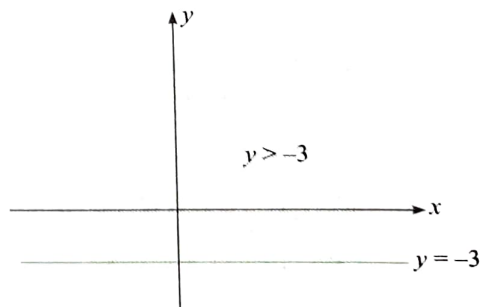


ILLUSTRATION 2.49

Sketch the region in which the points satisfying the following inequalities lie.

(i) $|x + y| < 2$

(ii) $|2x - y| > 3$

(iii) $|x| > |y|$

Sol.

(i) $|x + y| < 2$

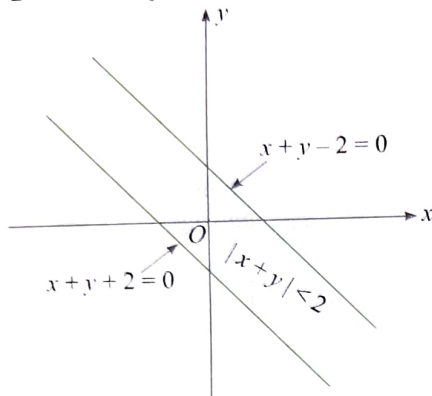
$\therefore -2 < x + y < 2$

$\Rightarrow x + y + 2 > 0 \text{ and } x + y - 2 < 0$

$L_1(x, y) = x + y + 2 \text{ and } L_2(x, y) = x + y - 2$

$L_1(0, 0) = 2 > 0 \text{ and } L_2(0, 0) = -2 < 0$

So, $x + y + 2 > 0$ represents the region where origin lies and $x + y - 2 < 0$ also represents the region where origin lies.



Thus, in general, $|ax + by + c| < d$ is the region of the points between the lines $ax + by + c = d$ and $ax + by + c = -d$.

(ii) $|2x - y| > 3$

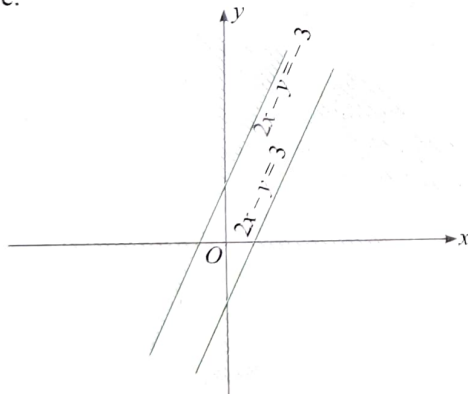
$\therefore 2x - y < -3 \text{ and } 2x - y > 3$

$\Rightarrow 2x - y + 3 < 0 \text{ and } 2x - y - 3 > 0$

$L_1(x, y) = 2x - y + 3 \text{ and } L_2(x, y) = 2x - y - 3$

$L_1(0, 0) = 3 > 0 \text{ and } L_2(0, 0) = -3 < 0$

So, $2x - y + 3 < 0$ represents where origin does not lie and $2x - y - 3 > 0$ also represents the region where origin does not lie.



Thus, in general $|ax + by + c| > d$ is the region of the points which does not lie between the lines $ax + by + c = d$ and $ax + by + c = -d$.

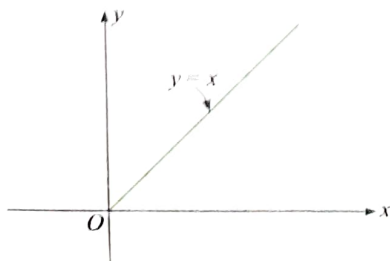
(iii) $|x| > |y|$

If $x > 0$, then we have $x > y$ or $x - y > 0$.

Let $L(x, y) = x - y$.

$\therefore L(1, 0) = 1 - 0 > 0$

Hence, points satisfying this inequality lie below the line $x - y = 0$ in first quadrant.



If $x > 0, y < 0$, then we have $x > -y$ or $x + y > 0$.

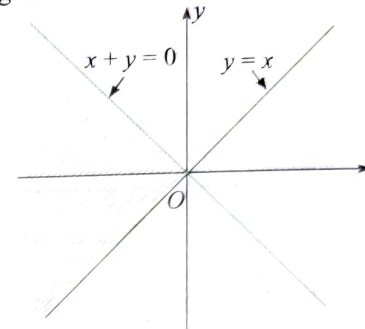
Let $L(x, y) = x + y$.

$\therefore L(1, 0) = 1 + 0 > 0$

Points satisfying this inequality lie above the line $x + y = 0$ in fourth quadrant.

Similarly, we have one region in second quadrant and one in third quadrant.

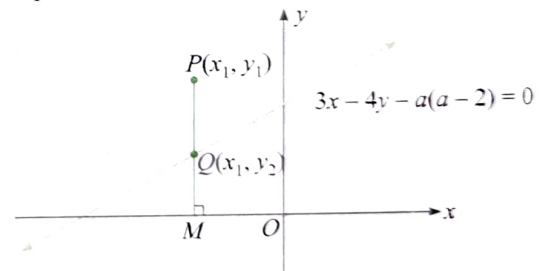
Combining all the cases, we have following region.

**ILLUSTRATION 2.50**

Find the values of b for which the point $(2b + 3, b^2)$ lies above the line $3x - 4y - a(a - 2) = 0 \quad \forall a \in \mathbb{R}$.

Sol. Given line is $3x - 4y - a(a - 2) = 0$

One of the possible lines is as shown in the following figure.



Consider the point $P(x_1, y_1)$ lying above this line.

Draw perpendicular from point P to the x -axis meeting the line at $Q(x_1, y_2)$ and x -axis at M .

Clearly, $PM > QM$

$\therefore y_1 > y_2$

$\Rightarrow -4y_1 < -4y_2$

$\Rightarrow 3x_1 - a(a - 2) - 4y_1 < 3x_1 - a(a - 2) - 4y_2$

$\Rightarrow 3x_1 - a(a - 2) - 4y_1 < 0 \quad (\text{As point } Q \text{ lies on the line})$

$\Rightarrow 3(2b + 3) - a(a - 2) - 4b^2 < 0, \quad \forall a \in \mathbb{R}$

$(\text{Putting } x_1 = 2b + 3 \text{ and } y_1 = b^2)$

$a^2 - 2a + 4b^2 - 6b - 9 > 0 \quad \forall a \in \mathbb{R}$

$\therefore \text{Discriminant} < 0$

$\Rightarrow 4 - 4(4b^2 - 6b - 9) < 0$

$\Rightarrow 1 - 4b^2 + 6b + 9 < 0$

$\Rightarrow 2b^2 - 3b - 5 > 0$

$\Rightarrow (2b - 5)(b + 1) > 0$

$\Rightarrow b \in (-\infty, -1) \cup \left(\frac{5}{2}, \infty\right)$

ILLUSTRATION 2.51

Plot the region of the points $P(x, y)$ satisfying $|x| + |y| < 1$.

Sol. Given that $|x| + |y| < 1$.
We have four cases.

If $x, y > 0$, then we have $x + y < 1$ or $x + y - 1 < 0$.

Points satisfying these inequalities lie in triangular region in first quadrant formed by axes and line $x + y - 1 = 0$.

Similarly, we have three triangular regions in all other quadrants.

Thus, points satisfying given inequality lie in a square as shown in the following figure.

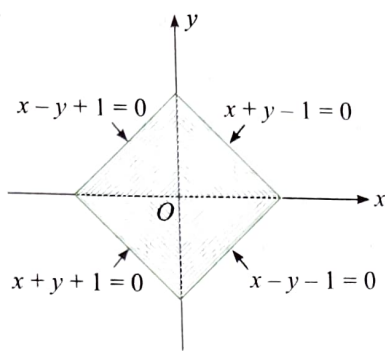


ILLUSTRATION 2.52

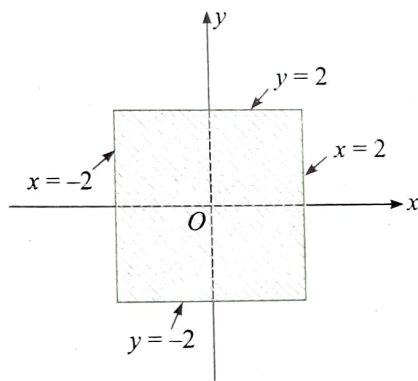
Plot the region of the points $P(x, y)$ satisfying $2 > \max\{|x|, |y|\}$.

Sol. We have $2 > \max\{|x|, |y|\}$

$\therefore 2 > |x|$ and $2 > |y|$

i.e., $-2 < x < 2$ and $-2 < y < 2$.

Thus, common region of these inequalities is the square formed by lines $x = \pm 2$ and $y = \pm 2$.



CONCEPT APPLICATION EXERCISE 2.4

- How the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ are positioned w.r.t. point $(8, -9)$?
- How the following pairs of points are placed w.r.t. the line $3x - 8y - 7 = 0$?
(i) $(-3, -4)$ and $(1, 2)$ (ii) $(-1, -1)$ and $(3, 7)$
- Find the range of α for which the points $(\alpha, 2 + \alpha)$ and $(\frac{3\alpha}{2}, \alpha^2)$ lie on opposite sides of the line $2x + 3y = 6$.
- If the point $P(a^2, a)$ lies in the region of acute angle between the lines $2y = x$ and $4y = x$, then find the values of a .
- If $(a, 3a)$ is a variable point lying above the straight line $2x + y + 4 = 0$ and below the line $x + 4y - 8 = 0$, then find the values of a .

- Find the values of α such that the variable point $(\alpha, \tan \alpha)$ lies inside the triangle whose sides are

$$y = x + \sqrt{3} - \frac{\pi}{3}, \quad x + y + \frac{1}{\sqrt{3}} + \frac{\pi}{6} = 0 \quad \text{and} \quad x - \frac{\pi}{2} = 0.$$

- Find the area of the region in which points satisfy $3 \leq |x| + |y| \leq 5$.
- Find the area of the region formed by the points satisfying $|x| + |y| + |x + y| \leq 2$.

ANSWERS

- Both the lines lie on the same side of point $(8, -9)$.
- (i) points are on opposite sides of the line
(ii) points are on same side of the line
- $\alpha \in (-\infty, -2) \cup (0, 1)$ 4. $a \in (2, 4)$
- $-4/5 < a < 8/13$ 6. $-\frac{\pi}{6} < \alpha < \frac{\pi}{3}$
- 32 sq. units 8. 3 sq. units

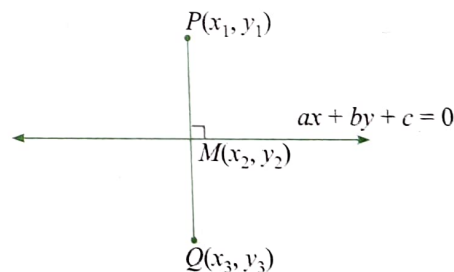
IMAGE OF A POINT IN A LINE AND EQUATION OF ANGLE BISECTORS

FOOT OF PERPENDICULAR DRAWN FROM A POINT ON A LINE AND IMAGE OF POINT IN A LINE

Earlier in one of the illustrations, we have discussed how to find the foot of perpendicular drawn from point on a line and the image of point in a line. In alternative method, we use the direct formula for both of these.

Consider a point $P(x_1, y_1)$ and a line $L \equiv ax + by + c = 0$.

Let the foot of the perpendicular from point P on the line be $M(x_2, y_2)$.



Slope of line L is $-\frac{a}{b}$.

Slope of PM is $\frac{y_2 - y_1}{x_2 - x_1}$

Since $PM \perp L$, we have

$$\frac{y_2 - y_1}{x_2 - x_1} \times \left(-\frac{a}{b}\right) = -1$$

$$\Rightarrow \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \lambda \text{ (say)}$$

$$x_2 = x_1 + a\lambda \text{ and } y_2 = y_1 + b\lambda$$

Point (x_2, y_2) lies on the line $ax + by + c = 0$.

$$\therefore a(x_1 + a\lambda) + b(y_1 + b\lambda) + c = 0$$

$$\Rightarrow \lambda = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

Putting this value in (1), we get

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2} \quad (2)$$

Using this formula, we can get coordinates of foot of perpendicular drawn from point P on the line.

Now, let image of point P in the line be $Q(x_3, y_3)$.

So, M will be the midpoint of PQ .

$$\therefore x_2 = \frac{x_1 + x_3}{2}, y_2 = \frac{y_1 + y_3}{2}$$

Putting these values in (2), we get

$$\begin{aligned} \frac{\frac{x_1 + x_3}{2} - x_1}{a} &= \frac{\frac{y_1 + y_3}{2} - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2} \\ \Rightarrow \frac{x_3 - x_1}{a} &= \frac{y_3 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2} \end{aligned}$$

Using this formula, we can get coordinates of image of point P in the line.

ILLUSTRATION 2.53

If one of the vertices of a square is $(3, 2)$ and one of the diagonals is along the line $3x + 4y + 8 = 0$, then find the centre of the square and other vertices.

Sol. We have square $ABCD$, where coordinates of A are $(3, 2)$.

Clearly, point $(3, 2)$ does not satisfy the line $3x + 4y + 8 = 0$.

Thus, diagonal BD is along this line.

$M(h, k)$ is centre of the square, which is the foot of perpendicular from vertex A on the diagonal BD .

$$\therefore \frac{h-3}{3} = \frac{k-2}{4} = -\frac{(3(3)+4(2)+8)}{3^2+4^2}$$

$$\Rightarrow \frac{h-3}{3} = \frac{k-2}{4} = -1$$

$\Rightarrow (h, k) \equiv (0, -2)$, which is the centre of the square.

M is midpoint of AC .

$$\therefore C \equiv (-3, -6)$$

$$\text{Now, } AM = \frac{|3(3)+4(2)+8|}{\sqrt{3^2+4^2}} = 1$$

$$\therefore MB = MD = 1$$

Slope of $BD = -\frac{3}{4} = \tan \theta$, where θ is inclination of BD with x -axis.

Therefore, coordinates of B and D are

$$(0 \pm 1 \cdot \cos \theta, -2 \pm 1 \cdot \sin \theta)$$

$$\equiv \left(0 \pm 1 \left(-\frac{4}{5} \right), -2 \pm 1 \left(\frac{3}{5} \right) \right)$$

$$\equiv \left(\mp \frac{4}{5}, -2 \pm \frac{3}{5} \right)$$

$$\equiv \left(-\frac{4}{5}, -\frac{7}{5} \right), \left(\frac{4}{5}, -\frac{13}{5} \right)$$

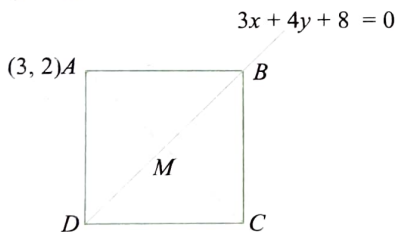


ILLUSTRATION 2.54

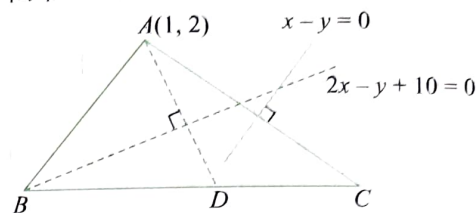
In $\triangle ABC$, vertex A is $(1, 2)$. If the internal angle bisector of $\angle B$ is $2x - y + 10 = 0$ and the perpendicular bisector of AC is $y = x$, then find the equation of BC .

Sol. The image D of A in the bisector of $\angle B$ lies on BC .

Now, the image of $A(1, 2)$ in the line $2x - y + 10 = 0$ is given by

$$\frac{x_1 - 1}{2} = \frac{y_1 - 2}{-1} = \frac{-2(2 - 2 + 10)}{4 + 1}$$

$$\therefore D(x_1, y_1) \equiv D(-7, 6)$$



Also, C is the image of A in the perpendicular bisector of AC . The image of $A(1, 2)$ in $y = x$ is $C(2, 1)$.

Hence, the equation of line BC is

$$y - 1 = \frac{6 - 1}{-7 - 2} (x - 2)$$

$$\text{or } 5x + 9y - 19 = 0$$

ILLUSTRATION 2.55

Find the locus of image of the variable point $(\lambda^2, 2\lambda)$ in the line mirror $x - y + 1 = 0$, where λ is a parameter.

Sol. Let the image of $(\lambda^2, 2\lambda)$ in the line mirror $x - y + 1 = 0$ be (h, k) .

$$\therefore \frac{h - \lambda^2}{1} = \frac{k - 2\lambda}{-1} = \frac{-2(\lambda^2 - 2\lambda + 1)}{2}$$

$$\therefore h + 1 = 2\lambda \quad (1)$$

$$\text{and } k = \lambda^2 + 1 \quad (2)$$

Putting the value of λ from (1) in (2), we get

$$k - 1 = \left(\frac{h+1}{2} \right)^2$$

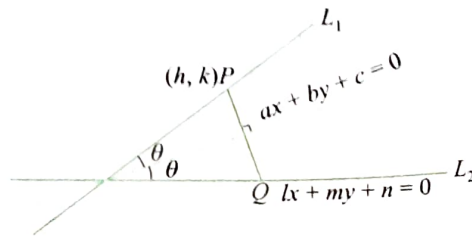
$$\text{or } 4(y - 1) = (x + 1)^2$$

This is the required locus.

ILLUSTRATION 2.56

Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at point P and make an angle θ with each other. Find the equation of a line different from L_2 which passes through P and makes the same angle θ with L_1 .

Sol.



From the figure, image of point $P(h, k)$ on required line in the line L_1 lies on line L_2 .

So, equation of required line is locus of point P such that its image in line L_1 lies on line L_2 .

Let image of point P in line L_1 be $Q(x_1, y_1)$.

$$\Rightarrow \frac{x_1 - h}{a} = \frac{y_1 - k}{b} = -\frac{2(ah + bk + c)}{a^2 + b^2}$$

$$\Rightarrow x_1 = -\frac{2a(ah + bk + c)}{a^2 + b^2} + h$$

$$\text{and } y_1 = -\frac{2b(ah + bk + c)}{a^2 + b^2} + k$$

Now, $Q(x_1, y_1)$ lies on the line L_2 .

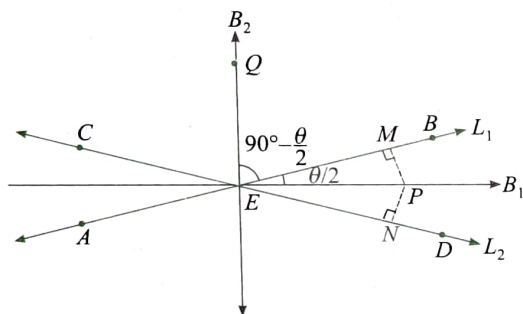
$$\therefore l \left[-\frac{2a(ah + bk + c)}{a^2 + b^2} + h \right] + m \left[-\frac{2b(ah + bk + c)}{a^2 + b^2} + k \right] + n = 0$$

Therefore, equation of required line is

$$2(al + mb)(ax + by + c) - (a^2 + b^2)(lx + my + n) = 0$$

EQUATIONS OF BISECTORS OF THE ANGLES BETWEEN THE LINES

Consider two lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$, which intersect at point E .



We have a pair of acute angles, viz. $\angle BED$ and $\angle AEC$ and a pair of obtuse angles, viz. $\angle CEM$ and $\angle AED$.

Also, we have a pair of bisectors: B_1 for pair of acute angles and B_2 for pair of obtuse angles.

In the figure, $\angle BED = \theta$.

$$\therefore \angle BEP = \angle PED = \frac{\theta}{2}$$

$$\text{Now, } \angle CEB = 180^\circ - \theta$$

$$\therefore \angle CEQ = \angle QEB = 90^\circ - \frac{\theta}{2}$$

$$\therefore \angle QEP = \angle QEB + \angle BEP = 90^\circ - \frac{\theta}{2} + \frac{\theta}{2} = 90^\circ$$

Thus, two bisectors are always perpendicular.

Now, consider point $P(x, y)$ on the bisector B_1 .

Draw perpendiculars PM and PN on the lines L_1 and L_2 .

Clearly, triangles PME and PNE are congruent.

$$\therefore PM = PN$$

Thus, bisector is locus of point which moves in the plane of lines L_1 and L_2 such that lengths of perpendiculars drawn from it to the two given lines (L_1 and L_2) are equal.

From $PM = PN$, we have

$$\frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2x + b_2y + c_2|}{\sqrt{a_2^2 + b_2^2}}$$

$$\therefore \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

$$\therefore \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (1)$$

$$\text{and } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (2)$$

These are equations of two bisectors.

To identify the acute angle and obtuse angle bisector, we choose one of the bisectors, say given in equation (1). Let the angle between this bisector and one of the given lines be $\alpha/2$, where α is the angle between lines containing this bisector.

$$\text{If } \left| \tan \frac{\alpha}{2} \right| < 1, \text{ then } \left| \tan \frac{\alpha}{2} \right| < \tan 45^\circ$$

$$\therefore \frac{\alpha}{2} < 45^\circ \Rightarrow \alpha < 90^\circ$$

Hence, chosen bisector corresponds to pair of acute angles.

$$\text{If we get } \left| \tan \frac{\alpha}{2} \right| > 1, \text{ then } \alpha > 90^\circ.$$

Then, chosen bisector is of pair of obtuse angles.

To identify the acute angle and obtuse angle bisectors, we can use the following shortcut method also.

Let the lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ be such that $c_1 > 0$ and $c_2 > 0$.

If $a_1a_2 + b_1b_2 < 0$, then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ will be acute angle bisector.}$$

If $a_1a_2 + b_1b_2 > 0$, then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ will be acute angle bisector.}$$

BISECTOR CONTAINING THE GIVEN POINT

Lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ divide the plane into four regions: two opposite regions where expressions $f_1(x, y) = a_1x + b_1y + c_1$ and $f_2(x, y) = a_2x + b_2y + c_2$ have same sign and two opposite regions where expressions $f_1(x, y)$ and $f_2(x, y)$ have opposite signs. So, angle bisector

$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$ goes through the regions where expressions $f_1(x, y)$ and $f_2(x, y)$ have same sign and

other angle bisector $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$ goes through the regions where these expressions have opposite signs.

From the above discussion, we understand that:

- (i) if $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ have same sign, then

$$\text{angle bisector } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ goes}$$

through the region which contains the point (α, β) .

- (ii) if $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ have opposite signs,

$$\text{then angle bisector } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ goes}$$

through the region which contains the point (α, β) .

Also, if c_1 and c_2 have same sign, then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \text{ is the bisector of the}$$

pair of angles which contains origin $(0, 0)$.

ILLUSTRATION 2.57

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the:

- bisector of the obtuse angle between them
- bisector of the acute angle between them
- bisector of the angle which contains $(1, 2)$
- bisector of the angle which contains $(0, 0)$

Sol. Equations of bisectors of the angles between the given lines are:

$$\frac{4x + 3y - 6}{\sqrt{4^2 + 3^2}} = \pm \frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$

$$\text{or } \frac{4x + 3y - 6}{5} = \pm \frac{5x + 12y + 9}{13}$$

$$\text{or } 9x - 7y - 41 = 0 \text{ and } 7x + 9y - 3 = 0$$

If $\alpha/2$ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector $9x - 7y - 41 = 0$, then

$$\tan \frac{\alpha}{2} = \left| \frac{-\frac{4}{3} - \frac{9}{7}}{1 + \left(\frac{-4}{3}\right)\left(\frac{9}{7}\right)} \right| = \frac{11}{3} > 1$$

$$\therefore \frac{\alpha}{2} > 45^\circ \text{ or } \alpha > 90^\circ$$

Hence, $9x - 7y - 41 = 0$ is obtuse angle bisector.

So, $7x + 9y - 3 = 0$ is acute angle bisector.

Alternatively, given lines are $-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$.

Here, $a_1a_2 + b_1b_2 = (-4)(5) + (-3)(12) < 0$

So, acute angle bisector is

$$\frac{-4x - 3y + 6}{\sqrt{4^2 + 3^2}} = \frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$

$$\text{or } \frac{-4x - 3y + 6}{5} = \frac{5x + 12y + 9}{13}$$

$$\text{or } 7x + 9y - 3 = 0$$

Now, $f_1(x, y) = 4x + 3y - 6$ and $f_2(x, y) = 5x + 12y + 9$

$$f_1(1, 2) = 4(1) + 3(2) - 6 > 0$$

$$f_2(1, 2) = 5(1) + 12(2) + 9 > 0$$

Hence, equation of the bisector of the angle containing the point $(1, 2)$ is

$$\frac{4x + 3y - 6}{5} = \frac{5x + 12y + 9}{13} \text{ or } 9x - 7y - 41 = 0$$

Also, in original equations, $c_1 = -6$ and $c_2 = 9$.

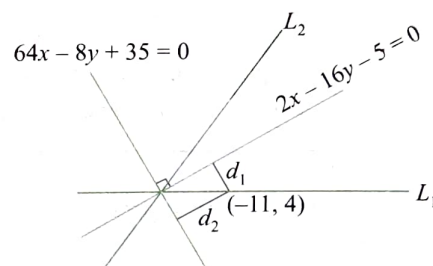
Thus, c_1 and c_2 have opposite signs.

Hence, angle bisector containing origin is $7x + 9y - 3 = 0$.

ILLUSTRATION 2.58

The equation of bisector of two lines L_1 and L_2 are $2x - 16y - 5 = 0$ and $64x + 8y + 35 = 0$. If the line L_1 passes through $(-11, 4)$, then identify the equation of acute angle bisector of L_1 and L_2 .

Sol.



From the figure,

$$d_1 = \frac{|2(-11) - 16(4) - 5|}{\sqrt{4 + 256}} = \frac{91}{2\sqrt{65}} \approx 5.6 \quad (1)$$

$$d_2 = \frac{|64(-11) + 8(4) + 35|}{\sqrt{(64)^2 + 8^2}} = \frac{637}{64.49} \approx 9.87 \quad (2)$$

$$d_1 < d_2$$

Therefore, $2x - 16y - 5 = 0$ is acute angle bisector and $64x + 8y + 35 = 0$ is obtuse angle bisector.

ILLUSTRATION 2.59

If $x + y = 0$ is the angle bisector of the angle containing the point $(1, 0)$, for the lines $3x + 4y + b = 0$ and $4x + 3y - b = 0$, then find the values of b .

Sol. Given lines are

$$3x + 4y + b = 0 \quad (1)$$

$$\text{and } 4x + 3y - b = 0 \quad (2)$$

Equation of bisectors of lines (1) and (2) are

$$(3x + 4y + b) = \pm (4x + 3y - b)$$

For bisector $x + y = 0$, we have to choose negative sign.

Thus, bisector $x + y = 0$ goes through the region where lines $3x + 4y + b$ and $4x + 3y - b$ have opposite signs.

For bisector of the angle containing the point $(1, 0)$, we must have

$$(3(1) + 4(0) + b)(4(1) + 3(0) - b) < 0$$

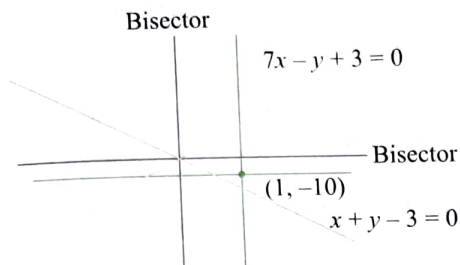
$$\Rightarrow (3 + b)(4 - b) < 0$$

$$\Rightarrow b > 4 \text{ or } b < -3$$

ILLUSTRATION 2.60

Two equal sides of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$. Its third side passes through the point $(1, -10)$. Determine the equation of the third side.

Sol. Since triangle is isosceles, the third side is equally inclined to the lines $7x - y + 3 = 0$ and $x + y - 3 = 0$. Hence, the third side is parallel to angle bisectors of the given lines.



The equation of the two bisectors of given lines are

$$\frac{7x - y + 3}{\sqrt{50}} = \pm \frac{x + y - 3}{\sqrt{2}}$$

or $3x + y - 3 = 0$ (1)

and $x - 3y + 9 = 0$ (2)

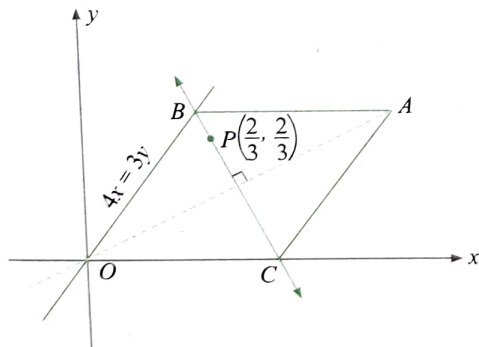
Equation of line through $(1, -10)$ and parallel to (1) is $3x + y + 7 = 0$.

Equation of line through $(1, -10)$ and parallel to (2) is $x - 3y - 31 = 0$.

ILLUSTRATION 2.61

The vertices B and C of a triangle ABC lie on the lines $3y = 4x$ and $y = 0$, respectively. The side BC passes through the point $(2/3, 2/3)$. If $ABOC$ is a rhombus lying in first quadrant, O being the origin, then find the equation of the line BC .

Sol.



Diagonals of rhombus are equally inclined to the sides.

So, diagonals are parallel to or coincident with the angle bisectors of sides.

Equations of bisectors of sides OC and OB of rhombus are

$$\frac{4x - 3y}{5} = \pm \frac{y}{1}$$

or $x - 2y = 0$ and $2x + y = 0$

Clearly, diagonal BC is parallel to the bisector $2x + y = 0$.

So, equation BC is

$$y - \frac{2}{3} = -2 \left[x - \frac{2}{3} \right]$$

or $2x + y - 2 = 0$

ILLUSTRATION 2.62

Two sides of a rhombus lying in the first quadrant are given by $3x - 4y = 0$ and $12x - 5y = 0$. If the length of the longer diagonal is 12, then find the equations of the other two sides of the rhombus.

Sol. OB is along the acute angle bisector of the given lines: $3x - 4y = 0$ and $12x - 5y = 0$.

The equation of acute angle bisector is

$$\frac{3x - 4y}{5} = -\frac{12x - 5y}{13}$$

or $9x - 7y = 0$

$$\therefore \text{Slope of } OB = \tan \theta = \frac{9}{7}$$

Now, $OB = 12$.

Therefore, the coordinates of B are

$$(12 \cos \theta, 12 \sin \theta) \equiv \left(12 \frac{7}{\sqrt{130}}, 12 \frac{9}{\sqrt{130}} \right) \equiv \left(\frac{84}{\sqrt{130}}, \frac{108}{\sqrt{130}} \right)$$

AB is passing through B and is parallel to OC .

Therefore, the equation of AB is

$$y - \frac{108}{\sqrt{130}} = \frac{12}{5} \left(x - \frac{84}{\sqrt{130}} \right)$$

BC is passing through B and is parallel to OA .

Therefore, the equation of BC is $y - \frac{108}{\sqrt{130}} = \frac{3}{4} \left(x - \frac{84}{\sqrt{130}} \right)$

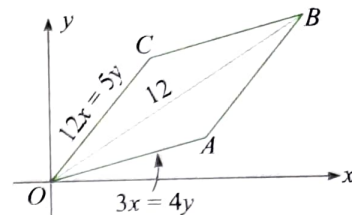


ILLUSTRATION 2.63

The line $ax + by = 1$ passes through the point of intersection of $y = x \tan \alpha + p \sec \alpha$ and $y \sin(30^\circ - \alpha) - x \cos(30^\circ - \alpha) = p$. If it is inclined at 30° with $y = (\tan \alpha)x$, then prove that

$$a^2 + b^2 = \frac{3}{4p^2}$$

Sol. Given lines are

$$x \sin \alpha - y \cos \alpha + p = 0 \quad (1)$$

and $x \cos(30^\circ - \alpha) - y \sin(30^\circ - \alpha) + p = 0$

or $x \sin(60^\circ + \alpha) - y \cos(60^\circ + \alpha) + p = 0 \quad (2)$

Clearly, angle between lines (1) and (2) is 60° .

Slope of line (1) is $\tan \alpha$ which is parallel to the line $y = (\tan \alpha)x$.

Given that line $ax + by = 1$ is inclined at 30° with $y = (\tan \alpha)x$.

So, line $ax + by + 1$ is also inclined at 30° with line (1).

Hence, line $ax + by + 1$ is along acute angle bisector of lines (1) and (2).

Now, acute angle bisector is

$$x \sin \alpha - y \cos \alpha + p = -(x \sin(60^\circ + \alpha) - y \cos(60^\circ + \alpha) + p)$$

or $(\sin \alpha + \sin(60^\circ + \alpha))x - (\cos \alpha + \cos(60^\circ + \alpha))y + 2p = 0$

Comparing this line with $ax + by = 1$, we get

$$\frac{a}{\sin \alpha + \sin(60^\circ + \alpha)} = \frac{b}{\cos \alpha + \cos(60^\circ + \alpha)} = \frac{1}{2p}$$

$$\therefore a = \frac{\sin \alpha + \sin(60^\circ + \alpha)}{2p} \text{ and } b = \frac{\cos \alpha + \cos(60^\circ + \alpha)}{2p}$$

Squaring and adding, we get

$$a^2 + b^2 = \frac{1 + 1 + 2 \cos 60^\circ}{4p^2} \Rightarrow a^2 + b^2 = \frac{3}{4p^2}$$

CONCEPT APPLICATION EXERCISE 2.5

- Find the equation of the bisector of the obtuse angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$.
- The incident ray is along the line $3x - 4y - 3 = 0$ and the reflected ray is along the line $24x + 7y + 5 = 0$. Find the equation of mirrors.
- If the two sides of rhombus are $x + 2y + 2 = 0$ and $2x + y - 3 = 0$, then find the slope of the longer diagonal.
- In triangle ABC , the equation of the right bisectors of the sides AB and AC are $x + y = 0$ and $y - x = 0$, respectively. If $A \equiv (5, 7)$, then find the equation of side BC .
- Show that the reflection of the line $ax + by + c = 0$ in the line $x + y + 1 = 0$ is the line $bx + ay + (a + b - c) = 0$, where $a \neq b$.
- Equations of two altitudes of equilateral triangle are $\sqrt{3}x - y + 8 - 4\sqrt{3} = 0$ and $-\sqrt{3}x - y + 12 + 4\sqrt{3} = 0$. Find the equation of the third altitude.
- The equations of the perpendicular bisectors of the sides AB and AC of triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$, respectively. If the point A is $(1, -2)$, then find the equation of the line BC .
- Two sides of a rhombus $ABCD$ are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point $(1, 2)$ and the vertex A is on the y -axis, then find the possible coordinates of A .

ANSWERS

- $21x + 77y - 101 = 0$
- $9x + 27y + 20 = 0$ or $39x - 13y - 10 = 0$
- -1
- $14y = 10x$
- $y = 10$
- $14x + 23y - 40 = 0$
- $(0, 0)$ or $(0, 5/2)$

CONCURRENCY OF THREE LINES AND FAMILY OF STRAIGHT LINES

CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if they pass through a common point, i.e., they meet at a point. Thus, if three lines are concurrent, then the point of intersection of two lines lies on the third line.

$$\begin{aligned} \text{Let } a_1x + b_1y + c_1 &= 0 & (1) \\ a_2x + b_2y + c_2 &= 0 & (2) \\ a_3x + b_3y + c_3 &= 0 & (3) \end{aligned}$$

be three concurrent lines.

Then the point of intersection of lines (2) and (3) must lie on line (1).

The coordinates of the point of intersection of (2) and (3) are

$$\left(\frac{b_3c_2 - b_2c_3}{a_3b_2 - a_2b_3}, \frac{c_3a_2 - c_2a_3}{a_3b_2 - a_2b_3} \right)$$

This point lies on line (1).

$$\begin{aligned} \Rightarrow a_1 \left(\frac{b_3c_2 - b_2c_3}{a_3b_2 - a_2b_3} \right) + b_1 \left(\frac{c_3a_2 - c_2a_3}{a_3b_2 - a_2b_3} \right) + c_1 &= 0 \\ \Rightarrow a_1(b_2c_3 - b_3c_2) - b_1(c_3a_2 - c_2a_3) + c_1(a_2b_3 - a_3b_2) &= 0 \\ \Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= 0 \end{aligned}$$

This is the required condition of concurrency of three lines.

Another condition for three lines to be concurrent is that if there exist constants λ_1, λ_2 and λ_3 not all zero such that

$\lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$, then lines are concurrent.

ILLUSTRATION 2.64

Find the value of λ , if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33$, and $2x - 3y + \lambda = 0$ are concurrent.

Sol. The given lines are concurrent if

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$$\begin{aligned} \text{or } 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) &= 0 \\ \text{or } -\lambda - 7 &= 0 \\ \text{or } \lambda &= -7 \end{aligned}$$

Alternative method:

The given equations are

$$3x - 4y - 13 = 0 \quad (1)$$

$$8x - 11y - 33 = 0 \quad (2)$$

$$\text{and } 2x - 3y + \lambda = 0 \quad (3)$$

Solving (1) and (2), we get

$$x = 11 \text{ and } y = 5$$

Thus, $(11, 5)$ is the point of intersection of (1) and (2).

The given lines will be concurrent if they pass through a common point, i.e., the point of intersection of any two lines lies on the third.

Therefore, $(11, 5)$ lies on (3), i.e.,

$$2 \times 11 - 3 \times 5 + \lambda = 0$$

$$\text{or } \lambda = -7$$

ILLUSTRATION 2.65

If the lines $a_1x + b_1y + 1 = 0$, $a_2x + b_2y + 1 = 0$, and $a_3x + b_3y + 1 = 0$ are concurrent, then show that the points (a_1, b_1) , (a_2, b_2) , and (a_3, b_3) are collinear.

Sol. The given lines are

$$a_1x + b_1y + 1 = 0 \quad (1)$$

$$a_2x + b_2y + 1 = 0 \quad (2)$$

$$\text{and } a_3x + b_3y + 1 = 0 \quad (3)$$

If these lines are concurrent, we must have

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$$

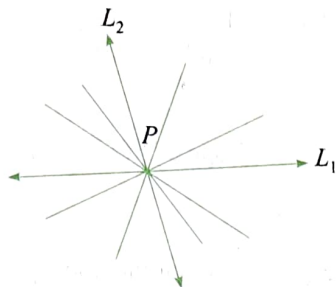
which is the condition of the collinearity of three points (a_1, b_1) , (a_2, b_2) , and (a_3, b_3) .
Hence, if the given lines are concurrent, the given points are collinear.

FAMILY OF LINES

We know that through one given point, innumerable straight lines can be drawn.

Let us get the general equation of straight line passing through given fixed point in terms of two particular straight lines through the same point.

Consider two straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ through fixed point $P(x_1, y_1)$.



General equation of line through point $P(x_1, y_1)$ is given by $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, $\lambda \in R$.

Since $a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$, above equation satisfies the point $P(x_1, y_1)$ for all real values of λ .

Note:

- Consider variable straight line $ax + by + c = 0$, where $a, b, c \in R$.

For randomly chosen values of a, b and c , lines obtained are not necessarily concurrent.

But if a, b, c are related by equation $al + bm + cn = 0$, where l, m, n are constants, then lines are concurrent for different values of a, b, c which satisfy the above equation. Here, only two of a, b and c can be chosen independently.

From the relation, $c = -\frac{al + bm}{n}$.

Putting this value of c in the equation of line, we get

$$ax + by - \frac{al + bm}{n} = 0$$

$$\therefore a\left(x - \frac{l}{n}\right) + b\left(y - \frac{m}{n}\right) = 0$$

This is the equation of family of straight lines concurrent at $\left(\frac{l}{n}, \frac{m}{n}\right)$.

- A straight line is such that the algebraic sum of the perpendiculars drawn on it from any number of fixed points is zero. Then the line always passes through a fixed point which is mean of the given points.

Proof:

Let the given fixed points be (x_r, y_r) ; $r = 1, 2, 3, \dots, n$.

Also, let the variable line be

$$ax + by + c = 0 \quad (1)$$

$$\text{Given that } \sum_{r=1}^n \frac{ax_r + by_r + c}{\sqrt{a^2 + b^2}} = 0$$

$$\Rightarrow \sum_{r=1}^n (ax_r + by_r + c) = 0$$

$$\Rightarrow a \sum_{r=1}^n x_r + b \sum_{r=1}^n y_r + nc = 0$$

$$\Rightarrow a(x_1 + x_2 + \dots + x_n) + b(y_1 + y_2 + \dots + y_n) + cn = 0$$

$$\Rightarrow a\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) + b\left(\frac{y_1 + y_2 + \dots + y_n}{n}\right) + c = 0 \quad (2)$$

Comparing (1) and (2), we find that

$$(x, y) \equiv \left(\frac{x_1 + x_2 + \dots + x_n}{n}, \frac{y_1 + y_2 + \dots + y_n}{n}\right).$$

This is the mean of given point through which variable line $ax + by + c = 0$ passes.

From the above discussion, it is clear that in a triangle, line passing through centroid is such that the sum of algebraic distances from three vertices of triangle is zero.

ILLUSTRATION 2.66

Show that the straight lines $x(a + 2b) + y(a + 3b) = a + b$ for different values of a and b pass through a fixed point. Find that point.

Sol. Given equation of variable line is

$$x(a + 2b) + y(a + 3b) = a + b$$

$$\text{or } a(x + y - 1) + b(2x + 3y - 1) = 0 \quad (1)$$

At least one of a and b will be non zero.

Let $a \neq 0$.

So, equation (1) reduces to

$$(x + y - 1) + \frac{b}{a}(2x + 3y - 1) = 0$$

$$\text{or } x + y - 1 + \lambda(2x + 3y - 1) = 0, \text{ where } \lambda = \frac{b}{a} \quad (2)$$

This is the equation of family of straight lines concurrent at point of intersection of lines

$$x + y - 1 = 0 \quad (3)$$

$$\text{and } 2x + 3y - 1 = 0 \quad (4)$$

Solving (3) and (4), we get $x = 2, y = -1$.

Hence, given variable lines pass through the fixed point $(2, -1)$ for all values of a and b .

ILLUSTRATION 2.67

Let $ax + by + c = 0$ be a variable straight line, where a, b and c are 1st, 3rd and 7th terms of an increasing A.P., respectively. Then prove that the variable straight line always passes through a fixed point and find that point.

Sol. Let the common difference of A.P. be d .

Then $b = a + 2d$ and $c = a + 6d$.

So, given variable straight line will be

$$ax + (a + 2d)y + a + 6d = 0$$

or $a(x + y + 1) + d(2y + 6) = 0$

This is the equation of family of straight lines concurrent at point of intersection of lines $x + y + 1 = 0$ and $2y + 6 = 0$ which is $(2, -3)$.

ILLUSTRATION 2.68

Prove that all the lines having sum of the intercepts on the axes equal to half of the product of the intercepts pass through a fixed point. Also, find that fixed point.

Sol. Equation of line in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1$$

or $bx + ay = ab$ (1)

Given that $a + b = \frac{ab}{2}$

$\therefore 2a + 2b = ab$

So, equation (1) reduces to

$$bx + ay = 2a + 2b$$

or $b(x - 2) + a(y - 2) = 0$,

This passes through fixed point $(2, 2)$ for all real values of a and b .

ILLUSTRATION 2.69

Find the straight line passing through the point of intersection of lines $2x + 3y + 5 = 0$ and $5x - 2y - 16 = 0$ and through the point $(-1, 3)$ using the concept of family of lines.

Sol. The equation family of lines through the point of intersection of the given lines is

$$2x + 3y + 5 + \lambda(5x - 2y - 16) = 0 \quad (1)$$

If a member of the above family of lines passes through the point $(-1, 3)$, then

$$2(-1) + 3(3) + 5 + \lambda(5(-1) - 2(3) - 16) = 0$$

or $\lambda = \frac{4}{9}$

Putting this value λ in equation (1), we get the required equation of line as

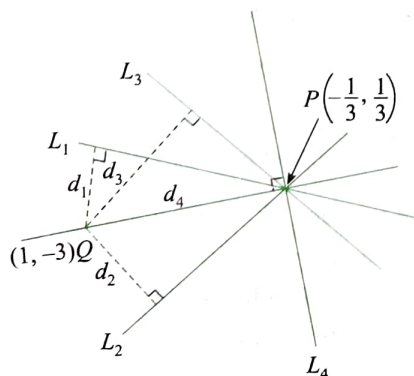
$$9(2x + 3y + 5) + 4(5x - 2y - 16) = 0$$

or $2x + y - 1 = 0$

ILLUSTRATION 2.70

Consider a family of straight lines $(x + y) + \lambda(2x - y + 1) = 0$. Find the equation of the straight line belonging to this family that is farthest from $(1, -3)$.

Sol.



Given lines are concurrent at point of intersection of lines

$$x + y = 0 \text{ and } 2x - y + 1 = 0 \text{ which is } P\left(-\frac{1}{3}, \frac{1}{3}\right).$$

We require a line through point P which is farthest from point $Q(1, -3)$, i.e., line whose perpendicular distance from point P is the greatest.

We observe that perpendicular distances $d_1, d_2, d_3 \dots$ of the lines $L_1, L_2, L_3, \dots$, respectively, from point Q are less than distance d_4 of the line L_4 from Q . Line L_4 is perpendicular to the segment joining P and Q . Line L_4 is at greatest distance from point Q .

$$\text{Slope of } PQ \text{ is } m_{PQ} = \frac{-3 - \frac{1}{3}}{1 + \frac{1}{3}} = -\frac{5}{2}$$

$\therefore$ Slope of the line $L_4 = \frac{2}{5}$

Therefore, equation of the required line is

$$y - \frac{1}{3} = \frac{2}{5}\left(x + \frac{1}{3}\right)$$

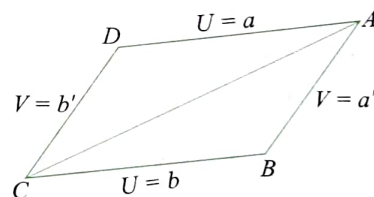
or $6x - 15y + 7 = 0$

ILLUSTRATION 2.71

Let the sides of a parallelogram be $U = a$, $U = b$, $V = a'$ and $V = b'$, where $U = lx + my + n$, $V = l'x + m'y + n'$. Show that the equation of the diagonal through the point of intersection of

$$U = a, V = a' \text{ and } U = b, V = b' \text{ is given by } \begin{vmatrix} U & V & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix} = 0.$$

Sol. Parallelogram is formed by lines $U = a$, $U = b$, $V = a'$ and $V = b'$, where $U = lx + my + n$, $V = l'x + m'y + n'$



Let the required diagonal be AC .

Since one end A of the diagonal AC passes through the point of intersection of lines $U - a = 0$ and $V - a' = 0$, its equation is given by

$$(U - a) + \lambda(V - a') = 0 \quad (1)$$

But the other end C of the same diagonal passes through the point of intersection of lines $U - b = 0$ and $V - b' = 0$.

So, its equation is given by

$$(U - b) + \mu(V - b') = 0 \quad (2)$$

Equation (1) and (2) represents the same straight lines.

$$\therefore 1 = \frac{\lambda}{\mu} = \frac{a + \lambda a'}{b + \mu b'}$$

$$\Rightarrow \lambda = \mu = -\left(\frac{a - b}{a' - b'}\right)$$

Putting the value of λ in (1), we get

$$(U-a) - \left(\frac{a-b}{a'-b'} \right) (V-a') = 0$$

or
$$\begin{vmatrix} U & V & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix} = 0$$

This is the required equation of diagonal.

ILLUSTRATION 2.72

Find the values of non-negative real numbers $h_1, h_2, h_3, k_1, k_2, k_3$ such that algebraic sum of the perpendiculars drawn from points $(2, k_1), (3, k_2), (7, k_3), (h_1, 4), (h_2, 5), (h_3, -3)$ on a variable line passing through $(2, 1)$ is zero.

Sol. Let the equation of variable line be

$$ax + by + c = 0 \quad (1)$$

It is given that

$$\sum_{i=1}^6 \frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}} = 0, \text{ where } (x_i, y_i); i = 1, 2, 3, 4, 5, 6 \text{ are given points.}$$

$$\Rightarrow a \left(\frac{\sum_{i=1}^6 x_i}{6} \right) + b \left(\frac{\sum_{i=1}^6 y_i}{6} \right) + c = 0 \quad (2)$$

Comparing (1) and (2), we get

$$\frac{\sum_{i=1}^6 x_i}{6} = \frac{\sum_{i=1}^6 y_i}{6} = 1$$

$$\therefore (x, y) \equiv \left(\frac{\sum_{i=1}^6 x_i}{6}, \frac{\sum_{i=1}^6 y_i}{6} \right)$$

This is the fixed point through which the variable (1) passes.

But it is given that the variable line is passing through the point $(2, 1)$.

$$\text{So, } \frac{\sum_{i=1}^6 x_i}{6} = 2 \text{ and } \frac{\sum_{i=1}^6 y_i}{6} = 1$$

$$\Rightarrow \frac{2+3+7+h_1+h_2+h_3}{6} = 2 \text{ and } \frac{k_1+k_2+k_3+4+5-3}{6} = 1$$

$$\Rightarrow h_1 + h_2 + h_3 = 0 \text{ and } k_1 + k_2 + k_3 = 0$$

Since $h_1, h_2, h_3, k_1, k_2, k_3$ are non-negative, $h_1 = h_2 = h_3 = 0$ and $k_1 = k_2 = k_3 = 0$.

CONCEPT APPLICATION EXERCISE 2.6

1. If a and b are two arbitrary constants, then prove that the straight line $(a-2b)x + (a+3b)y + 3a+4b = 0$ will pass through a fixed point. Find that point.
2. If a, b, c are in harmonic progression, then the straight line $(x/a) + (y/b) + (1/c) = 0$ always passes through a fixed point. Find that point.

3. If the algebraic sum of the distances of a variable line from the points $(2, 0), (0, 2),$ and $(-2, -2)$ is zero, then prove that the line passes through the fixed point. Find that point.
4. Consider the family of lines $5x + 3y - 2 + \lambda_1(3x - y - 4) = 0$ and $x - y + 1 + \lambda_2(2x - y - 2) = 0$. Find the equation of a straight line that belongs to both the families.
5. If the straight lines $x + y - 2 = 0, 2x - y + 1 = 0,$ and $ax + by - c = 0$ are concurrent, then at which point the family of lines $2ax + 3by + c = 0$ (a, b, c are nonzero) is concurrent?

ANSWERS

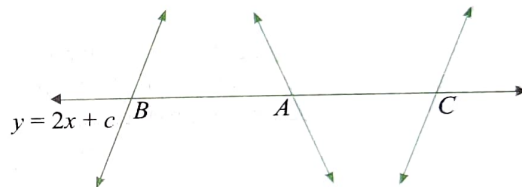
- | | | |
|----------------------|-------------------|-------------|
| 1. $(-1, -2)$ | 2. $(1, -2)$ | 3. $(0, 0)$ |
| 4. $5x - 2y - 7 = 0$ | 5. $(-1/6, -5/9)$ | |

Solved Examples

EXAMPLE 2.1

Show that the lines $4x + y - 9 = 0, x - 2y + 3 = 0, 5x - y - 6 = 0$ make equal intercepts on any line of slope 2.

Sol.



The given lines are

$$4x + y - 9 = 0 \quad (1)$$

$$x - 2y + 3 = 0 \quad (2)$$

$$5x - y - 6 = 0 \quad (3)$$

Now, the equation of any line having slope (gradient) 2 will be

$$y = 2x + c \quad (4)$$

Let line (4) cuts lines (1), (2), and (3) at $A, B,$ and $C,$ respectively.

Solving (1) and (4), we get

$$A \equiv \left(\frac{3}{2} - \frac{c}{6}, 3 + \frac{2c}{3} \right)$$

$$\text{Similarly, } B \equiv \left(1 - \frac{2c}{3}, 2 - \frac{c}{3} \right)$$

$$\text{and } C \equiv \left(2 + \frac{c}{3}, 4 + \frac{5c}{3} \right)$$

Clearly, A is the middle point of BC . Hence, $AB = AC$.

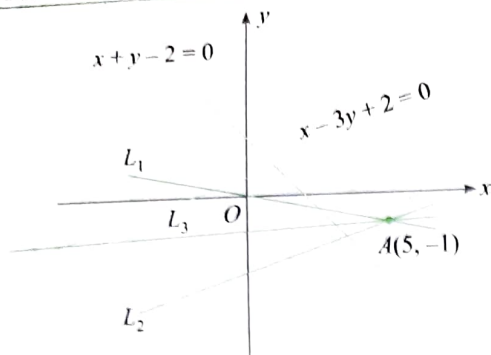
EXAMPLE 2.2

The equations of two sides of a triangle are $3y - x - 2 = 0$ and $y + x - 2 = 0$. The third side, which is variable, always passes through the point $(5, -1)$. Find the range of the values of the slope of the third side, so that the origin is an interior point of the triangle.

Sol. We have two sides of triangle as $3y - x - 2 = 0$ and $y + x - 2 = 0$.

The third side is passing through the point $A(5, -1)$.

Different lines through point A are drawn as shown in the following figure.



Line L_2 through A is parallel to the line $x - 3y + 2 = 0$, which has slope $1/3$.

In this case, triangle is not formed.

Line L_1 through A goes through origin, which has slope $-1/5$.

In this case triangle is formed but origin lies on the side of the triangle.

Line L_3 through A is such that triangle is formed and origin lies inside the triangle.

Thus, required values of slope of line are $(-1/5, 1/3)$.

EXAMPLE 2.3

Find the locus of the circumcenter of a triangle whose two sides are along the coordinate axes and the third side passes through the point of intersection of the lines $ax + by + c = 0$ and $lx + my + n = 0$.

Sol. Let the equation of the third line be

$$(ax + by + c) + \lambda(lx + my + n) = 0$$

where λ is a parameter. It meets the x -axis at $\left(-\frac{\lambda n + c}{\lambda l + a}, 0\right)$ and the y -axis at $B\left(0, -\frac{\lambda n + c}{\lambda m + b}\right)$.

The triangle OAB is a right-angled triangle. Its circumcenter is the midpoint of the hypotenuse. Let it be (α, β) . Then,

$$2\alpha = -\left(\frac{\lambda n + c}{\lambda l + a}\right)$$

$$\text{and } 2\beta = -\left(\frac{\lambda n + c}{\lambda m + b}\right)$$

$$\text{Hence, } \lambda = -\frac{2\alpha a + c}{2\alpha l + n} = -\frac{2\beta b + c}{2\beta m + n}$$

Hence, the locus of (α, β) is

$$\frac{c + 2\alpha a}{n + 2\alpha l} = \frac{c + 2\beta b}{n + 2\beta m}$$

$$\text{or } 2xy(bl - ma) = x(an - lc) + y(mc - bn)$$

EXAMPLE 2.4

Let ABC be a triangle with $AB = AC$. If D is the midpoint of BC , E is the foot of the perpendicular drawn from D to AC , and F is the midpoint of DE , then prove that AF is perpendicular to BE .

Sol. Let BC be taken as the x -axis with the origin at D . Let A be taken on y -axis. Let $BC = 2a$. Now $AB = AC$, then the coordinates of B and C are $(-a, 0)$ and $(a, 0)$, respectively. Let $DA = h$. Then the coordinates of A are $(0, h)$.

Hence, the equation of AC is

$$\frac{x}{a} + \frac{y}{h} = 1 \quad (1)$$

and the equation of DE perpendicular to AC and passing through the origin is

$$\frac{x}{h} - \frac{y}{a} = 0$$

$$\text{or } x = \frac{hy}{a} \quad (2)$$

Solving (1) and (2), we get the coordinates of E as follows:

$$\frac{hy}{a^2} + \frac{y}{h} = 1$$

$$\text{or } h^2y + a^2y = a^2h$$

$$\text{or } y = \frac{a^2h}{a^2 + h^2}$$

$$\therefore x = \frac{ah^2}{a^2 + h^2}$$

$$\therefore E \equiv \left(\frac{ah^2}{a^2 + h^2}, \frac{a^2h}{a^2 + h^2}\right)$$

Since F is the midpoint of DE , its coordinates are

$$\left(\frac{ah^2}{2(a^2 + h^2)}, \frac{a^2h}{2(a^2 + h^2)}\right)$$

The slope of AF is

$$\begin{aligned} m_1 &= \frac{h - \frac{a^2h}{2(a^2 + h^2)}}{0 - \frac{ah^2}{2(a^2 + h^2)}} \\ &= \frac{2h(a^2 + h^2) - a^2h}{-ah^2} = -\frac{a^2 + 2h^2}{ah} \end{aligned} \quad (1)$$

and the slope of BE is

$$\begin{aligned} m_2 &= \frac{\frac{a^2h}{2(a^2 + h^2)} - 0}{\frac{ah^2}{2(a^2 + h^2)} + a} \\ &= \frac{a^2h}{ah^2 + a^3 + ah^2} = \frac{ah}{a^2 + 2h^2} \end{aligned} \quad (2)$$

From (1) and (2), we have

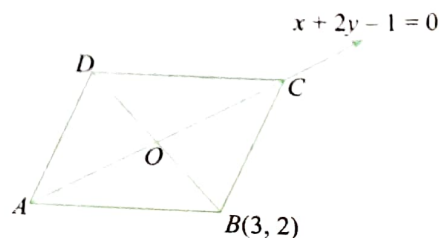
$$m_1m_2 = -1$$

$$\text{or } AF \perp BE$$

EXAMPLE 2.5

A diagonal of rhombus $ABCD$ is member of both the families of lines $(x + y - 1) + \lambda(2x + 3y - 2) = 0$ and $(x - y + 2) + \lambda(2x - 3y + 5) = 0$; one of the vertices of the rhombus is $(3, 2)$. If the area of the rhombus is $12\sqrt{5}$ sq. units, then find the remaining vertices of the rhombus.

Sol.



Since diagonal is member of both the given families of line, it will pass through the points $(1, 0)$ and $(-1, 1)$ which are points where given family of lines are concurrent.

So, equation of diagonal AC is $x + 2y - 1 = 0$.

Clearly, point $(3, 2)$ does not satisfy this diagonal.

$\therefore$ Let $B \equiv (3, 2)$.

Diagonal BD is perpendicular to AC .

So, equation of diagonal BD is $2x - y - 4 = 0$.

Point of intersection of diagonals AC and BD is $O\left(\frac{9}{5}, \frac{-2}{5}\right)$.

Since O is midpoint of BD , $D \equiv \left(\frac{3}{5}, -\frac{14}{5}\right)$.

Now, $OD = \frac{6}{\sqrt{5}}$. So, $BD = \frac{12}{\sqrt{5}} = d_1$.

Area of rhombus $= \frac{1}{2} d_1 \times d_2 = 12\sqrt{5}$ (where d_2 is length of diagonal AC)

$\therefore d_2 = 10$ units

This is the length of diagonal AC .

Slope of $AC = -\frac{1}{2} = \tan \theta$, where θ is inclination of AC with x -axis.

Since $OA = OC = 5$, using parametric form of straight line, we have

$$\begin{aligned} A, C &\equiv \left(\frac{9}{5} \pm 5 \cos \theta, -\frac{2}{5} \pm 5 \sin \theta\right) \\ &\equiv \left(\frac{9}{5} \pm 5 \left(-\frac{2}{\sqrt{5}}\right), -\frac{2}{5} \pm 5 \left(\frac{1}{\sqrt{5}}\right)\right) \\ &\equiv \left(\frac{9}{5} - 2\sqrt{5}, -\frac{2}{5} + \sqrt{5}\right), \left(\frac{9}{5} + 2\sqrt{5}, -\frac{2}{5} - \sqrt{5}\right) \end{aligned}$$

EXAMPLE 2.6

Let ABC be a given isosceles triangle with $AB = AC$. Sides AB and AC are extended up to E and F , respectively, such that $BE \times CF = AB^2$. Prove that the line EF always passes through a fixed point.

Sol. Let ABC be the triangle having vertices $(-a, 0)$, $(0, b)$, and $(a, 0)$. Now,

$$BE \times CF = AB^2$$

$$\text{or } \frac{BE}{AB} = \frac{CF}{AB} = \lambda \text{ (Let)}$$

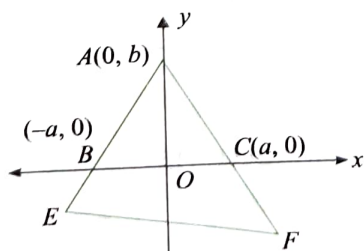
$$\text{or } \frac{BE}{AB} = \frac{CF}{AB} = \lambda$$

Hence, the coordinates of E and F are $(-a(\lambda + 1), -\lambda b)$ and $(a(1 + \lambda), -b/\lambda)$. The equation of line EF is

$$y + \lambda b = \frac{-\lambda b + \frac{b}{\lambda}}{-a(\lambda + 1) - \frac{a(\lambda + 1)}{\lambda}} [x + a(\lambda + 1)]$$

$$\text{or } y + \lambda b = \frac{\frac{b}{\lambda}(1 - \lambda^2)}{-\frac{a(\lambda + 1)}{\lambda} [1 + \lambda]} [x + a(\lambda + 1)]$$

$$\text{or } y + \lambda b = \frac{b(\lambda - 1)}{a(\lambda + 1)} [x + a(\lambda + 1)]$$



$$\text{or } a(\lambda + 1)y + ab\lambda(\lambda + 1) = b(\lambda - 1)x + ab(\lambda^2 - 1)$$

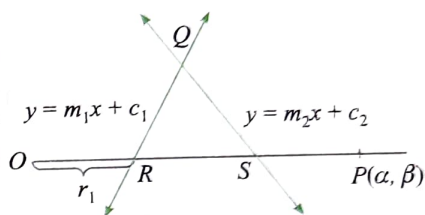
$$\text{or } (bx + ay + ab) - \lambda(bx - ay - ab) = 0$$

which is the equation of a family of lines passing through the point of intersection of the lines $bx + ay + ab = 0$ and $bx - ay - ab = 0$, the point of intersection being $(0, -b)$. Hence, the line EF passes through a fixed point.

EXAMPLE 2.7

Let $L_1 = 0$ and $L_2 = 0$ be two fixed lines. A variable line is drawn through the origin to cut the two lines at R and S . P is a point on the line AB such that $(m + n)/OP = m/OR + n/OS$. Show that the locus of P is a straight line passing through the point of intersection of the given lines (R, S, P are on the same side of O).

Sol. Let the variable line through O makes an angle θ with the positive direction of the x -axis. Any point on this variable line is $(0 + r \cos \theta, 0 + r \sin \theta)$, i.e., $(r \cos \theta, r \sin \theta)$



Let the fixed lines be

$$y = m_1x + c_1 \quad (1)$$

$$\text{and } y = m_2x + c_2 \quad (2)$$

Let $OR = r_1$, $OS = r_2$, $OP = r_3$. So, according to the question,

$$\frac{m + n}{r_3} = \frac{m}{r_1} + \frac{n}{r_2} \quad (3)$$

Since R and S lie on lines (1) and (2), respectively, we have

$$r_1 \sin \theta = m_1 (r_1 \cos \theta) + c_1 \quad (4)$$

$$\text{and } r_2 \sin \theta = m_2 (r_2 \cos \theta) + c_2 \quad (5)$$

Let $P \equiv (\alpha, \beta)$. Then,

$$\alpha = r_3 \cos \theta, \beta = r_3 \sin \theta \quad (6)$$

From (3), (4), and (5), we have

$$\frac{m + n}{r_3} = m \frac{\sin \theta - m_1 \cos \theta}{c_1} + n \frac{\sin \theta - m_2 \cos \theta}{c_2}$$

$$\text{or } m + n = \frac{m}{c_1} (\beta - m_1 \alpha) + \frac{n}{c_2} (\beta - m_2 \alpha) \quad [\text{From (6)}]$$

Hence, the locus of P is

$$m + n = \left(\frac{m}{c_1} + \frac{n}{c_2}\right) y - \left(\frac{mm_1}{c_1} + \frac{nm_2}{c_2}\right) x$$

$$\text{or } y - m_1x - c_1 + \frac{n}{m} \frac{c_1}{c_2} (y - m_2x - c_2) = 0 \quad (7)$$

Clearly, the locus of P is a straight line passing through the point of intersection of the given lines (1) and (2).

EXAMPLE 2.8

Let points A, B and C lie on lines $y - x = 0$, $2x - y = 0$ and $y - 3x = 0$, respectively. Also, AB passes through fixed point $P(1, 0)$ and BC passes through fixed point $Q(0, -1)$. Then prove that AC also passes through a fixed point and find that point.

Sol. Let the coordinates of points A, B and C be (α, α) , $(\beta, 2\beta)$ and $(\gamma, 3\gamma)$, respectively.

Points A, B, P are collinear.

$$\therefore \begin{vmatrix} 1 & 0 & 1 \\ \alpha & \alpha & 1 \\ \beta & 2\beta & 1 \end{vmatrix} = 0$$

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$$\Rightarrow \alpha - 2\beta + \alpha\beta = 0 \quad (1)$$

Also, points B, C, Q are collinear.

$$\therefore \begin{vmatrix} 0 & -1 & 1 \\ \beta & 2\beta & 1 \\ \gamma & 3\gamma & 1 \end{vmatrix} = 0$$

$$\Rightarrow \beta - \gamma + \beta\gamma = 0$$

$$\Rightarrow \beta = \frac{\gamma}{1+\gamma} \quad (2)$$

Putting value of β in equation (1), we get $\alpha + 2\alpha\gamma = 2\gamma$.

Let AC pass through fixed point $R(h, k)$.

Since C, A and R are collinear,

$$\begin{vmatrix} \alpha & \alpha & 1 \\ \gamma & 3\gamma & 1 \\ h & k & 1 \end{vmatrix} = 0$$

$$\Rightarrow h(\alpha - 3\gamma) - k(\alpha - \gamma) + 2\alpha\gamma = 0$$

$$\Rightarrow h(\alpha - 3\gamma) - k(\alpha - \gamma) + 2\gamma - \alpha = 0$$

$$\Rightarrow \alpha(h - k - 1) + \gamma(-3h + k + 2) = 0 \text{ for all } \alpha, \gamma.$$

$$\therefore h - k - 1 = 0 \text{ and } -3h + k + 2 = 0$$

$$\therefore h = \frac{1}{2}, k = -\frac{1}{2}$$

Thus, AC passes through the point $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

EXAMPLE 2.9

Consider two lines L_1 and L_2 given by $x - y = 0$ and $x + y = 0$, respectively, and a moving point $P(x, y)$. Let $d(P, L_i)$, $i = 1, 2$, represents the distance of point P from the line L_i . If point P moves in a certain region R in such a way that $2 \leq d(P, L_1) + d(P, L_2) \leq 4$, find the area of region R .

Sol. $d(P, L_1) = \frac{|x - y|}{\sqrt{2}}$

and $d(P, L_2) = \frac{|x + y|}{\sqrt{2}}$

Now, we have

$$2 \leq d(P, L_1) + d(P, L_2) \leq 4$$

or $2\sqrt{2} \leq |x - y| + |x + y| \leq 4\sqrt{2} \quad (1)$

Now, let us consider the four regions, namely, R_1, R_2, R_3 , and R_4 , in the lines L_1 and L_2 dividing the coordinate plane.

In R_1 , we have $y < x, y > -x$. In R_2 , we have $y > x, y > -x$. Similarly, in R_3 , we have $y > x, y < -x$. Finally, in R_4 , we have $y < x, y < -x$.

Thus, for R_1 , (1) becomes

$$2\sqrt{2} \leq x - y + x + y \leq 4\sqrt{2} \text{ or } \sqrt{2} \leq x \leq 2\sqrt{2}$$

Similarly, for R_2 , (1) becomes

$$2\sqrt{2} \leq y - x + x + y \leq 4\sqrt{2} \text{ or } \sqrt{2} \leq y \leq 2\sqrt{2}$$

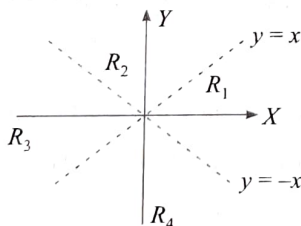
In R_3 , (1) will become

$$2\sqrt{2} \leq y - x - x - y \leq 4\sqrt{2} \text{ or } -\sqrt{2} \leq x \leq -2\sqrt{2}$$

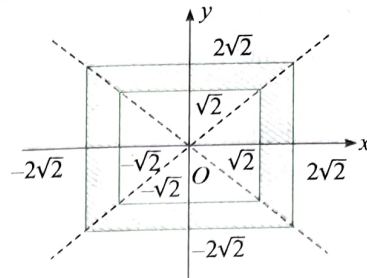
Finally, in R_4 , (1) will become

$$2\sqrt{2} \leq x - y - x - y \leq 4\sqrt{2}$$

or $-\sqrt{2} \leq y \leq -2\sqrt{2}$



Thus, region R will be the region between concentric squares formed by the lines $x = \pm\sqrt{2}, y = \pm\sqrt{2}$ and $x = \pm2\sqrt{2}, y = \pm2\sqrt{2}$. Thus the required area is $(4\sqrt{2})^2 - (2\sqrt{2})^2 = 24$ sq. units.

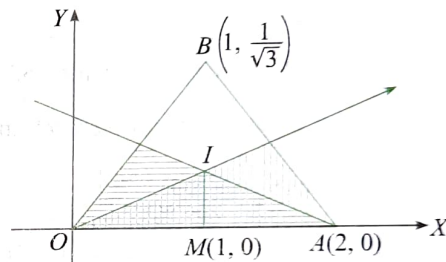


EXAMPLE 2.10

Let $O(0, 0)$, $A(2, 0)$ and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle.

Let R be the region consisting of all those points P inside $\triangle OAB$ which satisfy $d(P, OA) \leq \min[d(P, OB), d(P, AB)]$, where d denotes the distance from the point to the corresponding line. Sketch the region R and find its area.

Sol.



$$d(P, OA) \leq \min[d(P, OB), d(P, AB)]$$

$$\Rightarrow d(P, OA) \leq d(P, OB) \text{ and } d(P, OA) \leq d(P, AB)$$

When $d(P, OA) = d(P, OB)$, P is equidistant from OA and OB , i.e., P lies on angle bisector of lines OA and OB .

So, when $d(P, OA) < d(P, OB)$, point P is nearer to OA than to OB , i.e., it lies below bisector of OA and OB .

Similarly, when $d(P, OA) < d(P, AB)$, P is nearer to OA than to AB , i.e., it lies below bisector of OA and AB .

Common region satisfied by these two inequalities is the region formed by two bisectors and OA , i.e., triangle OIA .

$$\therefore \text{Required area} = \text{Area of triangle } OIA$$

$$\text{Now, } \angle BOA = \frac{1/\sqrt{3}}{1} = \frac{1}{\sqrt{3}}$$

$$\therefore \angle BOA = 30^\circ$$

$$\Rightarrow \angle IOA = 15^\circ$$

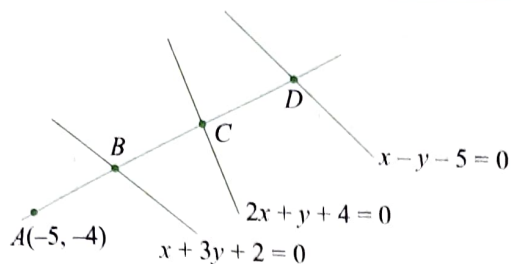
$$\Rightarrow IM = \tan 15^\circ = 2 - \sqrt{3}$$

$$\therefore \text{Area of triangle } OIA = \frac{1}{2} \times OA \times IM$$

$$= \frac{1}{2} \times 2 \times (2 - \sqrt{3}) = 2 - \sqrt{3} \text{ sq. units}$$

EXAMPLE 2.11

A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$, and $x - y - 5 = 0$ at the points B, C , and D , respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line.

Sol.

Let θ be the inclination of line through $A(-5, -4)$.

Therefore, equation of this line is $\frac{x+5}{\cos \theta} = \frac{y+4}{\sin \theta} = r$ (1)

Any point on this line at distance r from point $A(-5, -4)$ is given by $(-5 + r \cos \theta, -4 + r \sin \theta)$

If $AB = r_1$, $AC = r_2$ and $AD = r_3$, then

$$B(r_1 \cos \theta - 5, r_1 \sin \theta - 4)$$

$$C(r_2 \cos \theta - 5, r_2 \sin \theta - 4)$$

$$D(r_3 \cos \theta - 5, r_3 \sin \theta - 4)$$

But B lies on $x + 3y + 2 = 0$. Therefore,

$$r_1 \cos \theta - 5 + 3r_1 \sin \theta - 12 + 2 = 0$$

$$\text{or } \frac{15}{\cos \theta + 3 \sin \theta} = r_1$$

$$\text{or } \frac{15}{AB} = \cos \theta + 3 \sin \theta \quad (1)$$

As C lies on $2x + y + 4 = 0$, we have

$$2(r_2 \cos \theta - 5) + (r_2 \sin \theta - 4) + 4 = 0$$

$$\text{or } r_2 = \frac{10}{2 \cos \theta + \sin \theta} = AC$$

$$\text{or } \frac{10}{AC} = 2 \cos \theta + \sin \theta \quad (2)$$

Similarly, D lies on $x - y - 5 = 0$. Therefore,

$$r_3 \cos \theta - 5 - r_3 \sin \theta + 4 - 5 = 0$$

$$\text{or } r_3 = \frac{6}{\cos \theta - \sin \theta} = AD$$

$$\text{or } \frac{6}{AD} = \cos \theta - \sin \theta \quad (3)$$

Now, given that

$$\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$$

$$\text{or } (\cos \theta + 3 \sin \theta)^2 + (2 \cos \theta + \sin \theta)^2 = (\cos \theta - \sin \theta)^2$$

[Using (1), (2), and (3)]

$$\text{or } 4 \cos^2 \theta + 9 \sin^2 \theta + 12 \sin \theta \cos \theta = 0$$

$$\text{or } (2 \cos \theta + 3 \sin \theta)^2 = 0$$

$$\text{or } \tan \theta = -\frac{2}{3}$$

Hence, the equation of the required line is

$$y + 4 = -\frac{2}{3}(x + 5)$$

$$\text{or } 3y + 12 = -2x - 10$$

$$\text{or } 2x + 3y + 22 = 0$$

EXAMPLE 2.12

A rectangle $PQRS$ has its side PQ parallel to the line $y = mx$ and vertices P , Q , and S on the lines $y = a$, $x = b$, and $x = -b$, respectively. Find the locus of the vertex R .

Sol. Let the coordinate of Q be (b, α) and that of S be $(-b, \beta)$. Let PR and SQ intersect each other at T .

$\therefore T$ is the mid point of SQ

Since diagonals of a rectangle bisect each other, x co-ordinates of P is $-h$.

As P lies on $y = a$, therefore coordinates of P are $(-h, a)$.

Given that PQ is parallel to $y = mx$ and slope of $PQ = m$.

$$\therefore \frac{\alpha - a}{b + h} = m$$

$$\Rightarrow \alpha = a + m(b + h) \quad (1)$$

Also, $RQ \perp PQ \Rightarrow$ slope of $RQ = \frac{-1}{m}$

$$\therefore \frac{k - \alpha}{h - b} = \frac{-1}{m} \Rightarrow \alpha = k + \frac{1}{m}(h - b) \quad (2)$$

From (1) and (2), we get

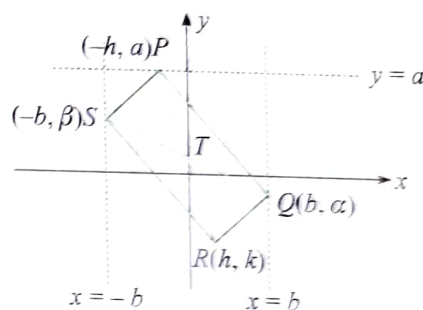
$$a + m(b + h) = k + \frac{1}{m}(h - b)$$

$$\Rightarrow am + m^2(b + h) = km + (h - b)$$

$$\Rightarrow (m^2 - 1)h - mk + b(m^2 + 1) + am = 0$$

Therefore, locus of (h, k) is

$$(m^2 - 1)x - my + b(m^2 + 1) + am = 0$$



Exercises

Single Correct Answer Type

- The equations of the diagonals of square formed by lines $x = 0$, $y = 0$, $x = 1$, and $y = 1$ are
 - $y = x$, $y + x = 1$
 - $y = x$, $x + y = 2$
 - $2y = x$, $y + x = 1/3$
 - $y = 2x$, $y + 2x = 1$
- The coordinates of two consecutive vertices A and B of a regular hexagon $ABCDEF$ are $(1, 0)$ and $(2, 0)$, respectively. The equation of the diagonal CE is
 - $\sqrt{3}x + y = 4$
 - $x + \sqrt{3}y + 4 = 0$
 - $x + \sqrt{3}y = 4$
 - none of these
- If each of the points $(x_1, 4)$, $(-2, y_1)$ lies on the line joining the points $(2, -1)$ and $(5, -3)$, then the point $P(x_1, y_1)$ lies on the line
 - $6(x + y) - 25 = 0$
 - $2x + 6y + 1 = 0$
 - $2x + 3y - 6 = 0$
 - $6(x + y) + 25 = 0$
- The equation to the straight line passing through the point $(a \cos^3 \theta, a \sin^3 \theta)$ and perpendicular to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ is
 - $x \cos \theta - y \sin \theta = a \cos 2\theta$
 - $x \cos \theta + y \sin \theta = a \cos 2\theta$
 - $x \sin \theta + y \cos \theta = a \cos 2\theta$
 - none of these
- The line PQ whose equation is $x - y = 2$ cuts the x -axis at P , and Q is $(4, 2)$. The line PQ is rotated about P through 45° in the anticlockwise direction. The equation of the line PQ in the new position is
 - $y = -\sqrt{2}$
 - $y = 2$
 - $x = 2$
 - $x = -2$
- A line moves in such a way that the sum of the intercepts made by it on the axes is always c (constant). The locus of midpoint of its intercept between the axes is
 - $x + y = 2c$
 - $x + y = c$
 - $2(x + y) = c$
 - $2x + y = c$
- If the x -intercept of the line $y = mx + 2$ is greater than $1/2$, then the gradient of the line lies in the interval
 - $(-1, 0)$
 - $\left(-\frac{1}{4}, 0\right)$
 - $(-\infty, -4)$
 - $(-4, 0)$
- The equation of a straight line on which the length of perpendicular from the origin is four units and the line makes an angle of 120° with the x -axis is
 - $x\sqrt{3} + y + 8 = 0$
 - $x\sqrt{3} - y = 8$
 - $x\sqrt{3} - y = 8$
 - $x - \sqrt{3}y + 8 = 0$
- $ABCD$ is a square. $A \equiv (1, 2)$, $B \equiv (3, -4)$. If line CD passes through $(3, 8)$, then the midpoint of CD is
 - $(2, 6)$
 - $(6, 2)$
 - $(2, 5)$
 - $(28/5, 1/5)$
- The equation of the straight line which passes through the point $(-4, 3)$ such that the portion of the line between the axes is divided internally by the point in the ratio $5 : 3$ is
 - $9x - 20y + 96 = 0$
 - $9x + 20y = 24$
 - $20x + 9y + 53 = 0$
 - none of these
- A square of side a lies above the x -axis and has one vertex at the origin. The side passing through the origin makes an angle α ($0 < \alpha < \pi/4$) with the positive direction of the x -axis. The equation of its diagonal not passing through the origin is
 - $y(\cos \alpha + \sin \alpha) + x(\sin \alpha - \cos \alpha) = a$
 - $y(\cos \alpha + \sin \alpha) + x(\sin \alpha + \cos \alpha) = a$
 - $y(\cos \alpha + \sin \alpha) + x(\cos \alpha - \sin \alpha) = a$
 - $y(\cos \alpha - \sin \alpha) - x(\sin \alpha - \cos \alpha) = a$
- Let $P \equiv (-1, 0)$, $Q \equiv (0, 0)$, and $R \equiv (3, 3\sqrt{3})$ be three points. Then the equation of the bisector of $\angle PQR$ is
 - $(\sqrt{3}/2)x + y = 0$
 - $x + \sqrt{3}y = 0$
 - $\sqrt{3}x + y = 0$
 - $x + (\sqrt{3}/2)y = 0$
- The equation of a line through the point $(1, 2)$ whose distance from the point $(3, 1)$ has the greatest value is
 - $y = 2x$
 - $y = x + 1$
 - $x + 2y = 5$
 - $y = 3x - 1$
- One diagonal of a square is along the line $8x - 15y = 0$ and one of its vertex is $(1, 2)$. Then the equations of the sides of the square passing through this vertex are
 - $7x - 8y + 9 = 0$, $8x + 7y - 22 = 0$
 - $9x - 8y + 7 = 0$, $8x + 9y - 26 = 0$
 - $23x - 7y - 9 = 0$, $7x + 23y - 53 = 0$
 - none of these
- The angle between the diagonals of a quadrilateral formed by the lines $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{b} + \frac{y}{a} = 1$, $\frac{x}{a} + \frac{y}{b} = 2$ and $\frac{x}{b} + \frac{y}{a} = 2$ is
 - $\frac{\pi}{4}$
 - $\frac{\pi}{2}$
 - $\frac{\pi}{3}$
 - $\frac{\pi}{6}$
- A line with positive rational slope, passes through the point $A(6, 0)$ and is at a distance of 5 units from $B(1, 3)$. The slope of line is
 - $\frac{15}{8}$
 - $\frac{8}{15}$
 - $\frac{5}{8}$
 - $\frac{8}{5}$
- The line $x + 3y - 2 = 0$ bisects the angle between a pair of straight lines of which one has equation $x - 7y + 5 = 0$. The equation of the other line is
 - $3x + 3y - 1 = 0$
 - $x - 3y + 2 = 0$
 - $5x + 5y - 3 = 0$
 - none of these

18. Given $A \equiv (1, 1)$ and AB is any line through it cutting the x -axis at B . If AC is perpendicular to AB and meets the y -axis in C , then the equation of the locus of midpoint P of BC is
- $x + y = 1$
 - $x + y = 2$
 - $x + y = 2xy$
 - $2x + 2y = 1$
19. The number of possible straight lines passing through $(2, 3)$ and forming a triangle with the coordinate axes, whose area is 12 sq. units, is
- one
 - two
 - three
 - four
20. Two parallel lines lying in the same quadrant make intercepts a and b on x and y -axes, respectively, between them. The distance between the lines is
- $\sqrt{a^2 + b^2}$
 - $\frac{ab}{\sqrt{a^2 + b^2}}$
 - $\frac{1}{\sqrt{a^2 + b^2}}$
 - $\frac{1}{a^2} + \frac{1}{b^2}$
21. The line $L_1 \equiv 4x + 3y - 12 = 0$ intersects the x - and y -axis at A and B , respectively. A variable line perpendicular to L_1 intersects the x - and the y -axis at P and Q , respectively. Then the locus of the circumcenter of triangle ABQ is
- $3x - 4y + 2 = 0$
 - $4x + 3y + 7 = 0$
 - $6x - 8y + 7 = 0$
 - none of these
22. A beam of light is sent along the line $x - y = 1$, which after refracting from the x -axis enters the opposite side by turning through 30° towards the normal at the point of incidence on the x -axis. Then the equation of the refracted ray is
- $(2 - \sqrt{3})x - y = 2 + \sqrt{3}$
 - $(2 + \sqrt{3})x - y = 2 + \sqrt{3}$
 - $(2 - \sqrt{3})x + y = (2 + \sqrt{3})$
 - $y = (2 - \sqrt{3})(x - 1)$
23. The number of integral values of m for which the x -coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is
- 2
 - 0
 - 4
 - 1
24. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is
- a square
 - a circle
 - a straight line
 - two intersecting lines
25. The equation of set of lines which are at a constant distance of 2 units from the origin is
- $x + y + 2 = 0$
 - $x + y + 4 = 0$
 - $x \cos \alpha + y \sin \alpha = 2$
 - $x \cos \alpha + y \sin \alpha = \frac{1}{2}$
26. The lines $y = m_1x$, $y = m_2x$, and $y = m_3x$ make equal intercepts on the line $x + y = 1$. Then
- $2(1 + m_1)(1 + m_3) = (1 + m_2)(2 + m_1 + m_3)$
 - $(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$
 - $(1 + m_1)(1 + m_2) = (1 + m_3)(2 + m_1 + m_3)$
 - $2(1 + m_1)(1 + m_3) = (1 + m_2)(1 + m_1 + m_3)$
27. The condition on a and b , such that the portion of the line $ax + by - 1 = 0$ intercepted between the lines $ax + y = 0$ and $x + by = 0$ subtends a right angle at the origin, is
- $a = b$
 - $a + b = 0$
 - $a = 2b$
 - $2a = b$
28. The area of the triangle formed by the lines $y = ax$, $x + y - a = 0$, and the y -axis is equal to
- $\frac{1}{2|1+a|}$
 - $\frac{a^2}{|1+a|}$
 - $\frac{1}{2} \left| \frac{a}{1+a} \right|$
 - $\frac{a^2}{2|1+a|}$
29. The line $\frac{x}{a} + \frac{y}{b} = 1$ meets the x -axis at A , the y -axis at B , and the line $y = x$ at C such that the area of $\triangle AOC$ is twice the area of $\triangle BOC$. Then the coordinates of C are
- $\left(\frac{b}{3}, \frac{b}{3}\right)$
 - $\left(\frac{2a}{3}, \frac{2a}{3}\right)$
 - $\left(\frac{2b}{3}, \frac{2b}{3}\right)$
 - none of these
30. The line $x/3 + y/4 = 1$ meets the y - and x -axis at A and B , respectively. A square $ABCD$ is constructed on the line segment AB away from the origin. The coordinates of the vertex of the square farthest from the origin are
- $(7, 3)$
 - $(4, 7)$
 - $(6, 4)$
 - $(3, 8)$
31. The area of a parallelogram formed by the lines $ax = bx = c = 0$ is
- $c^2/(ab)$
 - $2c^2/(ab)$
 - $c^2/2ab$
 - none of these
32. One diagonal of a square is $3x - 4y + 8 = 0$ and one vertex is $(-1, 1)$, then the area of square is
- $\frac{1}{50}$ sq. unit
 - $\frac{1}{25}$ sq. unit
 - $\frac{3}{50}$ sq. unit
 - $\frac{2}{25}$ sq. unit
33. In an isosceles triangle OAB , O is the origin and $OA = OB = 6$. The equation of the side AB is $x - y + 1 = 0$. The area of triangle is
- $2\sqrt{21}$
 - $\sqrt{142}$
 - $\frac{\sqrt{142}}{2}$
 - $\frac{\sqrt{71}}{2}$
34. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q , respectively. Then the point O divides the segment PQ in the ratio
- 1 : 2
 - 3 : 4
 - 2 : 1
 - 4 : 3
35. The coordinates of the foot of the perpendicular from the point $(2, 3)$ on the line $-y + 3x + 4 = 0$ are given by
- $(37/10, -1/10)$
 - $(-1/10, 37/10)$
 - $(10/37, -10)$
 - $(2/3, -1/3)$

36. The straight lines $7x - 2y + 10 = 0$ and $7x + 2y - 10 = 0$ form an isosceles triangle with the line $y = 2$. The area of this triangle is equal to
- (1) $15/7$ sq. units (2) $10/7$ sq. units
(3) $18/7$ sq. units (4) none of these
37. The equations of the sides of a triangle are $x + y - 5 = 0$, $x - y + 1 = 0$, and $y - 1 = 0$. Then the coordinates of the circumcenter are
- (1) (2, 1) (2) (1, 2)
(3) (2, -2) (4) (1, -2)
38. If the intercept made on the line $y = mx$ by lines $y = 2$ and $y = 6$ is less than 5, then the range of values of m is
- (1) $(-\infty, -4/3) \cup (4/3, +\infty)$
(2) $(-4/3, 4/3)$
(3) $(-3/4, 4/3)$
(4) none of these
39. The range of θ in the interval $(0, \pi)$ such that the points (3, 5) and $(\sin \theta, \cos \theta)$ lie on the same side of the line $x + y - 1 = 0$ is
- (1) $0 < \theta < \frac{\pi}{4}$ (2) $0 < \theta < \frac{\pi}{2}$
(3) $0 < \theta < \pi$ (4) $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$
40. The distance of origin from line $(1 + \sqrt{3})y + (1 - \sqrt{3})x = 10$ measured along the line $y = \sqrt{3}x + k$ is
- (1) $\frac{5}{\sqrt{2}}$ (2) $5\sqrt{2} + k$
(3) 10 (4) 5
41. Consider the points $A(0, 1)$ and $B(2, 0)$, and P be a point on the line $4x + 3y + 9 = 0$. The coordinates of P such that $|PA - PB|$ is maximum are
- (1) $(-12/5, 17/5)$ (2) $(-24/5, 17/5)$
(3) $(-6/5, 17/5)$ (4) (0, -3)
42. Consider the points $A(3, 4)$ and $B(7, 13)$. If P is a point on the line $y = x$ such that $PA + PB$ is minimum, then the coordinates of P are
- (1) $(12/7, 12/7)$ (2) $(-24/5, 17/5)$
(3) $(31/7, 31/7)$ (4) (0, 0)
43. The area enclosed by $2|x| + 3|y| \leq 6$ is
- (1) 3 sq. units (2) 4 sq. units
(3) 12 sq. units (4) 24 sq. units
44. ABC is a variable triangle such that A is (1, 2), and B and C lie on the line $y = x + \lambda$ (λ is a variable). Then the locus of the orthocenter of $\triangle ABC$ is
- (1) $x + y = 0$ (2) $x - y = 0$
(3) $x^2 + y^2 = 4$ (4) $x + y = 3$
45. In $\triangle ABC$, the coordinates of the vertex A are (4, -1), and lines $x - y - 1 = 0$ and $2x - y = 3$ are the internal bisectors of angles B and C . Then, the radius of the in circle of triangle ABC is
- (1) $4/\sqrt{5}$ (2) $3/\sqrt{5}$
(3) $6/\sqrt{5}$ (4) $7/\sqrt{5}$
46. P is a point on the line $y + 2x = 1$, and Q and R are two points on the line $3y + 6x = 6$ such that triangle PQR is an equilateral triangle. The length of the side of the triangle is
- (1) $2/\sqrt{15}$ (2) $3/\sqrt{5}$
(3) $4/\sqrt{5}$ (4) none of these
47. If the equation of base of an equilateral triangle is $2x - y = 1$ and the vertex is $(-1, 2)$, then the length of the sides of the triangle is
- (1) $\sqrt{20/3}$ (2) $2/\sqrt{15}$
(3) $\sqrt{8/15}$ (4) $\sqrt{15/2}$
48. The locus of a point that is equidistant from the lines $x + y - 2\sqrt{2} = 0$ and $x + y - \sqrt{2} = 0$ is
- (1) $x + y - 5\sqrt{2} = 0$ (2) $x + y - 3\sqrt{2} = 0$
(3) $2x + 2y - 3\sqrt{2} = 0$ (4) $2x + 2y - 5\sqrt{2} = 0$
49. If the quadrilateral formed by the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$, $ax + by + c' = 0$, $a'x + b'y + c = 0$ has perpendicular diagonals, then
- (1) $b^2 + c^2 = b'^2 + c'^2$ (2) $c^2 + a^2 = c'^2 + a'^2$
(3) $a^2 + b^2 = a'^2 + b'^2$ (4) none of these
50. A line segment of fixed length 2 units moves so that its ends are on the positive x -axis and on the part of the line $x + y = 0$ which lies in the second quadrant. Then the locus of the midpoint of the line has the equation
- (1) $x^2 + 5y^2 + 4xy - 1 = 0$ (2) $x^2 + 5y^2 + 4xy + 1 = 0$
(3) $x^2 + 5y^2 - 4xy - 1 = 0$ (4) $4x^2 + 5y^2 + 4xy + 1 = 0$
51. If the extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$, and the equation of one of the sides is $x = 2a$, then the area of the triangle is
- (1) $5a^2$ sq. units (2) $5a^2/2$ sq. units
(3) $25a^2/2$ sq. units (4) none of these
52. $A \equiv (-4, 0)$, $B \equiv (4, 0)$. M and N are the variable points of the y -axis such that M lies below N and $MN = 4$. Lines AM and BN intersect at P . The locus of P is
- (1) $2xy - 16 - x^2 = 0$ (2) $2xy + 16 - x^2 = 0$
(3) $2xy + 16 + x^2 = 0$ (4) $2xy - 16 + x^2 = 0$
53. The number of triangles that the four lines $y = x + 3$, $y = 2x + 3$, $y = 3x + 2$ and $y + x = 3$ form is
- (1) 4 (2) 2 (3) 3 (4) 1
54. A variable line $\frac{x}{a} + \frac{y}{b} = 1$ moves in such a way that the harmonic mean of a and b is 8. The least area of triangle made by the line with the coordinate axes is ($a, b > 0$)
- (1) 8 sq. unit (2) 16 sq. unit
(3) 32 sq. unit (4) 64 sq. unit
55. Given $A(0, 0)$ and $B(x, y)$ with $x \in (0, 1)$ and $y > 0$. Let the slope of line AB be m_1 . Point C lies on line $x = 1$ such that the slope of BC is equal to m_2 , where $0 < m_2 < m_1$. If the area of triangle ABC can be expressed as $(m_1 - m_2)f(x)$, then the largest possible value of $f(x)$ is
- (1) 1 (2) $1/2$ (3) $1/4$ (4) $1/8$

56. A triangle is formed by the lines $x + y = 0$, $x - y = 0$, and $lx + my = 1$. If l and m vary subject to the condition $l^2 + m^2 = 1$, then the locus of its circumcenter is
- $(x^2 - y^2)^2 = x^2 + y^2$
 - $(x^2 + y^2)^2 = (x^2 - y^2)$
 - $(x^2 + y^2) = 4x^2y^2$
 - $(x^2 - y^2)^2 = (x^2 + y^2)^2$
57. Let P be $(5, 3)$ and a point R on $y = x$ and Q on the x -axis be such that $PQ + QR + RP$ is minimum. Then the coordinates of Q are
- $(17/4, 0)$
 - $(17, 0)$
 - $(17/2, 0)$
 - none of these
58. A pair of perpendicular straight lines drawn through the origin form an isosceles triangle with line $2x + 3y = 6$. The area of triangle so formed is
- $\frac{36}{13}$ sq. unit
 - $\frac{12}{17}$ sq. unit
 - $\frac{13}{5}$ sq. unit
 - $\frac{17}{13}$ sq. unit
59. A point $P(x, y)$ moves such that the sum of its distances from the lines $2x - y - 3 = 0$ and $x + 3y + 4 = 0$ is 7. The area bounded by locus of P is (in sq. unit)
- 70
 - $70\sqrt{2}$
 - $35\sqrt{2}$
 - 140
60. If AD , BE and CF are the altitudes of $\triangle ABC$ whose vertex A is $(-4, 5)$. The coordinates of points E and F are $(4, 1)$ and $(-1, -4)$, respectively. Equation of BC is
- $3x - 4y + 28 = 0$
 - $4x + 3y + 28 = 0$
 - $3x - 4y - 28 = 0$
 - $x + 2y + 7 = 0$
61. The vertex A of $\triangle ABC$ is $(3, -1)$. The equations of median BE and angle bisector CF are $x - 4y + 10 = 0$ and $6x + 10y - 59 = 0$, respectively. Equation of AC is
- $5x + 18y = 37$
 - $15x + 8y = 37$
 - $15x - 8y = 37$
 - $15x + 8y + 37 = 0$
62. Suppose A and B are two points on $2x - y + 3 = 0$ and $P(1, 2)$ is such that $PA = PB$. Then the midpoint of AB is
- $\left(\frac{-1}{5}, \frac{13}{5}\right)$
 - $\left(\frac{-7}{5}, \frac{9}{5}\right)$
 - $\left(\frac{7}{5}, \frac{-9}{5}\right)$
 - $\left(\frac{-7}{5}, \frac{-9}{5}\right)$
63. Triangle formed by variable lines $(a+b)x + (a-b)y - 2ab = 0$ and $(a-b)x + (a+b)y - 2ab = 0$ and $x + y = 0$ is (where $a, b \in \mathbb{R}$)
- equilateral
 - right angled
 - scalene
 - none of these
64. A light ray coming along the line $3x + 4y = 5$ gets reflected from the line $ax + by = 1$ and goes along the line $5x - 12y = 10$. Then
- $a = \frac{64}{115}, b = \frac{112}{15}$
 - $a = \frac{14}{15}, b = -\frac{8}{115}$
 - $a = \frac{64}{115}, b = -\frac{8}{115}$
 - $a = \frac{64}{15}, b = \frac{14}{15}$
65. The point $A(2, 1)$ is translated parallel to the line $x - y = 3$ by a distance of 4 units. If the new position A' is in the third quadrant, then the coordinates of A' are
- $(2 + 2\sqrt{2}, 1 + 2\sqrt{2})$
 - $(-2 + \sqrt{2}, -1 - 2\sqrt{2})$
 - $(2 - 2\sqrt{2}, 1 - 2\sqrt{2})$
 - none of these
66. One of the diagonals of a square is the portion of the line $x/2 + y/3 = 2$ intercepted between the axes. Then the extremities of the other diagonal are
- $(5, 5), (-1, 1)$
 - $(0, 0), (4, 6)$
 - $(0, 0), (-1, 1)$
 - $(5, 5), (4, 6)$
67. The point $P(2, 1)$ is shifted through a distance $3\sqrt{2}$ units measured parallel to the line $x + y = 1$ in the direction of decreasing ordinates, to reach at Q . The image of Q with respect to given line is
- $(3, -4)$
 - $(-3, 2)$
 - $(0, -1)$
 - none of these
68. Let O be the origin. If $A(1, 0)$ and $B(0, 1)$ and $P(x, y)$ are points such that $xy > 0$ and $x + y < 1$, then
- P lies either inside the triangle OAB or in the third quadrant
 - P cannot lie inside the triangle OAB
 - P lies inside the triangle OAB
 - P lies in the first quadrant only
69. In a triangle ABC , the bisectors of angles B and C lie along the lines $x = y$ and $y = 0$. If A is $(1, 2)$, then the equation of line BC is
- $2x + y = 1$
 - $3x - y = 5$
 - $x - 2y = 3$
 - $x + 3y = 1$
70. Line $ax + by + p = 0$ makes angle $\pi/4$ with $x \cos \alpha + y \sin \alpha = p$, $p \in \mathbb{R}^+$. If these lines and the line $x \sin \alpha - y \cos \alpha = 0$ are concurrent, then
- $a^2 + b^2 = 1$
 - $a^2 + b^2 = 2$
 - $2(a^2 + b^2) = 1$
 - none of these
71. The equation of the line segment AB is $y = x$. If A and B lie on the same side of the line mirror $2x - y = 1$, then the image of AB has the equation
- $x + y = 2$
 - $8x + y = 9$
 - $7x - y = 6$
 - none of these
72. The equation of the bisector of the acute angle between the lines $2x - y + 4 = 0$ and $x - 2y = 1$ is
- $x + y + 5 = 0$
 - $x - y + 1 = 0$
 - $x - y = 5$
 - none of these
73. The straight lines $4ax + 3by + c = 0$, where $a + b + c = 0$, are concurrent at the point
- $(4, 3)$
 - $(1/4, 1/3)$
 - $(1/2, 1/3)$
 - none of these
74. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$, and $x + y + c = 0$ (a, b, c being distinct and different from 1) are concurrent, then
- $$\left(\frac{1}{1-a}\right) + \left(\frac{1}{1-b}\right) + \left(\frac{1}{1-c}\right) =$$
- 0
 - 1
 - $1/(a + b + c)$
 - none of these

Multiple Correct Answers Type

75. If lines $x + 2y - 1 = 0$, $ax + y + 3 = 0$, and $bx - y + 2 = 0$ are concurrent, and S is the curve denoting the locus of (a, b) , then the least distance of S from the origin is

- (1) $5/\sqrt{57}$ (2) $5/\sqrt{51}$
(3) $5/\sqrt{58}$ (4) $5/\sqrt{59}$

76. The straight lines $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$, and $ax + by - 1 = 0$ are concurrent, if the straight line $35x - 22y + 1 = 0$ passes through the point

- (1) (a, b) (2) (b, a)
(3) $(-a, -b)$ (4) none of these

77. If the straight lines $2x + 3y - 1 = 0$, $x + 2y - 1 = 0$, and $ax + by - 1 = 0$ form a triangle with the origin as orthocenter, then (a, b) is given by

- (1) $(6, 4)$ (2) $(-3, 3)$
(3) $(-8, 8)$ (4) $(0, 7)$

78. If $\frac{a}{\sqrt{bc}} - 2 = \sqrt{b/c} + \sqrt{c/b}$, where $a, b, c > 0$, then the family of lines $\sqrt{a}x + \sqrt{b}y + \sqrt{c} = 0$ passes through the fixed point given by

- (1) $(1, 1)$ (2) $(1, -2)$
(3) $(-1, 2)$ (4) $(-1, 1)$

79. If it is possible to draw a line which belongs to all the given family of lines $y - 2x + 1 + \lambda_1(2y - x - 1) = 0$, $3y - x - 6 + \lambda_2(y - 3x + 6) = 0$, $ax + y - 2 + \lambda_3(6x + ay - a) = 0$, then

- (1) $a = 4$ (2) $a = 3$
(3) $a = -2$ (4) $a = 2$

80. If two members of family $(2 + \lambda)x + (1 + 2\lambda)y - 3(1 + \lambda) = 0$ and line $x + y = 0$ make an equilateral triangle, then the incentre of triangle so formed is

- (1) $\left(\frac{1}{3}, \frac{1}{3}\right)$ (2) $\left(\frac{7}{6}, -\frac{5}{6}\right)$
(3) $\left(\frac{5}{6}, \frac{5}{6}\right)$ (4) $\left(-\frac{3}{2}, -\frac{3}{2}\right)$

81. The set of lines $x \tan^{-1} a + y \sin^{-1} \frac{1}{\sqrt{1+a^2}} + 2 = 0$, where $a \in (0, 1)$, is concurrent at

- (1) $\left(\frac{1}{\pi}, \frac{1}{\pi}\right)$ (2) $\left(-\frac{4}{\pi}, -\frac{4}{\pi}\right)$
(3) (π, π) (4) none of these

82. If $\sin(\alpha + \beta) \sin(\alpha - \beta) = \sin \gamma (2 \sin \beta + \sin \gamma)$, where $0 < \alpha, \beta, \gamma < \pi$, then the straight line whose equation is $x \sin \alpha + y \sin \beta - \sin \gamma = 0$ passes through the point

- (1) $(1, 1)$ (2) $(-1, 1)$
(3) $(1, -1)$ (4) none of these

1. If P is a point (x, y) on the line $y = -3x$ such that P and the point $(3, 4)$ are on the opposite sides of the line $3x - 4y = 8$, then

- (1) $x > 8/15$ (2) $x > 8/5$
(3) $y < -8/5$ (4) $y < -8/15$

2. If (x, y) is a variable point on the line $y = 2x$ lying between the lines $2(x + 1) + y = 0$ and $x + 3(y - 1) = 0$, then

- (1) $x \in (-1/2, 6/7)$ (2) $x \in (-1/2, 3/7)$
(3) $y \in (-1, 3/7)$ (4) $y \in (-1, 6/7)$

3. Let $P(\sin \theta, \cos \theta)$, $(0 \leq \theta \leq 2\pi)$, be a point in a triangle with vertices $(0, 0)$, $(\sqrt{3}/2, 0)$, and $(0, \sqrt{3}/2)$. Then,

- (1) $0 < \theta < \pi/12$ (2) $5\pi/2 < \theta < \pi/2$
(3) $0 < \theta < 5\pi/2$ (4) $5\pi/2 < \theta < \pi$

4. The lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$, and $2x - y - 4 = 0$ are the sides of a square. The equation of the remaining side of the square can be

- (1) $2x - y + 6 = 0$ (2) $2x - y + 8 = 0$
(3) $2x - y - 10 = 0$ (4) $2x - y - 14 = 0$

5. Angle made with the x -axis by a straight line drawn through $(1, 2)$ so that it intersects $x + y = 4$ at a distance $\sqrt{6}/3$ from $(1, 2)$ is

- (1) 105° (2) 75° (3) 60° (4) 15°

6. Given three straight lines $2x + 11y - 5 = 0$, $24x + 7y - 20 = 0$, and $4x - 3y - 2 = 0$. Then,

- (1) they form a triangle
(2) they are concurrent
(3) one line bisects the angle between the other two
(4) two of them are parallel

7. A triangle is formed by the lines whose equations are $AB: x + y - 5 = 0$, $BC: x + 7y - 7 = 0$ and $CA: 7x + y + 14 = 0$. Then

- (1) angle at A is acute
(2) angle at C is acute
(3) internal angle bisector at angle B is $3x + 6y - 16 = 0$
(4) external angle bisector at angle C is $8x + 8y + 7 = 0$

8. If the points $(a^3/(a-1), (a^2-3)/(a-1))$, $(b^3/(b-1), (b^2-3)/(b-1))$, and $(c^3/(c-1), (c^2-3)/(c-1))$, where a, b, c are different from 1, lie on the line $lx + my + n = 0$, then

- (1) $a + b + c = -\frac{m}{l}$
(2) $ab + bc + ca = \frac{n}{l}$

- (3) $abc = \frac{(m+n)}{l}$

- (4) $abc - (bc + ca + ab) + 3(a + b + c) = 0$

9. Two sides of a rhombus $OABC$ (lying entirely in the first or third quadrant) of area equal to 2 sq. units are $y = x/\sqrt{3}$, $y = \sqrt{3}x$. Then the possible coordinates of B is/are (O being the origin)
- (1) $(1 + \sqrt{3}, 1 + \sqrt{3})$ (2) $(-1 - \sqrt{3}, -1 - \sqrt{3})$
 (3) $(3 + \sqrt{3}, 3 + \sqrt{3})$ (4) $(\sqrt{3} - 1, \sqrt{3} - 1)$
10. If $(x/a) + (y/b) = 1$ and $(x/c) + (y/d) = 1$ intersect the axes at four concyclic points and $a^2 + c^2 = b^2 + d^2$, then these lines can intersect at, ($a, b, c, d > 0$),
- (1) $(1, 1)$ (2) $(1, -1)$
 (3) $(2, -2)$ (4) $(3, 3)$
11. The straight line $3x + 4y - 12 = 0$ meets the coordinate axes at A and B . An equilateral triangle ABC is constructed. The possible coordinates of vertex C are
- (1) $\left(2\left(1 - \frac{3\sqrt{3}}{4}\right), \frac{3}{2}\left(1 - \frac{4}{\sqrt{3}}\right)\right)$
 (2) $\left(-2(1 + \sqrt{3}), \frac{3}{2}(1 - \sqrt{3})\right)$
 (3) $\left(2(1 + \sqrt{3}), \frac{3}{2}(1 + \sqrt{3})\right)$
 (4) $\left(2\left(1 + \frac{3\sqrt{3}}{4}\right), \frac{3}{2}\left(1 + \frac{4}{\sqrt{3}}\right)\right)$
12. The equation of the lines passing through the point $(1, 0)$ and at a distance $\sqrt{3}/2$ from the origin is
- (1) $\sqrt{3}x + y - \sqrt{3} = 0$ (2) $x + \sqrt{3}y - \sqrt{3} = 0$
 (3) $\sqrt{3}x - y - \sqrt{3} = 0$ (4) $x - \sqrt{3}y - \sqrt{3} = 0$
13. The sides of a triangle are the straight lines $x + y = 1$, $7y = x$, and $\sqrt{3}y + x = 0$. Then which of the following is an interior point of the triangle?
- (1) Circumcenter (2) Centroid
 (3) Incenter (4) Orthocenter
14. If the straight line $ax + cy = 2b$, where $a, b, c > 0$, makes a triangle of area 2 sq. units with the coordinate axes, then
- (1) a, b, c are in GP (2) $a, -b, c$ are in GP
 (3) $a, 2b, c$ are in GP (4) $a, -2b, c$ are in GP
15. Consider the equation $y - y_1 = m(x - x_1)$. If m and x_1 are fixed and different lines are drawn for different values of y_1 , then
- (1) the lines will pass through a fixed point
 (2) there will be a set of parallel lines
 (3) all the lines intersect the line $x = x_1$
 (4) all the lines will be parallel to the line $y = x_1$
16. Equation(s) of the straight line(s), inclined at 30° to the x -axis such that the length of its (each of their) line segment(s) between the coordinate axes is 10 units, is (are)
- (1) $x + \sqrt{3}y + 5\sqrt{3} = 0$ (2) $x - \sqrt{3}y + 5\sqrt{3} = 0$
 (3) $x + \sqrt{3}y - 5\sqrt{3} = 0$ (4) $x - \sqrt{3}y - 5\sqrt{3} = 0$
17. The lines $x + y - 1 = 0$, $(m - 1)x + (m^2 - 7)y - 5 = 0$, and $(m - 2)x + (2m - 5)y = 0$ are
- (1) concurrent for three values of m
 (2) concurrent for one value of m
 (3) concurrent for no value of m
 (4) parallel for $m = 3$
18. The equation of a straight line passing through the point $(2, 3)$ and inclined at an angle of $\tan^{-1}(1/2)$ with the line $y + 2x = 5$ is
- (1) $y = 3$ (2) $x = 2$
 (3) $3x + 4y - 18 = 0$ (4) $4x + 3y - 17 = 0$
19. The equation of the lines on which the perpendiculars from the origin make 30° angle with the x -axis and which form a triangle of area $50/\sqrt{3}$ with the axes are
- (1) $\sqrt{3}x + y - 10 = 0$ (2) $\sqrt{3}x + y + 10 = 0$
 (3) $x + \sqrt{3}y - 10 = 0$ (4) $x - \sqrt{3}y - 10 = 0$
20. A line is drawn perpendicular to line $y = 5x$, meeting the coordinate axes at A and B . If the area of triangle OAB is 10 sq. unit where O is the origin, then the sum of intercepts of the line is
- (1) 12 (2) -12 (3) 10 (4) -10
21. If $x - 2y + 4 = 0$ and $2x + y - 5 = 0$ are the sides of an isosceles triangle having area 10 sq. units, then possible equation of the third side is
- (1) $x + 3y = -1$ (2) $x + 3y = 19$
 (3) $3x - y = -9$ (4) $3x - y = 11$
22. Find the value(s) of a for which the given lines are concurrent.
- $2x + y - 1 = 0;$
 $ax + 3y - 3 = 0;$
 $3x + 2y - 2 = 0.$
- (1) -3 (2) -1 (3) 1 (4) 4
23. Three lines $px + qy + r = 0$, $qx + ry + p = 0$, and $rx + py + q = 0$ are concurrent if
- (1) $p + q + r = 0$ (2) $p^2 + q^2 + r^2 = pr + rp + pq$
 (3) $p^3 + q^3 + r^3 = 3pqr$ (4) none of these
24. θ_1 and θ_2 are the inclination of lines L_1 and L_2 , respectively, with x -axis. If L_1 and L_2 pass through $P(x_1, y_1)$, then equations of angle bisectors of lines are
- (1) $\frac{x - x_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}$
 (2) $\frac{x - x_1}{-\sin\left(\frac{\theta_1 - \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$
 (3) $\frac{x - x_1}{\sin\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$
 (4) $\frac{x - x_1}{-\sin\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$
25. Consider the lines $L_1 \equiv 3x - 4y + 2 = 0$ and $L_2 \equiv 3y - 4x - 5 = 0$. Now, choose the correct statement(s).
- (1) The line $x + y + 3 = 0$ bisects the acute angle between L_1 and L_2 containing the origin.
 (2) The line $x - y + 1 = 0$ bisects the obtuse angle between L_1 and L_2 not containing the origin.

- (3) The line $x + y + 3 = 0$ bisects the obtuse angle between L_1 and L_2 containing the origin.
- (4) The line $x - y + 1 = 0$ bisects the acute angle between L_1 and L_2 not containing the origin.

26. The sides of a rhombus are parallel to the lines $x + y - 1 = 0$ and $7x - y - 5 = 0$. It is given that the diagonals of the rhombus intersect at $(1, 3)$ and one vertex, A of the rhombus lies on the line $y = 2x$. Then the coordinates of vertex A are

- (1) $(8/5, 16/5)$ (2) $(7/15, 14/15)$
(3) $(6/5, 12/5)$ (4) $(4/15, 8/15)$

27. Two straight lines $u = 0$ and $v = 0$ pass through the origin and the angle between them is $\tan^{-1}(7/9)$. If the ratio of the slope of $v = 0$ and $u = 0$ is $9/2$, then their equations are

- (1) $y + 3x = 0$ and $3y + 2x = 0$
(2) $2y + 3x = 0$ and $3y + x = 0$
(3) $2y = 3x$ and $3y = x$
(4) $y = 3x$ and $3y = 2x$

28. Let $u \equiv ax + by + a\sqrt{b} = 0$, $v \equiv bx - ay + b\sqrt{a} = 0$, $a, b \in R$, be two straight lines. The equations of the bisectors of the angle formed by $k_1u - k_2v = 0$ and $k_1u + k_2v = 0$, for nonzero and real k_1 and k_2 , are

- (1) $u = 0$ (2) $k_2u + k_1v = 0$
(3) $k_2u - k_1v = 0$ (4) $v = 0$

29. Two sides of a triangle are parallel to the coordinate axes. If the slopes of the medians through the acute angles of the triangle are 2 and m , then $m =$

- (1) $1/2$ (2) 2 (3) 4 (4) 8

30. A line which makes an acute angle θ with the positive direction of the x -axis is drawn through the point $P(3, 4)$ to meet the line $x = 6$ at R and $y = 8$ at S . Then,

- (1) $PR = 3 \sec \theta$
(2) $PS = 4 \operatorname{cosec} \theta$
(3) $PR + PS = \frac{2(3\sin\theta + 4\cos\theta)}{\sin 2\theta}$
(4) $\frac{9}{(PR)^2} + \frac{16}{(PS)^2} = 1$

Linked Comprehension Type

For Problems 1–3

Let L be the line belonging to the family of straight lines $(a + 2b)x + (a - 3b)y + a - 8b = 0$, $a, b \in R$, which is the farthest from the point $(2, 2)$.

1. The equation of line L is

- (1) $x + 4y + 7 = 0$ (2) $2x + 3y + 4 = 0$
(3) $4x - y - 6 = 0$ (4) none of these

2. Area enclosed by the line L and the coordinate axes is

- (1) $4/3$ sq. units (2) $9/2$ sq. units
(3) $49/8$ sq. units (4) none of these

3. If L is concurrent with the lines $x - 2y + 1 = 0$ and $3x - 4y + \lambda = 0$, then the value of λ is

- (1) 2 (2) 1
(3) -4 (4) 5

For Problems 4–6

The equation of an altitude of an equilateral triangle is $\sqrt{3}x + y = 2\sqrt{3}$, and one of the vertices is $(3, \sqrt{3})$.

4. The possible number of triangles is

- (1) 1 (2) 2 (3) 3 (4) 4

5. Which of the following is not one of the possible vertices of the triangle?

- (1) $(0, 0)$ (2) $(0, 2\sqrt{3})$
(3) $(3, -\sqrt{3})$ (4) None of these

6. Which of the following is one of the orthocenters of the triangle?

- (1) $(1, \sqrt{3})$ (2) $(0, \sqrt{3})$
(3) $(0, 2)$ (4) none of these

For Problems 7–9

A variable line L is drawn through $O(0, 0)$ to meet the lines L_1 and L_2 given by $y - x - 10 = 0$ and $y - x - 20 = 0$ at points A and B , respectively.

7. A point P is taken on L such that $2/OP = 1/OA + 1/OB$. Then the locus of P is

- (1) $3x + 3y = 40$ (2) $3x + 3y + 40 = 0$
(3) $3x - 3y = 40$ (4) $3y - 3x = 40$

8. Locus of P , if $OP^2 = OA \times OB$, is

- (1) $(y - x)^2 = 100$ (2) $(y + x)^2 = 50$
(3) $(y - x)^2 = 200$ (4) none of these

9. Locus of P , if $(1/OP^2) = (1/OB^2) + (1/OA^2)$, is

- (1) $(y - x)^2 = 80$ (2) $(y - x)^2 = 100$
(3) $(y - x)^2 = 64$ (4) none of these

For Problems 10–12

The line $6x + 8y = 48$ intersects the coordinate axes at A and B , respectively. A line L bisects the area and the perimeter of triangle OAB , where O is the origin.

10. The number of such lines possible is

- (1) 1 (2) 2
(3) 3 (4) more than 3

11. The slope of line L can be

- (1) $(10 + 5\sqrt{6})/10$ (2) $(10 - 5\sqrt{6})/10$
(3) $(8 + 3\sqrt{6})/10$ (4) none of these

12. Line L

- (1) does not intersect AB
(2) does not intersect OB
(3) does not intersect OA
(4) can intersect all the sides

For Problems 13–15

$A(1, 3)$ and $C(-2/5, -2/5)$ are the vertices of a triangle ABC and the equation of the internal angle bisector of $\angle ABC$ is $x + y = 2$

13. The equation of side BC is

- (1) $7x + 3y - 4 = 0$ (2) $7x + 3y + 4 = 0$
 (3) $7x - 3y + 4 = 0$ (4) $7x - 3y - 4 = 0$

14. The coordinates of vertex B are

- (1) $(3/10, 17/10)$ (2) $(17/10, 3/10)$
 (3) $(-5/2, 9/2)$ (4) $(1, 1)$

15. The equation of side AB is

- (1) $3x + 7y = 24$ (2) $3x + 7y + 24 = 0$
 (3) $13x + 7y + 8 = 0$ (4) $13x - 7y + 8 = 0$

For Problems 16–18

Let $ABCD$ be a parallelogram whose equations for the diagonals AC and BD are $x + 2y = 3$ and $2x + y = 3$, respectively.

16. If length of diagonal AC is 4 units and the area of parallelogram $ABCD$ is 8 sq. units, then the length of other diagonal BD is

- (1) $10/3$ (2) 2
 (3) $20/3$ (4) none of these

17. The length of side AB is equal to

- (1) $2\sqrt{58}/3$ (2) $4\sqrt{58}/9$
 (3) $3\sqrt{58}/9$ (4) $4\sqrt{58}/9$

18. The length of BC is equal to

- (1) $2\sqrt{10}/3$ (2) $4\sqrt{10}/3$
 (3) $8\sqrt{10}/3$ (4) none of these

For Problems 19–21

Consider a triangle PQR with coordinates of its vertices as $P(-8, 5)$, $Q(-15, -19)$, and $R(1, -7)$. The bisector of the interior angle of P has the equation which can be written in the form $ax + 2y + c = 0$.

19. The distance between the orthocenter and the circumcenter of triangle PQR is

- (1) $25/2$ (2) $29/2$
 (3) $37/2$ (4) $51/2$

20. The radius of the in circle of triangle PQR is

- (1) 4 (2) 5
 (3) 6 (4) 8

21. The sum $a + c$ is

- (1) 129 (2) 78
 (3) 89 (4) none of these

For Problems 22–24

The base of an isosceles triangle measures 4 units and its base angle is equal to 45° . A straight line cuts the extension of the base at a point M at the angle θ and bisects the lateral side of the triangle which is nearest to M .

22. The area of quadrilateral which the straight line cuts off from the given triangle is

- (1) $\frac{3 + \tan \theta}{1 + \tan \theta}$ (2) $\frac{3 + 5 \tan \theta}{1 + \tan \theta}$
 (3) $\frac{3 + \tan \theta}{1 - \tan \theta}$ (4) $\frac{3 + 2 \tan \theta}{1 + \tan \theta}$

23. The possible range of values in which area of quadrilateral which straight line cuts off from the given triangle lie in

- (1) $\left(\frac{5}{2}, \frac{7}{2}\right)$ (2) $(4, 3)$
 (3) $(4, 5)$ (4) $(3, 4)$

24. The length of portion of straight line inside the triangle may lie in the range

- (1) $(2, 4)$ (2) $\left(\frac{3}{2}, \sqrt{3}\right)$
 (3) $(\sqrt{2}, 2)$ (4) $(\sqrt{2}, \sqrt{3})$

For Problems 25–27

Consider point $A(6, 30)$, point $B(24, 6)$ and line $AB: 4x + 3y = 114$. Point $P(0, \lambda)$ is a point on y -axis such that $0 < \lambda < 38$ and point $Q(0, \lambda)$ is a point on y -axis such that $\lambda > 38$.

25. For all positions of point P , angle APB is maximum when point P is

- (1) $(0, 12)$ (2) $(0, 15)$
 (3) $(0, 18)$ (4) $(0, 21)$

26. The maximum value of angle APB is

- (1) $\frac{\pi}{3}$ (2) $\frac{\pi}{2}$ (3) $\frac{2\pi}{3}$ (4) $\frac{3\pi}{4}$

27. For all positions of point Q , angle AQB is maximum when point Q is

- (1) $(0, 54)$ (2) $(0, 58)$ (3) $(0, 60)$ (4) $(0, 1)$

Matrix Match Type

1. Consider the lines represented by equation $(x^2 + xy - x) \times (x - y) = 0$ forming a triangle. Then match the following lists:

List I	List II
a. Orthocenter of triangle	p. $(1/6, 1/2)$
b. Circumcenter	q. $(1/(2 + 2\sqrt{2}), 1/2)$
c. Centroid	r. $(0, 1/2)$
d. Incenter	s. $(1/2, 1/2)$

2. Consider the triangle formed by the lines $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$, $4y + x - 14 = 0$. Match the following lists:

List I	List II
a. Values of α if $(0, \alpha)$ lies inside the triangle	p. $(-\infty, 7/3) \cup (13/4, \infty)$
b. Values of α if $(\alpha, 0)$ lies inside the triangle	q. $-4/3 < \alpha < 1/2$
c. Values of α if $(\alpha, 2)$ lies inside the triangle	r. No value of α
d. Value of α if $(1, \alpha)$ lies outside the triangle	s. $5/3 < \alpha < 7/2$

3. Match the following lists:

List I	List II
a. A straight line with negative slope passing through (1, 4) meets the coordinate axes at A and B. The minimum length of $OA + OB$, O being the origin, is	p. $5\sqrt{2}$
b. If the point P is symmetric to the point Q(4, -1) with respect to the bisector of the first quadrant, then the length of PQ is	q. $3\sqrt{2}$
c. On the portion of the straight line $x + y = 2$ between the axis a square is constructed away from the origin, with this portion as one of its sides. If d denotes the perpendicular distance of a side of this square from the origin then the maximum value of d is	r. $9/2$
d. If the parametric equation of a line is given by $x = 4 + \lambda/\sqrt{2}$ and $y = -1 + \sqrt{2}\lambda$, where λ is a parameter, then the intercept made by the line on the x-axis is	s. 9

4. Match the following lists:

List I	List II
a. If lines $3x + y - 4 = 0$, $x - 2y - 6 = 0$, and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then the value of λ is	p. -4
b. If the points $(\lambda + 1, 1)$, $(2\lambda + 1, 3)$, and $(2\lambda + 2, 2\lambda)$ are collinear, then the value of λ is	q. $-1/2$
c. If the line $x + y - 1 - \lambda/2 = 0$, passing through the intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$, is perpendicular to one of them, then the value of λ is	r. 4
d. If the line $y - x - 1 + \lambda = 0$ is equidistant from the points (1, -2) and (3, 4), then λ is	s. 2

5. Match the following lists:

List I	List II
a. Four lines $x + 3y - 10 = 0$, $x + 3y - 20 = 0$, $3x - y + 5 = 0$, and $3x - y - 5 = 0$ form a figure which is	p. a quadrilateral which is neither a parallelogram nor a trapezium
b. The points A(1, 2), B(2, -3), C(-1, -5), and D(-2, 4) in order are the vertices of	q. a parallelogram
c. The lines $7x + 3y - 33 = 0$, $3x - 7y + 19 = 0$, $3x - 7y - 10 = 0$, and $7x + 3y - 4 = 0$ form a figure which is	r. a rectangle of area 10 sq. units
d. Four lines $4y - 3x - 7 = 0$, $3y - 4x + 7 = 0$, $4y - 3x - 21 = 0$, $3y - 4x + 14 = 0$ form a figure which is	s. a square

6. Match the following lists:

List I	List II
a. The lines $y = 0$; $y = 1$; $x - 6y + 4 = 0$, and $x + 6y - 9 = 0$ constitute a figure which is	p. a cyclic quadrilateral
b. The points A(a, 0), B(0, b), C(c, 0), and D(0, d) are such that $ac = bd$ and a, b, c, d are all positive. The points A, B, C, and D always constitute	q. a rhombus
c. The figure formed by the four lines $ax \pm by \pm c = 0$, $a \neq b$, is	r. a square
d. The line pairs $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$ constitute a figure which is	s. a trapezium

7. Consider the lines given by

$L_1: x + 3y - 5 = 0$

$L_2: 3x - ky - 1 = 0$

$L_3: 5x + 2y - 12 = 0$

Match the following lists.

List I	List II
a. L_1, L_2, L_3 are concurrent if	p. $k = -9$
b. One of L_1, L_2, L_3 is parallel to at least one of the other two if	q. $k = -6/5$
c. L_1, L_2, L_3 form a triangle if	r. $k = 5/6$
d. L_1, L_2, L_3 do not form a triangle if	s. $k = 5$

8. Consider a ΔABC in which sides AB and AC are perpendicular to $x - y - 4 = 0$ and $2x - y - 5 = 0$, respectively. Vertex A is (-2, 3) and the circumcenter of ΔABC is $(3/2, 5/2)$. The equation of line in List I is of the form $ax + by + c = 0$, where $a, b, c \in I$. Match it with the corresponding value of c in List II and then choose the correct code.

List I	List II
a. Equation of the perpendicular bisector of side AB	p. -1
b. Equation of the perpendicular bisector of side AC.	q. 1
c. Equation of side AC	r. -16
d. Equation of the median through A	s. -4

Codes:

	a	b	c	d
(1)	r	s	p	q
(2)	s	r	q	p
(3)	q	p	s	r
(4)	r	p	s	q

Numerical Value Type

- A straight line L with negative slope passes through the point (8, 2) and cuts the positive coordinate axes at points P and Q. The absolute minimum value of $OP + OQ$ as L varies, where O is the origin, is _____.
- The number of values of k for which the lines $(k + 1)x + 8y = 4k$ and $kx + (k + 3)y = 3k - 1$ are coincident is _____.
- The sides of a triangle ABC lie on the lines $3x + 4y = 0$, $4x + 3y = 0$, and $x = 3$. Let (h, k) be the center of the circle inscribed in ΔABC . The value of (h + k) equals _____.
- The absolute value of the sum of the abscissas of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$ is _____.
- Two sides of a rectangle are $3x + 4y + 5 = 0$, $4x - 3y + 15 = 0$ and one of its vertices is (0, 0). The area of rectangle is _____.
- The line $x = C$ cuts the triangle with vertices (0, 0), (1, 1), and (9, 1) into two regions. For the areas of the two regions to be the same, C must be equal to _____.
- For all real values of a and b, lines $(2a + b)x + (a + 3b)y + (b - 3a) = 0$ and $mx + 2y + 6 = 0$ are concurrent. Then m is equal to _____.

8. The line $3x + 2y = 24$ meets the y -axis at A and the x -axis at B . The perpendicular bisector of AB meets the line through $(0, -1)$ parallel to the x -axis at C . The area of triangle ABC is _____.
9. Consider a $\triangle ABC$ whose sides AB , BC , and CA are represented by the straight lines $2x + y = 0$, $x + py = q$, and $x - y = 3$, respectively. The point $P(2, 3)$ is the orthocenter. The value of $(p + q)$ is _____.
10. Triangle ABC with $AB = 13$, $BC = 5$, and $AC = 12$ slides on the coordinate axes with A and B on the positive x -axis and positive y -axis respectively. The locus of vertex C is a line $12x - ky = 0$. Then the value of k is _____.

11. The line $y = 3x/4$ meets the lines $x - y + 1 = 0$ and $2x - y - 5 = 0$ at points A and B , respectively. If P on the line $y = 3x/4$ satisfies the condition $PA \cdot PB = 25$, then the number of possible coordinates of P is _____.
12. In a plane there are two families of lines: $y = x + r$, $y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. The number of the squares of the diagonal of length 2 formed by these lines is _____.
13. If $5a + 5b + 20c = t$, then the value of t for which the line $ax + by + c - 1 = 0$ always passes through a fixed point is _____.

Archives

JEE MAIN

Single Correct Answer Type

1. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point $(13, 32)$. The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is

- (1) $\frac{23}{\sqrt{17}}$ (2) $\frac{23}{\sqrt{15}}$
(3) $\sqrt{17}$ (4) $\frac{17}{\sqrt{15}}$ (AIEEE 2010)

2. The lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q , respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R .

Statement 1: The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.

Statement 2: In any triangle, bisector of an angle divides the triangle into two similar triangles.

- (1) Statement 1 is true, statement 2 is false.
(2) Statement 1 is true, statement 2 is true; statement 2 is the correct explanation of statement 1.
(3) Statement 1 is true, statement 2 is true; statement 2 is not the correct explanation of statement 1.
(4) Statement 1 is false, statement 2 is true.

(AIEEE 2011)

3. A line is drawn through the point $(1, 2)$ to meet the coordinate axes at P and Q such that it forms a triangle OPQ , where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is:

- (1) $-\frac{1}{4}$ (2) -4
(3) -2 (4) $-\frac{1}{2}$ (AIEEE 2012)

4. The x -coordinate of the in centre of the triangle that has the coordinates of midpoints of its sides as $(0, 1)$, $(1, 1)$ and $(1, 0)$ is:

- (1) $2 + \sqrt{2}$ (2) $2 - \sqrt{2}$
(3) $1 + \sqrt{2}$ (4) $1 - \sqrt{2}$ (JEE Main 2013)

5. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x -axis, the equation of the reflected ray is:

- (1) $y = x + \sqrt{3}$ (2) $\sqrt{3}y = x - \sqrt{3}$
(3) $y = \sqrt{3}x - \sqrt{3}$ (4) $\sqrt{3}y = x - 1$

(JEE Main 2013)

6. Let a , b , c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then

- (1) $2bc - 3ad = 0$ (2) $2bc + 3ad = 0$
(3) $3bc - 2ad = 0$ (4) $3bc + 2ad = 0$

(JEE Main 2014)

7. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

- (1) $4x - 7y - 1 = 0$ (2) $2x + 9y + 7 = 0$
(3) $4x + 7y + 3 = 0$ (4) $2x - 9y - 11 = 0$

(JEE Main 2014)

8. Locus of the image of the point $(2, 3)$ in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a

- (1) straight line parallel to x -axis
(2) straight line parallel to y -axis
(3) circle of radius $\sqrt{2}$
(4) circle of radius 3

(JEE Main 2015)

9. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus?

- (1) $(-3, -8)$ (2) $\left(\frac{1}{3}, -\frac{8}{3}\right)$
(3) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$ (4) $(-3, -9)$

(JEE Main 2016)

JEE ADVANCED

Single Correct Answer Type

1. The locus of the orthocenter of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$, is

- (1) a hyperbola (2) a parabola
(3) an ellipse (4) a straight line

(IIT-JEE 2009)

2. A straight line L through the point $(3, -2)$ is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x -axis, then the equation of L is

(1) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$

(2) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(3) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$

(4) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

(IIT-JEE 2011)

3. For $a > b > c > 0$, the distance between $(1, 1)$ and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then

(1) $a + b - c > 0$

(3) $a - b + c > 0$

(2) $a - b + c < 0$

(4) $a + b - c < 0$

(JEE Advanced 2013)

Numerical Value Type

1. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$, respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is _____.

(JEE Advanced 2014)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (1) | 2. (3) | 3. (2) | 4. (1) | 5. (3) |
| 6. (3) | 7. (4) | 8. (1) | 9. (4) | 10. (1) |
| 11. (3) | 12. (3) | 13. (1) | 14. (3) | 15. (2) |
| 16. (2) | 17. (3) | 18. (1) | 19. (3) | 20. (2) |
| 21. (3) | 22. (4) | 23. (1) | 24. (1) | 25. (3) |
| 26. (1) | 27. (2) | 28. (4) | 29. (3) | 30. (2) |
| 31. (2) | 32. (4) | 33. (4) | 34. (2) | 35. (2) |
| 36. (3) | 37. (1) | 38. (1) | 39. (2) | 40. (4) |
| 41. (2) | 42. (3) | 43. (3) | 44. (4) | 45. (3) |
| 46. (1) | 47. (1) | 48. (3) | 49. (3) | 50. (1) |
| 51. (2) | 52. (4) | 53. (3) | 54. (3) | 55. (4) |
| 56. (1) | 57. (1) | 58. (1) | 59. (2) | 60. (3) |
| 61. (2) | 62. (1) | 63. (4) | 64. (3) | 65. (3) |
| 66. (1) | 67. (1) | 68. (1) | 69. (2) | 70. (2) |
| 71. (3) | 72. (2) | 73. (2) | 74. (2) | 75. (3) |
| 76. (1) | 77. (3) | 78. (4) | 79. (1) | 80. (1) |
| 81. (2) | 82. (3) | | | |

Multiple Correct Answers Type

- | | |
|------------------------|------------------------|
| 1. (1), (3) | 2. (2), (4) |
| 3. (1), (2) | 4. (1), (4) |
| 5. (2), (4) | 6. (2), (3) |
| 7. (1), (3), (4) | 8. (1), (2), (4) |
| 9. (1), (2) | 10. (1), (2), (3), (4) |
| 11. (1), (4) | 12. (1), (3) |
| 13. (2), (3) | 14. (1), (2) |
| 15. (2), (3) | 16. (2), (4) |
| 17. (3), (4) | 18. (2), (3) |
| 19. (1), (2) | 20. (1), (2) |
| 21. (1), (2), (3), (4) | 22. (1), (2), (3), (4) |
| 23. (1), (2), (3) | 24. (1), (4) |
| 25. (1), (2) | 26. (1), (3) |
| 27. (1), (2), (3), (4) | 28. (1), (4) |
| 29. (1), (4) | 30. (1), (2), (3), (4) |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (1) | 2. (3) | 3. (4) | 4. (2) | 5. (4) |
| 6. (1) | 7. (4) | 8. (3) | 9. (1) | 10. (1) |
| 11. (2) | 12. (3) | 13. (2) | 14. (3) | 15. (1) |
| 16. (3) | 17. (1) | 18. (1) | 19. (1) | 20. (2) |
| 21. (3) | 22. (2) | 23. (4) | 24. (3) | 25. (3) |
| 26. (2) | 27. (2) | | | |

Matrix Match Type

- $a \rightarrow s; b \rightarrow r; c \rightarrow p; d \rightarrow q$
- $a \rightarrow s; b \rightarrow r; c \rightarrow q; d \rightarrow p$
- $a \rightarrow s; b \rightarrow p; c \rightarrow q; d \rightarrow r$
- $a \rightarrow p, s; b \rightarrow q, s; c \rightarrow p, r; d \rightarrow s$
- $a \rightarrow q, r, s; b \rightarrow p; c \rightarrow q, s; d \rightarrow q$
- $a \rightarrow p, s; b \rightarrow p; c \rightarrow q; d \rightarrow p, q, r$
- $a \rightarrow s; b \rightarrow p, q; c \rightarrow r; d \rightarrow p, q, s$
- (3)

Numerical Value Type

- | | | | | |
|---------|---------|----------|---------|---------|
| 1. (18) | 2. (1) | 3. (0) | 4. (4) | 5. (3) |
| 6. (3) | 7. (-2) | 8. (91) | 9. (50) | 10. (5) |
| 11. (3) | 12. (9) | 13. (20) | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (1) | 2. (1) | 3. (3) | 4. (2) | 5. (2) |
| 6. (3) | 7. (2) | 8. (3) | 9. (2) | |

JEE ADVANCED

Single Correct Answer Type

- | | | |
|--------|--------|--------|
| 1. (4) | 2. (2) | 3. (1) |
|--------|--------|--------|

Numerical Value Type

1. (6)

3

Pair of Straight Lines

EQUATION OF PAIR OF STRAIGHT LINES

If equations of two lines are $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$, then their joint equation is expressed as $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$. This is the equation of pair of straight lines having component straight lines L_1 and L_2 .

For example, the joint equation of lines $2x + y + 3 = 0$ and $x - y + 4 = 0$ is $(2x + y + 3)(x - y + 4) = 0$ or $2x^2 - xy - y^2 + 11x + y - 12 = 0$.

Conversely, if combined equation of two straight lines or equation of pair of straight lines is $6x^2 + 5xy - 4y^2 = 0$, then this equation can be reduced to $(2x - y)(3x + 4y) = 0$. So, component lines are $2x - y = 0$ and $3x + 4y = 0$.

Note:

In order to find the combined equation of two lines, make the RHS of each equation of straight line equal to zero and then multiply the two equations.

ILLUSTRATION 3.1

Find the equations of two straight lines whose combined equation is $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$.

Sol. The given equation of pair of straight lines is

$$6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$$

$$\text{or } 6x^2 + (5y + 7)x - (4y^2 - 13y + 3) = 0$$

Solving it as a quadratic in x , we get

$$x = \frac{-(5y + 7) \pm \sqrt{(5y + 7)^2 + 24(4y^2 - 13y + 3)}}{12}$$

$$= \frac{-(5y + 7) \pm \sqrt{121y^2 - 242y + 121}}{12}$$

$$= \frac{-(5y + 7) \pm 11(y - 1)}{12}$$

$$= \frac{6y - 18}{12}, \frac{-16y + 4}{12} \text{ or } \frac{y - 3}{2}, \frac{-4y + 1}{3}$$

The two straight lines are $2x - y + 3 = 0$ and $3x + 4y - 1 = 0$.

ILLUSTRATION 3.2

Find the distance between the pair of parallel lines

$$x^2 + 4xy + 4y^2 + 3x + 6y - 4 = 0.$$

Sol. Given lines are

$$(x + 2y)^2 + 3(x + 2y) - 4 = 0$$

$$\therefore x + 2y = \frac{-3 \pm \sqrt{9 + 16}}{2} = \frac{-3 \pm 5}{2} = -4, 1$$

Therefore, the lines are

$$x + 2y + 4 = 0$$

$$x + 2y - 1 = 0$$

$$\begin{aligned} \text{Required distance} &= \frac{|4 - (-1)|}{\sqrt{1 + 4}} \\ &= \frac{5}{\sqrt{5}} = \sqrt{5} \end{aligned}$$

GENERAL EQUATION OF THE SECOND DEGREE IN TWO VARIABLES

The most general form of a quadratic equation in variables x and y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (1)$$

Since it is an equation in variables x and y , it must represent the equation of a locus in a plane. It may represent a pair of straight lines, circle, parabola, ellipse or hyperbola. If it represents pair of straight lines, then expression on L.H.S. must get factorized into two linear expressions. Let us find the condition for this.

If $a \neq 0$, we can write equation (1) as quadratic in x .

$$ax^2 + 2x(hy + g) + (by^2 + 2fy + c) = 0$$

Solving for x , we get

$$\begin{aligned} x &= \frac{-2(hy + g) \pm \sqrt{4(hy + g)^2 - 4a(by^2 + 2fy + c)}}{2a} \\ &= \frac{-(hy + g) \pm \sqrt{(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)}}{a} \end{aligned}$$

We must have x as linear expression in y . For that, the expression under the square root must be a perfect square of some linear expression in y . Thus, the discriminant of expression $(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)$ must be zero.

$$\text{So, } 4(gh - af)^2 - 4(h^2 - ab)(g^2 - ac) = 0$$

$$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad (2)$$

This is condition for which equation (1) represents the pair of straight lines.

Equation (2) can be put in the form of determinant as follows:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Point of Intersection of Pair of Straight Lines

The elementary method of finding the point of intersection of pair of straight lines is getting the component straight lines and solving them.

We can also use the direct formula for the point of intersection which is $\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$.

The point of intersection can also be determined with the help of partial differentiation as follows:

Differentiating (1) w.r.t. x , keeping y constant, we get

$$2ax + 2hy + 2g = 0 \quad (3)$$

Now, differentiating (1) w.r.t. y , keeping x constant, we get

$$2hx + 2by + 2f = 0 \quad (4)$$

Now, solving equations (3) and (4), we get the point of intersection.

It should be noted that equations (3) and (4) are not equations of component straight lines.

ILLUSTRATION 3.3

Find the value of λ if $2x^2 + 7xy + 3y^2 + 8x + 14y + \lambda = 0$ represents a pair of straight lines.

Sol. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines if $abc + 2fgh - af^2 - bg^2 - bc^2 = 0$.

So, for given equation,

$$6\lambda + 2(7)(4)\left(\frac{7}{2}\right) - 2(7)^2 - 3(4)^2 - \lambda\left(\frac{7}{2}\right)^2 = 0$$

$$\text{or } 6\lambda + 196 - 98 - 48 - \frac{49\lambda}{4} = 0$$

$$\text{or } \frac{49\lambda}{4} - 6\lambda = 196 - 146 = 50$$

$$\text{or } \frac{25\lambda}{4} = 50$$

$$\text{or } \lambda = \frac{200}{25} = 8$$

ILLUSTRATION 3.4

Does equation $x^2 + 2y^2 - 2\sqrt{3}x - 4y + 5 = 0$ satisfies the condition $abc + 2gh - af^2 - bg^2 - ch^2 = 0$? Does it represent a pair of straight lines?

Sol. Given equation is $x^2 + 2y^2 - 2\sqrt{3}x - 4y + 5 = 0$.

Here, $a = 1, b = 2, h = 0, g = -\sqrt{3}, f = -2, c = 5$.

But, $abc + 2gh - af^2 - bg^2 - ch^2 = 10 + 0 - 4 - 6 = 0$.

The given equation can be written as

$$(x - \sqrt{3})^2 + 2(y - 1)^2 = 0$$

Hence, it denotes only a point $P(\sqrt{3}, 1)$ but not a pair of straight lines.

ILLUSTRATION 3.5

If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y -axis, then prove that $2fgh = bg^2 + ch^2$.

Sol. Let the line intersect on y -axis at $P(0, y_1)$.

Putting this point in the equation of pair of straight lines, we get

$$by_1^2 + 2fy_1 + c = 0$$

Above equation must have equal roots.

$$\therefore 4f^2 - 4bc = 0$$

$$\text{or } f^2 = bc$$

Also, given equation represents pair of straight lines.

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

From (1), putting the value of f^2 in (2), we get

$$2fgh = bg^2 + ch^2$$

ILLUSTRATION 3.6

Find the coordinates of points where pair of lines given equation $2x^2 - 6y^2 + xy - 2x + 17y - 12 = 0$ intersect line $x = 1$.

Sol. Let the given pair of straight lines intersect line $x = 1$ at $(1, y)$.

Putting this point in the given equation, we get

$$2 - 6y^2 + y - 2 + 17y - 12 = 0$$

$$\Rightarrow y^2 - 3y + 2 = 0$$

$$\Rightarrow (y - 1)(y - 2) = 0$$

$$\Rightarrow y = 1, y = 2$$

Thus, required points of intersection are $(1, 1)$ and $(1, 2)$.

ILLUSTRATION 3.7

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents the pair of parallel straight lines, then prove that $h^2 = ab$ and $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

Sol. Let the given equation represent pair of parallel straight lines $lx + my + n = 0$ and $lx + my + n' = 0$

$$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

$$= (lx + my + n)(lx + my + n')$$

$$= (lx + my)^2 + l(n + n')x + m(n + n')y + nn' = 0$$

Thus, expression $(lx + my)^2$ is same as $ax^2 + by^2 + 2hxy$.

Therefore, $ax^2 + 2hxy + by^2$ must be perfect square of line expression in x and y .

$$\text{So, } (2h)^2 - 4ab = 0$$

$$\therefore h^2 = ab$$

$$\text{Also, } abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

CONCEPT APPLICATION EXERCISE 3.1

- Find the combined equation of the pair of lines through the point $(1, 0)$ and parallel to the lines represented by $2x^2 - xy - y^2 = 0$.
- Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the coordinates of their point of intersection and also the angle between them.
- If one of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angle between the positive direction of the axes, then find the relation for a, b , and h .
- If the pair of lines $\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0$ is rotated about the origin by $\pi/6$ in the anticlockwise sense, then find the equation of the pair of lines in the new position.

5. If the equation $2x^2 + kxy + 2y^2 = 0$ represents a pair of real and distinct lines, then find the values of k .
6. Find the point of intersection of the pair of straight lines represented by the equation $6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$.

ANSWERS

1. $2x^2 - xy - y^2 - 4x + y + 2 = 0$ 2. $(1, -2), \tan^{-1} 1/5$
 3. $a + b = -2h$ 4. $\sqrt{3}x^2 - xy = 0$
 5. $k \in (-\infty, -4) \cup (4, \infty)$ 6. $(-32/23, 17/32)$

SECOND-DEGREE HOMOGENEOUS EQUATION

The homogenous equation is that equation in which each term has same degree. General second-degree homogenous equation is given by

$$ax^2 + by^2 + 2hxy = 0 \quad (1)$$

This equation represents pair of straight lines passing through origin.

Consider one such equation of pair of straight lines.

$$(2x - y)(3x + 2y) = 0$$

$$\text{or } 6x^2 - 2y^2 + xy = 0 \quad (2)$$

Now, consider another equation of pair of straight lines.

$$(2x - y + 3)(3x + 2y - 4) = 0$$

$$\text{or } 6x^2 - 2y^2 + xy + x + 10y - 12 = 0 \quad (3)$$

The component lines of above equation are parallel to the component lines of equation (2).

We observe that second degree terms in both the equations (2) and (3) are same or ratio of coefficients of second-degree terms is same.

Now, general equation of pair of straight line is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (4)$$

Then corresponding equation of pair of straight lines parallel to component lines of above equation and passing through origin is given by

$$ax^2 + by^2 + 2hxy = 0 \quad (5)$$

Let the component lines of equation (5) be $y - m_1x = 0$ and $y - m_2x = 0$.

$$\therefore by^2 + 2hxy + ax^2 = b(y - m_1x)(y - m_2x) \\ = by^2 - b(m_1 + m_2)xy + bm_1m_2x^2$$

Comparing coefficients of both sides, we get

$$2h = -b(m_1 + m_2) \text{ and } a = bm_1m_2$$

$$\therefore m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

ILLUSTRATION 3.8

Find the equations of component lines whose combined equation is $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$ without solving for x or y .

Sol. Given equation of pair of straight lines is:

$$6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0 \quad (1)$$

Homogenous part of given equation is

$$6x^2 + 5xy - 4y^2 = 0 \quad (2)$$

$$\text{or } (3x + 4y)(2x - y) = 0$$

This represents equations of pair of straight lines parallel to the component lines of given equation.

Therefore, given equation of pair of straight lines can be put as

$$(3x + y + A)(2x - y + B) = 0 \quad (3)$$

Comparing coefficients of (1) and (3), we get

$$A = -1 \text{ and } B = 3$$

Hence, the required component lines are $2x - y + 3 = 0$, $3x + 4y - 1 = 0$.

ILLUSTRATION 3.9

If the component lines whose combined equation is $px^2 - qxy - y^2 = 0$ make the angles α and β with x -axis, then find the value of $\tan(\alpha + \beta)$.

Sol. Given equation of pair of straight lines is:

$$px^2 - qxy - y^2 = 0 \quad (1)$$

Let the component lines be $y = m_1x$ and $y = m_2x$.

$$\therefore m_1 = \tan \alpha \text{ and } m_2 = \tan \beta \quad (\text{Given})$$

$$\text{Also, } m_1 + m_2 = -q \text{ and } m_1m_2 = -p$$

$$\therefore \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ = \frac{m_1 + m_2}{1 - m_1m_2} = \frac{-q}{1 + p}$$

ILLUSTRATION 3.10

Find the joint equation of pair of lines which passes through origin and are perpendicular to the lines represented by the equation $y^2 + 3xy - 6x + 5y - 14 = 0$.

Sol. Homogeneous part of the given equation is $y^2 + 3xy = 0$.

So, component lines are $y = 0$ and $y + 3x = 0$.

Lines perpendicular to these lines are $x = 0$ and $x - 3y = 0$.

So, combined equation of above lines is $x^2 - 3xy = 0$.

ILLUSTRATION 3.11

If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then find the value of c .

Sol. Given equation of pair of lines is $x^2 - 2cxy - 7y^2 = 0$.

Let the slopes of the component lines be m_1 and m_2 .

It is given that

$$m_1 + m_2 = 4m_1m_2 \\ \Rightarrow \frac{-2c}{7} = \frac{-4}{7} \\ \therefore c = 2$$

ILLUSTRATION 3.12

If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is rotated about the origin through 90° , then find its equation in the new position.

Sol. Given equation of pair of straight lines is

$$ax^2 + 2hxy + by^2 = 0.$$

Let the component lines be $y = m_1x$ and $y = m_2x$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

Now, equations of lines perpendicular to above component lines are

$$y = -\frac{1}{m_1}x \text{ and } y = -\frac{1}{m_2}x$$

$$\text{or } m_1y + x = 0 \text{ and } m_2y + x = 0$$

Combined equation of above lines is

$$(m_1y + x)(m_2y + x) = 0$$

$$\text{or } m_1m_2y^2 + xy(m_1 + m_2) + x^2 = 0$$

$$\text{or } \frac{a}{b}y^2 + xy\left(\frac{-2h}{b}\right) + x^2 = 0$$

$$\text{or } bx^2 - 2hxy + ay^2 = 0$$

Alternative method:

We can write the given equation of pair of straight lines as:

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

The roots of this equation are m_1 and m_2 which are slopes of component lines.

Now, we require equation whose roots are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$.

So, in above equation, replacing $\frac{y}{x}$ by $-\frac{x}{y}$, we get

$$b\left(-\frac{x}{y}\right)^2 + 2h\left(-\frac{x}{y}\right) + a = 0$$

$$\text{or } bx^2 - 2hxy + ay^2 = 0$$

which is the required equation of pair of straight lines.

ANGLE BETWEEN PAIR OF STRAIGHT LINES

Consider general equation of pair of straight lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (1)$$

The corresponding equation of pair of straight lines parallel to component lines of above equation and passing through origin is

$$ax^2 + by^2 + 2hxy = 0 \quad (2)$$

Clearly, angle between component lines of (1) and (2) is same.

Let the component lines of equation (2) be $y - m_1x = 0$ and $y - m_2x = 0$.

$$\therefore m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

If angle between lines is θ , then

$$\begin{aligned} \tan \theta &= \left| \frac{m_1 - m_2}{1 + m_1m_2} \right| = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{|1 + m_1m_2|} \\ &= \frac{\sqrt{\frac{4h^2}{b^2} - 4\frac{a}{b}}}{\left|1 + \frac{a}{b}\right|} = \frac{2\sqrt{h^2 - ab}}{|a + b|} \end{aligned}$$

This is the formula for finding the acute angle between component lines of equation (1) or (2).

Special cases:

If lines are parallel or coincident, then $\theta = 0$. So, $h^2 = ab$.

If lines are perpendicular, then $\theta = 90^\circ$. So, $a + b = 0$.

ILLUSTRATION 3.13

Find acute and obtuse angle between component lines whose combined equation is $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$.

Sol. For the given equation of pair of straight lines, we have $a = 2$, $b = 3$, $h = 5/2$

Thus, acute angle between lines is given by

$$\theta = \tan^{-1} \frac{2\sqrt{h^2 - ab}}{|a + b|} = \frac{2\sqrt{\frac{25}{4} - 6}}{2 + 3} = \tan^{-1} \frac{1}{5}$$

Obtuse angle between lines = $\left(\pi - \tan^{-1} \frac{1}{5}\right)$

ILLUSTRATION 3.14

Find the value of a for which the lines represented by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular.

Sol. The lines given by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular if $a + 2 = 0$, i.e., $a = -2$.

EQUATION OF PAIR OF ANGLE BISECTORS OF GIVEN PAIR OF STRAIGHT LINES

Let the given equation of pair of straight lines is

$$ax^2 + by^2 + 2hxy = 0 \quad (1)$$

Let the component lines be $y = m_1x$ and $y = m_2x$.

$$\therefore m_1 + m_2 = \frac{-2h}{b}, m_1m_2 = \frac{a}{b}$$

Equations of angle bisectors of component lines are

$$\frac{y - m_1x}{\sqrt{1 + m_1^2}} = \pm \frac{y - m_2x}{\sqrt{1 + m_2^2}}$$

$\therefore$ Combined equation of bisector lines is

$$\left(\frac{y - m_1x}{\sqrt{1 + m_1^2}} - \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right)\left(\frac{y - m_1x}{\sqrt{1 + m_1^2}} + \frac{y - m_2x}{\sqrt{1 + m_2^2}}\right) = 0$$

$$\text{or } \frac{(y - m_1x)^2}{1 + m_1^2} - \frac{(y - m_2x)^2}{1 + m_2^2} = 0$$

$$\text{or } (m_1 + m_2)(x^2 - y^2) + 2(m_1m_2 - 1)xy = 0$$

$$\text{or } \frac{-2h}{b}(x^2 - y^2) + 2\left(\frac{a}{b} - 1\right)xy = 0$$

$$\text{or } \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

ILLUSTRATION 3.15

If the pair of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ are such that each pair bisects the angle between the other pair, then prove that $pq = -1$.

Sol. Combined equation of the bisectors of the angles between pair of lines $x^2 - 2pxy - y^2 = 0$ is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}$$

or $px^2 + 2xy - py^2 = 0$

This must be same as the given equation of pair of bisectors i.e., $x^2 - 2qxy - y^2 = 0$.

Comparing ratios of coefficients, we get

$$\frac{p}{1} = \frac{1}{-q} = \frac{p}{1}$$

$$\therefore pq = -1$$

HOMOGENIZATION OF SECOND DEGREE EQUATION

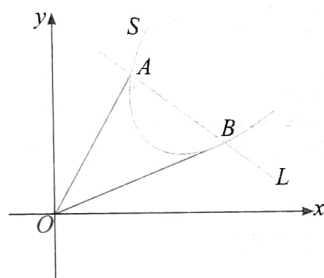
Consider general second degree equation

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (1)$$

This equation represents pair of straight lines, circle, parabola, ellipse or hyperbola.

Clearly, this is not homogenous equation.

Now, let the line $L \equiv lx + my + n = 0$ intersect the curve (1) in two points A and B .



The combined equation of pair of straight lines OA and OB must be homogenous and must contain only second-degree terms.

From the equation of line, we have

$$1 = \frac{lx + my}{-n} \quad \dots(2)$$

Now, equation of curve is

$$(ax^2 + 2hxy + by^2) + (2gx + 2fy) \cdot 1 + c \cdot 1^2 = 0$$

Putting the value of '1' from (2) in above equation, we have

$$ax^2 + 2hxy + by^2 + (2gx + 2fy) \left(\frac{lx + my}{-n} \right) + c \left(\frac{lx + my}{-n} \right)^2 = 0$$

This is homogenous equation of second degree.

Since points A and B satisfy above equation, it represents the pair of straight lines OA and OB . This process of making a second-degree equation homogeneous is called homogenization through which we get equation of pair of straight lines joining the points of intersection of the given line and curve with the origin.

ILLUSTRATION 3.16

Prove that the straight lines joining the origin to the points of intersection of the straight line $hx + ky = 2hk$ and the curve $(x - k)^2 + (y - h)^2 = c^2$ are at right angle if $h^2 + k^2 = c^2$.

Sol. From the equation of line, we have

$$1 = \frac{hx + ky}{2hk} \quad (1)$$

Equation of the curve is

$$x^2 + y^2 - 2kx - 2hy + h^2 + k^2 - c^2 = 0$$

Making above equation homogeneous with the help of (1), we get

$$x^2 + y^2 - 2(kx + hy) \left(\frac{hx + ky}{2hk} \right) + (h^2 + k^2 - c^2) \left(\frac{hx + ky}{2hk} \right)^2 = 0$$

This is combined equation of the pair of lines joining the origin to the points of intersection of the given line and the curve. The component lines are perpendicular if sum of coefficient of x^2 and coefficient of y^2 is zero.

$$\therefore (h^2 + k^2)(h^2 + k^2 - c^2) = 0$$

$$\Rightarrow h^2 + k^2 = c^2 \quad (\text{as } h^2 + k^2 \neq 0)$$

ILLUSTRATION 3.17

Find the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$.

Sol. The equation of straight line is

$$\frac{y - 3x}{2} = 1$$

Making given equation of curve homogeneous, we get

$$x^2 + 2xy + 3y^2 + (4x + 8y) \left(\frac{y - 3x}{2} \right) - 11 \left(\frac{y - 3x}{2} \right)^2 = 0$$

$$\text{or } 7x^2 - 2xy - y^2 = 0 \quad (1)$$

This is the equation of the pair of lines joining origin to the point of intersection of the given line and curve.

Comparing this equation with the equation $ax^2 + 2hxy + by^2 = 0$, we get

$$a = 7, b = -1 \text{ and } h = -1$$

If θ is the acute angle between component lines of (1), then

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \left| \frac{2\sqrt{1 + 7}}{7 - 1} \right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$

$$\therefore \theta = \tan^{-1} \frac{2\sqrt{2}}{3}$$

CONCEPT APPLICATION EXERCISE 3.2

1. If the slope of one line is double the slope of another line and the combined equation of the pair of lines is $(x^2/a) + (2xy/h) + (y^2/b) = 0$, then find the ratio $ab : h^2$.
2. Find the angle between the lines represented by $x^2 + 2xy \sec \theta + y^2 = 0$.

- Find the angle between the straight lines joining the origin to the points of intersection of $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y = 1$.
- If θ is the angle between the lines given by the equation $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$, then find the equation of the line passing through the point of intersection of these lines and making an angle θ with the positive x -axis.
- Show that the equation of the pair of lines bisecting the angles between the pair of bisectors of the angles between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $(a - b)(x^2 - y^2) + 4hxy = 0$.
- Find the equation of the bisectors of the angles between the lines joining the origin to the point of intersection of the straight line $x - y = 2$ with the curve $5x^2 + 11xy - 8y^2 + 8x - 4y + 12 = 0$.
- Show that the pairs of straight lines $2x^2 + 6xy + y^2 = 0$ and $4x^2 + 18xy + y^2 = 0$ have the same set of angular bisector.

ANSWERS

- 9 : 8
- θ
- $\pi/2$
- $11x - 2y + 13 = 0$
- $x^2 + 30xy - y^2 = 0$

Solved Examples**EXAMPLE 3.1**

Show that straight lines $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$ form with the line $Ax + By + C = 0$ an equilateral triangle of area $\frac{C^2}{\sqrt{3}(A^2 + B^2)}$.

Sol. Let the line through origin making an angle of 60° with the line $Ax + By + C = 0$ be $y = mx$.

$$\therefore \tan 60^\circ = \left| \frac{m - (-A/B)}{1 + m(-A/B)} \right|$$

$$\text{or } 3(B - Am)^2 = (mB + A)^2$$

Putting $m = y/x$ in above, we get

$$(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$$

This is pair of straight lines through origin in which each component line makes 60° with line $Ax + By + C = 0$.

Also, length of perpendicular from origin to line $Ax + By + C = 0$ is

$$\frac{|C|}{\sqrt{A^2 + B^2}}$$

$$\therefore \text{Length of side of triangle} = \frac{|2C|}{\sqrt{3(A^2 + B^2)}}$$

$$\therefore \text{Area of triangle} = \frac{\sqrt{3}}{4} \left(\frac{2C}{\sqrt{3(A^2 + B^2)}} \right)^2 = \frac{C^2}{\sqrt{3}(A^2 + B^2)}$$

EXAMPLE 3.2

Prove that the product of the perpendiculars from (α, β) to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a-b)^2 + 4h^2}}$$

Sol. Let the lines be $y = m_1x$ and $y = m_2x$
or $m_1x - y = 0$ and $m_2x - y = 0$

$$\text{where } m_1 + m_2 = -\frac{2h}{b}, \quad m_1m_2 = \frac{a}{b}$$

Now, if the lengths of perpendicular from $(0, 0)$ on these lines are d_1 and d_2 , respectively, then

$$\begin{aligned} d_1d_2 &= \frac{|m_1\alpha - \beta|}{\sqrt{m_1^2 + 1}} \cdot \frac{|m_2\alpha - \beta|}{\sqrt{m_2^2 + 1}} \\ &= \frac{|m_1m_2\alpha^2 - (m_1 + m_2)\alpha\beta + \beta^2|}{\sqrt{m_1^2m_2^2 + m_1^2 + m_2^2 + 1}} \\ &= \frac{|m_1m_2\alpha^2 - (m_1 + m_2)\alpha\beta + \beta^2|}{\sqrt{m_1^2m_2^2 + (m_1 + m_2)^2 - 2m_1m_2 + 1}} \\ &= \frac{\left| \frac{a}{b}\alpha^2 + \frac{2h}{b}\alpha\beta + \beta^2 \right|}{\sqrt{\frac{a^2}{b^2} + \frac{4h^2}{b^2} - 2\frac{a}{b} + 1}} \\ &= \frac{|a\alpha^2 + 2h\alpha\beta + b\beta^2|}{\sqrt{a^2 + 4h^2 - 2ab + b^2}} \\ &= \frac{|a\alpha^2 + 2h\alpha\beta + b\beta^2|}{\sqrt{(a-b)^2 + 4h^2}} \end{aligned}$$

EXAMPLE 3.3

The distance of a point (x_1, y_1) from each of the two straight lines which pass through the origin of coordinates is p . Find the combined equation of these straight lines.

Sol. Let the line through the origin be $y = mx$ or $y - mx = 0$. Distance of line $mx - y = 0$ from (x_1, y_1) is

$$\frac{mx_1 - y_1}{\sqrt{1 + m^2}} = p \text{ (given)}$$

$$\text{or } m^2x_1^2 + y_1^2 - 2mx_1y_1 = p^2 + p^2m^2$$

$$\text{or } (x_1^2 - p^2)m^2 - 2mx_1y_1 + y_1^2 - p^2 = 0 \quad (1)$$

Equation (1) has two roots: m_1 and m_2 .

Now, the pair of lines is

$$(y - m_1x)(y - m_2x) = 0$$

$$\text{or } y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0 \quad (2)$$

Now, from (1),

$$m_1 + m_2 = \frac{2x_1y_1}{(x_1^2 - p^2)}$$

and $m_1 m_2 = \frac{(y_1^2 - p^2)}{(x_1^2 - p^2)}$

Putting these values in (2), we get

$$y^2 - [2x_1 y_1 / (x_1^2 - p^2)] xy + [(y_1^2 - p^2) / (x_1^2 - p^2)] x^2 = 0$$

EXAMPLE 3.4

A point moves so that the distance between the foot of perpendiculars from it on the lines $ax^2 + 2hxy + by^2 = 0$ is a constant $2d$. Show that the equation to its locus is $(x^2 + y^2)(h^2 - ab) = d^2\{(a - b)^2 + 4h^2\}$.

Sol. From point $P(\alpha, \beta)$, perpendiculars are dropped on the given pair of lines.

According to the question, $AB = 2d$.

Clearly, $OAPB$ is a cyclic quadrilateral and OP will be the diameter of the circumcircle of this quadrilateral.

$$OP = \sqrt{\alpha^2 + \beta^2}$$

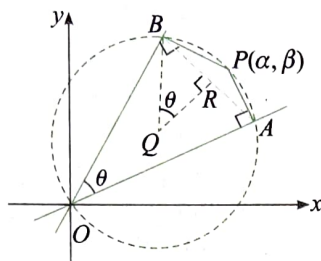
Let Q be the center of the circle.

Therefore, in $\triangle QRB$,

$$\sin \theta = \frac{BR}{QB} = \frac{2d}{\sqrt{\alpha^2 + \beta^2}} \quad (1)$$

Also, angle between the lines is

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b} \quad (2)$$



Therefore, from (1) and (2), we get

$$\frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2d}{\sqrt{\alpha^2 + \beta^2 - 4d^2}}$$

Therefore, the locus of $P(\alpha, \beta)$ is

$$(x^2 + y^2)(h^2 - ab) = d^2\{(a - b)^2 + 4h^2\}$$

EXAMPLE 3.5

Show that all chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin, pass through a fixed point. Find the coordinates of the point.

Sol. The given curve is

$$3x^2 - y^2 - 2x + 4y = 0 \quad (1)$$

Let $y = mx + c$ be the chord of curve (1) which subtends an angle of 90° at the origin. Then the combined equations of lines joining the points of intersection of curve (1) and chord $y = mx + c$ to the origin can be obtained by making the equation of curve homogeneous with the help of the equation of chord as

$$3x^2 - y^2 - 2x\left(\frac{y - mx}{c}\right) + 4y\left(\frac{y - mx}{c}\right) = 0$$

$$\text{or } 3cx^2 - cy^2 - 2xy + 2mx^2 + 4y^2 - 4mxy = 0$$

$$\text{or } (3c + 2m)x^2 - 2(1 + 2m)xy + (4 - c)y^2 = 0$$

As the lines represented by this pair are perpendicular to each other, we must have

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\text{Hence, } 3c + 2m + 4 - c = 0$$

$$\text{or } -2 = m + c$$

Comparing this result with $y = mx + c$, we can see that $y = mx + c$ passes through $(1, -2)$.

Exercises

Single Correct Answer Type

- The angle between the pair of lines whose equation is $4x^2 + 10xy + my^2 + 5x + 10y = 0$ is
 - $\tan^{-1}(3/8)$
 - $\tan^{-1}(3/4)$
 - $\tan^{-1}\{2\sqrt{25-4m}/(m+4)\}, m \in R$
 - none of these
- The two lines represented by $3ax^2 + 5xy + (a^2 - 2)y^2 = 0$ are perpendicular to each other for
 - two values of a
 - a
 - for one value of a
 - for no value of a
- The distance between the two lines represented by the equation $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$ is
 - $8/5$
 - $6/5$
 - $11/5$
 - none of these
- The equations $x - y = 4$ and $x^2 + 4xy + y^2 = 0$ represent the sides of
 - an equilateral triangle
 - a right-angled triangle
 - an isosceles triangle
 - none of these
- The straight lines represented by $(y - mx)^2 = a^2(1 + m^2)$ and $(y - nx)^2 = a^2(1 + n^2)$ form a
 - rectangle
 - rhombus
 - trapezium
 - none of these
- If the pairs of lines $x^2 + 2xy + ay^2 = 0$ and $ax^2 + 2xy + y^2 = 0$ have exactly one line in common, then the joint equation of the other two lines is given by
 - $3x^2 + 8xy - 3y^2 = 0$
 - $3x^2 + 10xy + 3y^2 = 0$
 - $y^2 + 2xy - 3x^2 = 0$
 - $x^2 + 2xy - 3y^2 = 0$
- The condition that one of the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ may coincide with one of those given by the equation $a'x^2 + 2h'xy + b'y^2 = 0$ is
 - $(ab' - a'b)^2 = 4(ha' - h'a)(bh' - b'h)$
 - $(ab' - a'b)^2 = (ha' - h'a)(bh' - b'h)$
 - $(ha' - h'a)^2 = 4(ab' - a'b)(bh' - b'h)$
 - $(bh' - b'h)^2 = 4(ab' - a'b)(ha' - h'a)$
- If the lines represented by the equation $3y^2 - x^2 + 2\sqrt{3}x - 3 = 0$ are rotated about the point $(\sqrt{3}, 0)$ through an angle of 15° , one in clockwise direction and the other in anticlockwise direction, so that they become perpendicular, then the equation of the pair of lines in the new position is
 - $y^2 - x^2 + 2\sqrt{3}x + 3 = 0$
 - $y^2 - x^2 + 2\sqrt{3}x - 3 = 0$
 - $y^2 - x^2 - 2\sqrt{3}x + 3 = 0$
 - $y^2 - x^2 + 3 = 0$
- The equation of a line which is parallel to the line common to the pair of lines given by $6x^2 - xy - 12y^2 = 0$ and $15x^2 + 14xy - 8y^2 = 0$ and at a distance of 7 units from it is
 - $3x - 4y = -35$
 - $5x - 2y = 7$
 - $3x + 4y = 35$
 - $2x - 3y = 7$
- The equation $x^2y^2 - 9y^2 + 6x^2y + 54y = 0$ represents
 - a pair of straight lines and a circle
 - a pair of straight lines and a parabola
 - a set of four straight lines forming a square
 - none of these
- The equations $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$ represent
 - two pairs of perpendicular straight lines
 - two pairs of parallel straight lines
 - two pairs of straight lines which are equally inclined to each other
 - none of these
- If the equation of the pair of straight lines passing through the point $(1, 1)$, one making an angle θ with the positive direction of the x -axis and the other making the same angle with the positive direction of the y -axis, is $x^2 - (a+2)xy + y^2 + a(x+y-1) = 0$, $a \neq -2$, then the value of $\sin 2\theta$ is
 - $a - 2$
 - $a + 2$
 - $2/(a+2)$
 - $2/a$
- If two lines represented by $x^4 + x^3y + cx^2y^2 - xy^3 + y^4 = 0$ bisect the angle between the other two, then the value of c is
 - 0
 - 1
 - 1
 - 6
- Through a point A on the x -axis, a straight line is drawn parallel to the y -axis so as to meet the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ at B and C . If $AB = BC$, then
 - $h^2 = 4ab$
 - $8h^2 = 9ab$
 - $9h^2 = 8ab$
 - $4h^2 = ab$
- The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $y = 0$ is
 - $ax^2 - 2hxy - by^2 = 0$
 - $bx^2 - 2hxy + ay^2 = 0$
 - $bx^2 + 2hxy + ay^2 = 0$
 - $ax^2 - 2hxy + by^2 = 0$
- The straight lines represented by the equation $135x^2 - 136xy + 33y^2 = 0$ are equally inclined to the line
 - $x - 2y = 7$
 - $x + 2y = 7$
 - $x - 2y = 4$
 - $3x + 2y = 4$
- If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is the square of the other, then $\frac{a+b}{h} + \frac{8h^2}{ab} =$
 - 4
 - 6
 - 8
 - none of these
- $x + y = 7$ and $ax^2 + 2hxy + ay^2 = 0$, ($a \neq 0$), are three real distinct lines forming a triangle. Then the triangle is
 - isosceles
 - scalene
 - equilateral
 - right-angled

19. **Statement 1:** If $-2h = a + b$, then one line of the pair of lines $ax^2 + 2hxy + by^2 = 0$ bisects the angle between the coordinate axes in the positive quadrant.

Statement 2: If $ax + y(2h + a) = 0$ is a factor of $ax^2 + 2hxy + by^2 = 0$, then $b + 2h + a = 0$.

- (1) Both the statements are true but statement 2 is the correct explanation of statement 1.
 (2) Both the statements are true but statement 2 is not the correct explanation of statement 1.
 (3) Statement 1 is true and statement 2 is false.
 (4) Statement 1 is false and statement 2 is true.
20. The orthocenter of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is
 (1) $(1/2, 1/2)$ (2) $(1/3, 1/3)$
 (3) $(0, 0)$ (4) $(1/4, 1/4)$
21. Let PQR be a right-angled isosceles triangle, right angled at $P(2, 1)$. If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is
 (1) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
 (2) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
 (3) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
 (4) $3x^2 - 3y^2 - 8xy - 15y - 20 = 0$
22. Area of the triangle formed by the line $x + y = 3$ and the angle bisectors of the pairs of straight lines $x^2 - y^2 + 2y = 1$ is
 (1) 2 sq. units (2) 4 sq. units
 (3) 6 sq. units (4) 8 sq. unit
23. The orthocentre of the triangle formed by the lines $2x^2 + 3xy - 2y^2 - 9x + 7y - 5 = 0$ with $4x + 5y - 3 = 0$ is
 (1) $(3/5, 11/5)$ (2) $(6/5, 11/5)$
 (3) $(5/6, 11/5)$ (4) $(3/5, 6/5)$

Multiple Correct Answers Type

1. The equation $x^3 + x^2y - xy^2 = y^3$ represents
 (1) three real straight lines
 (2) lines in which two of them are perpendicular to each other
 (3) lines in which two of them are coincident
 (4) none of these
2. The straight lines represented by $x^2 + mxy - 2y^2 + 3y - 1 = 0$ meet at
 (1) $(-1/3, 2/3)$ (2) $(-1/3, -2/3)$
 (3) $(1/3, 2/3)$ (4) none of these
3. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then m is
 (1) 1 (2) 2 (3) $-1/2$ (4) -1
4. If $x^2 + 2hxy + y^2 = 0$ ($h > 1$) represents the equation of the straight lines through the origin which make an angle α with the straight line $y + x = 0$, then

- (1) $\sec 2\alpha = h$ (2) $\cos \alpha = \sqrt{(1+h)/(2h)}$
 (3) $2 \sin \alpha = \sqrt{(1+h)/h}$ (4) $\cot \alpha = \sqrt{(h+1)/(h-1)}$

5. The combined equation of three sides of a triangle is $(x^2 - y^2)(2x + 3y - 6) = 0$. If $(-2, a)$ is an interior point and $(b, 1)$ is an exterior point of the triangle, then
 (1) $2 < a < 10/3$ (2) $-2 < a < 10/3$
 (3) $-1 < b < 9/2$ (4) $-1 < b < 1$
6. If one of the lines given by the equation $2x^2 + pxy + 3y^2 = 0$ coincide with one of those given by $2x^2 + qxy - 3y^2 = 0$ and the other lines represented by them are perpendicular, then value of $p + q$ is
 (1) 6 (2) -6 (3) -7 (4) 7
7. The lines joining the origin to the point of intersection of $3x^2 + mxy - 4x + 1 = 0$ and $2x + y - 1 = 0$ are at right angles. Then which of the following is a possible value of m ?
 (1) -4 (2) 4 (3) 7 (4) 3
8. If the equation $ax^2 - 6xy + y^2 + bx + cy + d = 0$ represents a pair of lines whose slopes are m and m^2 , then the value(s) of a is/are
 (1) $a = -8$ (2) $a = 8$ (3) $a = 27$ (4) $a = -27$
9. Two pairs of straight lines have the equations $y^2 + xy - 12x^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$. One line will be common among them if
 (1) $a + 8h - 16b = 0$ (2) $a - 8h + 16b = 0$
 (3) $a - 6h + 9b = 0$ (4) $a + 6h + 9b = 0$

Linked Comprehension Type

For Problems 1–3

Consider the equation of a pair of straight lines as $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$.

1. The value of λ is
 (1) 1 (2) 2 (3) $3/2$ (4) 3
2. The point of intersection of lines is (α, β) . Then the value of $\alpha\beta$ is
 (1) 35 (2) 45 (3) 20 (4) 15
3. The angle between the lines is θ . Then the value of $\tan \theta$ is
 (1) $1/5$ (2) $2/9$ (3) $1/7$ (4) $3/4$

For Problems 4–6

Consider a pair of perpendicular straight lines $ax^2 + 3xy - 2y^2 - 5x + 5y + c = 0$.

4. The value of a is
 (1) 1 (2) 3 (3) 2 (4) -2
5. The value of c is
 (1) -3 (2) 3 (3) -1 (4) 1
6. Distance between the orthocenter and the circumcenter of triangle ABC is
 (1) 4 (2) $9/2$ (3) $8/3$ (4) $7/4$

Numerical Value Type

- The distance between the lines $(x + 7y)^2 + 4\sqrt{2}(x + 7y) - 42 = 0$ is _____.
- Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 6$ is _____.
- The value k for which $4x^2 + 8xy + ky^2 = 9$ is the equation of a pair of straight lines is _____.

- If the gradient of one of the lines $x^2 + hxy + 2y^2 = 0$ is twice that of the other, then sum of possible values of h is _____.
- One of the bisectors of the angle between the lines $a(x-1)^2 + 2h(x-1)(y-2) + b(y-2)^2 = 0$ is $x + 2y - 5 = 0$. Then the value of $\frac{b-a}{h}$ is _____.

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (2) | 2. (1) | 3. (1) | 4. (1) | 5. (2) |
| 6. (2) | 7. (1) | 8. (2) | 9. (3) | 10. (3) |
| 11. (3) | 12. (3) | 13. (4) | 14. (2) | 15. (4) |
| 16. (2) | 17. (2) | 18. (1) | 19. (2) | 20. (3) |
| 21. (2) | 22. (1) | 23. (1) | | |

Multiple Correct Answers Type

- | | |
|------------------|------------------|
| 1. (1), (2), (3) | 2. (1), (3) |
| 3. (1), (4) | 4. (1), (2), (4) |

5. (1), (4)

7. (1), (2), (3), (4)

9. (2), (4)

6. (1), (2)

8. (2), (4)

Linked Comprehension Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (2) | 2. (1) | 3. (3) | 4. (3) | 5. (1) |
| 6. (4) | | | | |

Numerical Value Type

- | | | | | |
|--------|--------|--------|--------|----------|
| 1. (2) | 2. (3) | 3. (2) | 4. (0) | 5. (1.5) |
|--------|--------|--------|--------|----------|

DEFINITION

A circle is the locus of a point which moves in a plane such that its distance from a fixed point in the same plane is always constant. The fixed point is called the centre and the constant distance is called the radius of the circle.

EQUATION OF CIRCLE IN CENTRE–RADIUS FORM

If the centre of the circle is $C(x_1, y_1)$ and radius is r , then for variable point $P(x, y)$ lying on the circle, $CP = r$.

$$\therefore (x - x_1)^2 + (y - y_1)^2 = r^2$$

This is the equation of circle.

If the centre of the circle is origin, then its equation is

$$x^2 + y^2 = r^2$$

Circles having same centre and different radii are called concentric circles.

Here, the common centre of two circles is $C(x_1, y_1)$ and their radii are r_1 and r_2 .

The equations of circles are

$$(x - x_1)^2 + (y - y_1)^2 = r_1^2$$

and $(x - x_1)^2 + (y - y_1)^2 = r_2^2$

Expanding the equations, we find that equations of concentric circles differ by constant only.

Any line passing through centre of the circle is called diameter of the circle. A circle has infinite diameters.

ILLUSTRATION 4.1

Find the equation of the circle with radius 5 whose center lies on the x -axis and passes through the point $(2, 3)$.

Sol. Since the radius of the circle is 5 and its center lies on the x -axis, the equation of the circle is $(x - h)^2 + y^2 = 25$.

It is given that the circle passes through the point $(2, 3)$. Therefore,

$$(2 - h)^2 + 3^2 = 25$$

or $(2 - h)^2 = 16$

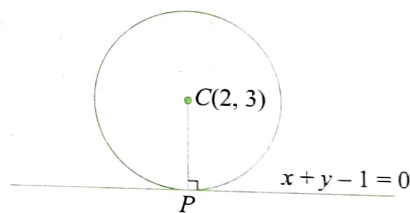
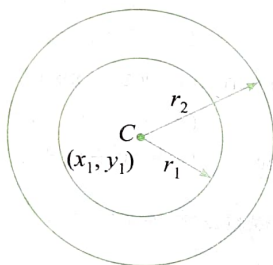
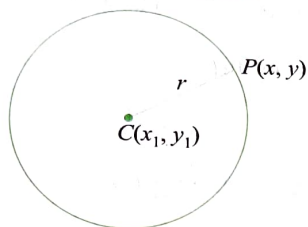
or $2 - h = \pm 4$

If $2 - h = 4$, then $h = -2$.

If $2 - h = -4$, then $h = 6$.

Therefore, the equation of circle is $(x + 2)^2 + y^2 = 25$ or $(x - 6)^2 + y^2 = 25$.

Hence, $x^2 + y^2 + 4x - 21 = 0$ or $x^2 + y^2 - 12x + 11 = 0$.

**ILLUSTRATION 4.2**

If the lines $x + y = 6$ and $x + 2y = 4$ are diameters of the circle which passes through the point $(2, 6)$, then find its equation.

Sol. Here center will be the point of intersection of the diameter, i.e., $C(8, -2)$.

Also, the circle passes through the point $P(2, 6)$. Then radius is $CP = 10$.

Hence, the required equation is

$$(x - 8)^2 + (y + 2)^2 = 10^2$$

or $x^2 + y^2 - 16x + 4y - 32 = 0$

ILLUSTRATION 4.3

Find the equation of circle having centre at $(2, 3)$ and touching the line $x + y = 1$.

Sol. The centre of the circle is $C(2, 3)$.

Also, line $x + y - 1 = 0$ is tangent to the circle.

Hence, the radius of the circle is the perpendicular distance of the centre from the tangent.

Therefore, radius $r = CP = \frac{|2 + 3 - 1|}{\sqrt{1^2 + 1^2}} = 2\sqrt{2}$

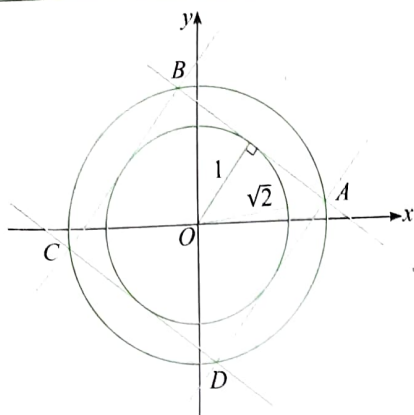
Hence, the equation of circle is $(x - 2)^2 + (y - 3)^2 = 8$.

ILLUSTRATION 4.4

Consider the quadrilateral formed by four lines $3x + 4y - 5 = 0$; $4x - 3y - 5 = 0$; $3x + 4y + 5 = 0$ and $4x - 3y + 5 = 0$. Find the equation of the circle inscribed in the quadrilateral and circumscribing the quadrilateral.

Sol. Distance between parallel lines $3x + 4y - 5 = 0$ and $3x + 4y + 5 = 0$ is 2.

Distance between parallel lines $4x - 3y - 5 = 0$ and $4x - 3y + 5 = 0$ is 2.



Clearly, lines form a square of side length 2.

Inscribed circle has radius 1.

Circumscribed circle has radius $\sqrt{2}$.

Also, centre of the square is $(0, 0)$.

So, centre of both the circles is $(0, 0)$.

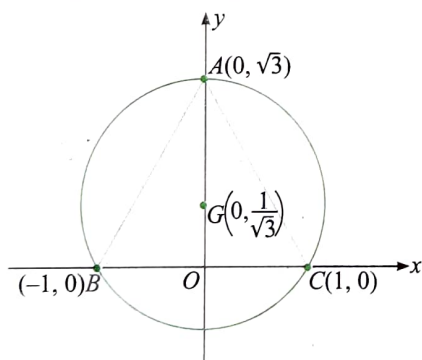
Therefore, the equation of the circle inscribed in the quadrilateral is $x^2 + y^2 = 1$.

And the equation of the circle circumscribing the quadrilateral is $x^2 + y^2 = 2$.

ILLUSTRATION 4.5

Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$, and its third vertex lies above the x -axis. Find the equation of circumcircle of triangle.

Sol. We have equilateral triangle ABC .



Let $B \equiv (-1, 0)$, $C \equiv (1, 0)$.

Then A lies on y -axis.

$$\therefore OA = AB \sin 60^\circ = \sqrt{3}$$

$$\therefore A \equiv (0, \sqrt{3})$$

In an equilateral triangle, circumcentre coincides with centroid.

$$\text{Thus, circumcentre of triangle } ABC \text{ is } G \equiv \left(\frac{-1+1+0}{3}, \frac{\sqrt{3}+0+0}{3} \right) \\ \equiv \left(0, \frac{1}{\sqrt{3}} \right).$$

$$\text{Circumradius} = AG = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

Therefore, equation of circumcircle of triangle ABC is

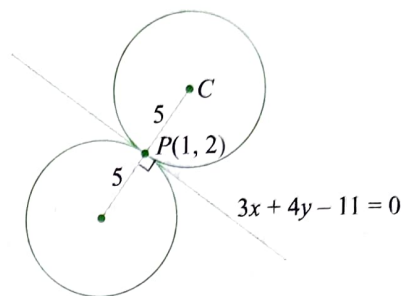
$$(x-0)^2 + \left(y - \frac{1}{\sqrt{3}} \right)^2 = \left(\frac{2}{\sqrt{3}} \right)^2.$$

ILLUSTRATION 4.6

Find the equations of circles each having radius 5 and touching the line $3x + 4y - 11 = 0$ at point $(1, 2)$.

Sol. As shown in the figure, the two circles touch the line $3x + 4y - 11 = 0$ at point $P(1, 2)$.

$\therefore$ Centre lies on the line perpendicular to the tangent at P at distance 5 from it on either side.



Slope of given tangent line is $-\frac{3}{4}$.

Thus, the slope of CP is $\frac{4}{3}$.

$$\therefore \tan \theta = \frac{4}{3}$$

$$\therefore \text{Centre, } C \equiv (1 \pm 5 \cos \theta, 2 \pm 5 \sin \theta)$$

$$\therefore C \equiv \left(1 \pm 5 \frac{3}{5}, 2 \pm 5 \frac{4}{5} \right)$$

$$\therefore C \equiv (4, 6) \text{ or } (-2, -2)$$

Therefore, equations of circles are

$$(x-4)^2 + (y-6)^2 = 25$$

$$\text{or } (x+2)^2 + (y+2)^2 = 25$$

ILLUSTRATION 4.7

Prove that for all values of θ , the locus of the point of intersection of the lines $x \cos \theta + y \sin \theta = a$ and $x \sin \theta - y \cos \theta = b$ is a circle.

Sol. Since the point of intersection satisfies both the given lines, we can find the locus by eliminating θ from the given equation.

Therefore, by squaring and adding, we get equation

$$x^2 + y^2 = a^2 + b^2$$

which is the equation of circle.

ILLUSTRATION 4.8

Prove that the maximum number of points with rational coordinates on a circle whose center is $(\sqrt{3}, 0)$ is two.

Sol. There cannot be three points on the circle with rational coordinates as for then the center of the circle, being the circumcenter of a triangle whose vertices have rational coordinates, must have rational coordinates (since the coordinates will be obtained by solving two linear equations in x, y having rational coefficients). But the point $(\sqrt{3}, 0)$ does not have rational coordinates.

Also, the equation of the circle is

$$(x - \sqrt{3})^2 + y^2 = r^2$$

$$x = \sqrt{3} \pm \sqrt{r^2 - y^2}$$

or For suitable r and x , where x is rational, y may have two rational values.

For example, $r = 2$, $x = 0$, $y = 1, -1$ satisfy $x = \sqrt{3} \pm \sqrt{r^2 - y^2}$.

So, we get two points $(0, 1)$ and $(0, -1)$ which have rational coordinates.

ILLUSTRATION 4.9

Find the locus of the midpoint of the chords of circle $x^2 + y^2 = a^2$ having fixed length l .

Sol. As shown in the figure, AB is one of the variable chords having length l .

Let the midpoint of AB is $M(h, k)$.

The foot of perpendicular from centre C on AB is M .

In right angled triangle CMA , we have

$$AC^2 = CM^2 + AM^2$$

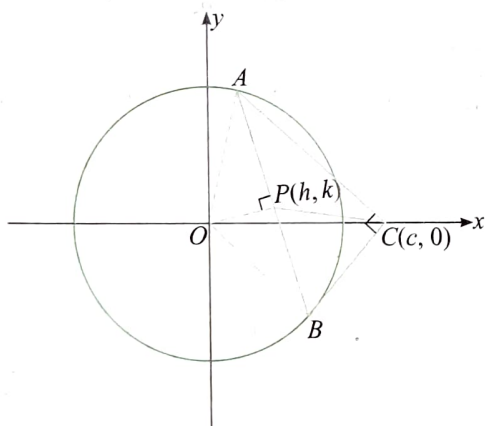
$$\therefore a^2 = (h^2 + k^2) + \left(\frac{l}{2}\right)^2$$

Hence, equation of required locus is $x^2 + y^2 = a^2 - \frac{l^2}{4}$.

ILLUSTRATION 4.10

Find the locus of midpoint of the chords of the circle $x^2 + y^2 = a^2$ which subtends a right angle at the point $(c, 0)$.

Sol. Let $P(h, k)$ be the midpoint of a chord AB of the circle which subtends a right angle at $C(c, 0)$.



In right angled triangle ACB , we have

$$PA = PC = PB$$

$$PC = \sqrt{(h-c)^2 + k^2}$$

$$PB = PA = \sqrt{OA^2 - OP^2} \\ = \sqrt{a^2 - (h^2 + k^2)}$$

Now, $PC = PA$

$$\therefore \sqrt{(h-c)^2 + k^2} = \sqrt{a^2 - (h^2 + k^2)}$$

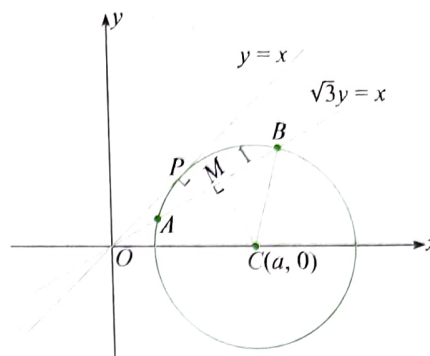
$$\Rightarrow (h-c)^2 + k^2 = a^2 - (h^2 + k^2)$$

Therefore, the equation of required locus is $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$.

ILLUSTRATION 4.11

Find the equation of the circle which is touched by $y = x$, has its centre on the positive direction of the x -axis and cuts off a chord of length 2 units along the line $x - \sqrt{3}y = 0$.

Sol.



Centre C lies on x -axis.

$$\therefore C \equiv (a, 0)$$

Line $y = x$ touches the circle at point P .

So, radius of circle is the length of perpendicular (CP) from the centre C on the line $y = x$.

Also, length of chord (AB) of circle on line $x - \sqrt{3}y = 0$ is 2 units.

$$\therefore MB = 1 \text{ (where } m \text{ is the midpoint of } AB\text{).}$$

Now, $CP = CB$ (radii of circle)

$$\therefore CP^2 = CB^2$$

$$\Rightarrow CP^2 = CM^2 + MB^2$$

$$\Rightarrow \left(\frac{|a-0|}{\sqrt{1^2 + (-1)^2}}\right)^2 = \left(\frac{|a - \sqrt{3}(0)|}{\sqrt{(\sqrt{3})^2 + (1)^2}}\right)^2 + 1^2$$

$$\Rightarrow \frac{a^2}{2} = \frac{a^2}{4} + 1$$

$$\Rightarrow \frac{a^2}{4} = 1$$

$$\Rightarrow a = 2 \text{ (as centre lies in first quadrant)}$$

$$\text{Also, radius } CP = \frac{a}{\sqrt{2}} = \sqrt{2}$$

Therefore, equation of the circle is

$$(x-2)^2 + (y-0)^2 = 2$$

$$\text{or } x^2 + y^2 - 4x + 2 = 0$$

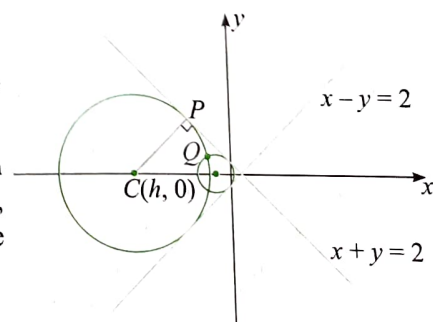
ILLUSTRATION 4.12

Find the equations of the circles passing through the point $(-4, 3)$ and touching the lines $x + y = 2$ and $x - y = 2$.

Sol. Since circles touch the lines $x + y = 2$ and $x - y = 2$, centre lies on x -axis or y -axis.

But circle passes through the point $Q(-4, 3)$. So, centre lies on negative x -axis.

Let the centre be $C(h, 0)$.



Also, radius, CP = distance of centre from the line $x - y - 2 = 0$ (or $x + y - 2 = 0$)

$$= \frac{|h - 0 - 2|}{\sqrt{1^2 + (-1)^2}} = \frac{|h - 2|}{\sqrt{2}}$$

Also, radius, $CQ = \sqrt{(h + 4)^2 + (k - 3)^2}$

$$\therefore CQ = CP$$

$$\Rightarrow \frac{(h - 2)^2}{2} = (h + 4)^2 + (0 - 3)^2$$

$$\Rightarrow h^2 + 20h + 46 = 0$$

$$\Rightarrow h = \frac{-20 \pm \sqrt{20^2 - 4(46)}}{2}$$

$$\Rightarrow h = -10 \pm 3\sqrt{6}$$

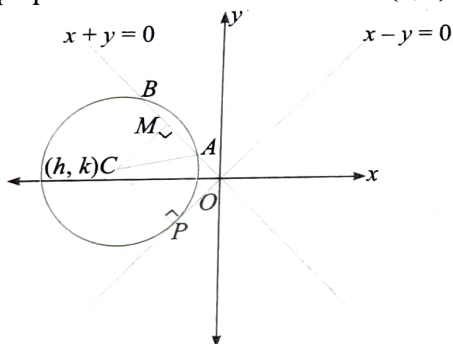
Thus, equation of the circle is $(x - h)^2 + y^2 = \frac{(h - 2)^2}{2}$, where $h = -10 \pm 3\sqrt{6}$.

ILLUSTRATION 4.13

A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle.

Sol. Let AB be the length of chord intercepted by the circle on $y + x = 0$.

Let CM be perpendicular to BA from center $C(h, k)$.



Also, $y - x = 0$ and $y + x = 0$ are perpendicular to each other.

Therefore, $OPCM$ is a rectangle. So,

$$CM = OP = 4\sqrt{2}.$$

Let r be the radius of the circle.

Also, $AM = 3\sqrt{2}$.

Therefore, in $\triangle CAM$,

$$AC^2 = AM^2 + MC^2$$

$$\text{or } r^2 = (3\sqrt{2})^2 + (4\sqrt{2})^2$$

$$\text{or } r^2 = (5\sqrt{2})^2$$

$$\text{or } r = 5\sqrt{2}$$

Also, the coordinates of P are

$$(0 - 4\sqrt{2} \cos 45^\circ, 0 - 4\sqrt{2} \sin 45^\circ) \text{ or } (-4, -4)$$

The slope of PC is -1 and $CP = 5\sqrt{2}$.

Therefore, the coordinates of C are

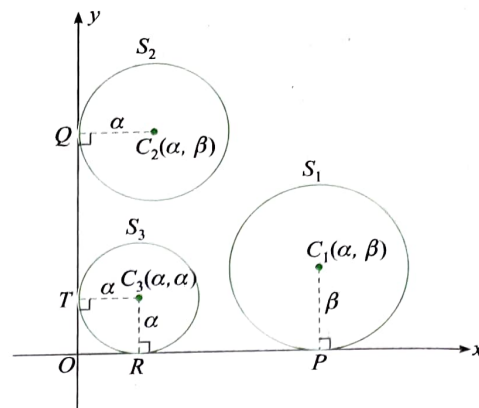
$$(-4 + 5\sqrt{2} \cos 135^\circ, -4 + 5\sqrt{2} \sin 135^\circ) \text{ or } (-9, 1)$$

Hence, equation of the circle is

$$(x + 9)^2 + (y - 1)^2 = 50$$

$$\text{or } x^2 + y^2 + 18x - 2y + 32 = 0$$

EQUATION OF CIRCLE TOUCHING ONE OR BOTH THE COORDINATE AXES

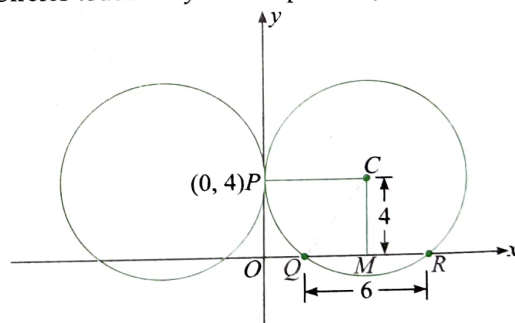


- Circle S_1 touches x -axis.
Centre of the circle is $C_1(\alpha, \beta)$. Clearly, radius is $C_1P = |\beta|$.
Therefore, equation of circle is $(x - \alpha)^2 + (y - \beta)^2 = \beta^2$
- Circle S_2 touches y -axis.
Centre of the circle is $C_2(\alpha, \beta)$. Clearly, radius is $C_2Q = |\alpha|$.
Therefore, equation of circle is $(x - \alpha)^2 + (y - \beta)^2 = \alpha^2$
- Circle S_3 touches both the axes.
Centre of the circle is $C_3(\alpha, \alpha)$. Clearly, radius is $C_3R = C_3T = |\alpha|$.
Therefore, equation of circle is $(x - \alpha)^2 + (y - \alpha)^2 = \alpha^2$

ILLUSTRATION 4.14

Find the equations of circles touching y -axis at the point $(0, 4)$ and cutting the x -axis in a chord of length 6 units.

Sol. Circles touch the y -axis at point $P(0, 4)$.



Intercept QR of circle on x -axis is 6 units.

$$\therefore QM = 3$$

Also, C is centre of one of the circles.

$$\therefore CM = 4$$

In triangle CMQ , we have

$$CQ^2 = QM^2 + CM^2 = 3^2 + 4^2$$

$$\therefore CQ = 5 = CP = \text{radius}$$

Therefore, centre of the circle lying in first quadrant is $C(5, 4)$.

Thus, equation of circle is $(x - 5)^2 + (y - 4)^2 = 25$.

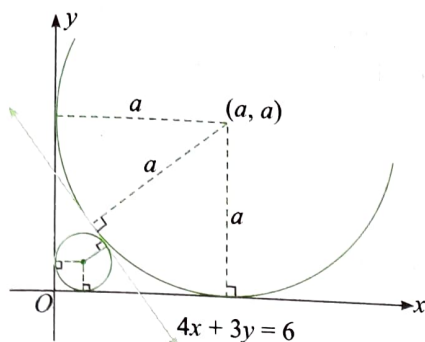
Equation of circle having centre $C(-5, 4)$ in second quadrant is $(x + 5)^2 + (y - 4)^2 = 25$.

ILLUSTRATION 4.15

Find the possible equation of circle which touches both the axes and the straight line $4x + 3y = 6$.

Sol. Since the circle touches both the axes, its equation takes the form

$$(x - a)^2 + (y - a)^2 = a^2$$



Also, circle touches the line $4x + 3y - 6 = 0$.

So, the distance of centre of the circle from the line is equal to its radius.

$$\therefore \frac{|4a + 3a - 6|}{\sqrt{16 + 9}} = a$$

$$\Rightarrow 7a - 6 = \pm 5a$$

$$\Rightarrow a = 3 \text{ or } 1/2$$

Therefore, possible equations of circle are

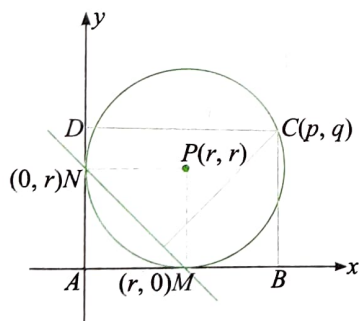
$$(x - 1/2)^2 + (y - 1/2)^2 = 1/4$$

$$\text{and } (x - 3)^2 + (y - 3)^2 = 9$$

ILLUSTRATION 4.16

A circle passing through the vertex C of a rectangle $ABCD$ and touching its sides AB and AD at M and N , respectively. If the distance from C to the line segment MN is equal to 5 units, then find the area of the rectangle $ABCD$.

Sol. Let us take AB and AD as coordinate axes.



If r is the radius of circle, then equation of circle touching both the axes is

$$(x - r)^2 + (y - r)^2 = r^2$$

$$\text{or } x^2 + y^2 - 2rx - 2ry + r^2 = 0$$

Let point C be (p, q) .

Equation of line MN is $x + y - r = 0$.

Distance of MN from C is 5 units.

$$\therefore \frac{|p + q - r|}{\sqrt{2}} = 5$$

$$\Rightarrow (p + q - r)^2 = 50$$

Since (p, q) lies on the circle, we have

$$p^2 + q^2 - 2rp - 2rq + r^2 = 0$$

$$\Rightarrow (p + q - r)^2 - 2pq = 0$$

$$\Rightarrow 50 - 2pq = 0$$

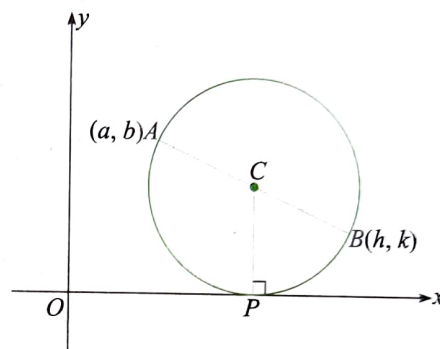
$$\Rightarrow pq = 25$$

Therefore, area of rectangle = 25 sq. units.

ILLUSTRATION 4.17

A variable circle passes through the point $A(a, b)$ and touches the x -axis. Show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.

Sol. As shown in the figure, the variable circle passes through the point $A(a, b)$ and touches the x -axis at point P .



Let the other end of diameter of the circle through A be $B(h, k)$.

So, centre of the circle is $C\left(\frac{h+a}{2}, \frac{k+b}{2}\right)$.

Radius of the circle, $CP = \frac{AB}{2}$

$$\Rightarrow \left|\frac{k+b}{2}\right| = \frac{\sqrt{(h-a)^2 + (k-b)^2}}{2}$$

Squaring and simplifying, we get $(h - a)^2 = 4bk$.

Therefore, required equation of the locus is $(x - a)^2 = 4by$.

GENERAL FORM OF EQUATION OF CIRCLE

The general equation of circle is taken as

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

where g, f and c are constants.

This equation can be put in centre-radius form as

$$(x + g)^2 + (y + f)^2 = (g^2 + f^2 - c)$$

Thus, centre of the circle is

$$(-g, -f) \equiv \left(-\frac{\text{coefficient of } x}{2}, -\frac{\text{coefficient of } y}{2}\right)$$

However, for this formula of centre, coefficients of x^2 and y^2 must be unity.

Also, radius is $\sqrt{g^2 + f^2 - c}$.

If $g^2 + f^2 - c > 0$, then we have real circle.

If $g^2 + f^2 - c = 0$, then radius of circle is zero. Hence, equation of circle becomes $(x + g)^2 + (y + f)^2 = 0$, which represents the point $(-g, -f)$ or point circle having radius 0.

If $g^2 + f^2 - c < 0$, then we have no real circle.

Now, recall general second-degree equation in variables x and y , which is

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

This equation represents circle if $a = b$ and $h = 0$.

Also, we must have $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$.

However, for point circle (having radius 0), $a = b$, $h = 0$ and $\Delta = 0$.

ILLUSTRATION 4.18

If the equation $px^2 + (2 - q)xy + 3y^2 - 6qx + 30y + 6q = 0$ represents a circle, then find the values of p and q .

Sol. In the equation of circle, the xy term does not occur and the coefficients of x^2 and y^2 are equal. Therefore,

$$2 - q = 0 \text{ or } q = 2$$

$$\text{and } p = 3$$

Also, for these values of p and q , the circle is real.

ILLUSTRATION 4.19

If $x^2 + y^2 - 2x + 2ay + a + 3 = 0$ represents the real circle with nonzero radius, then find the values of a .

Sol. For circle $x^2 + y^2 - 2x + 2ay + a + 3 = 0$, we have $g = -1$, $f = a$, and $c = a + 3$

For real circle with nonzero radius,

$$g^2 + f^2 - c > 0$$

$$\text{or } 1 + a^2 - (a + 3) > 0$$

$$\text{or } a^2 - a - 2 > 0$$

$$\text{or } (a - 2)(a + 1) > 0$$

$$\text{i.e., } a < -1 \text{ or } a > 2$$

ILLUSTRATION 4.20

A point P moves in such a way that the ratio of its distance from two coplanar points is always a fixed number ($\neq 1$). Then, identify the locus of the point.

Sol. Let two coplanar points be $A(0, 0)$ and $B(a, 0)$. Let $P(x, y)$ be the variable point.

According to question, $\frac{AP}{BP} = \lambda$

$$\therefore \frac{\sqrt{x^2 + y^2}}{\sqrt{(x - a)^2 + y^2}} = \lambda, \text{ (where } \lambda \text{ is any fixed number and } \lambda \neq 1)$$

$$\text{or } x^2 + y^2 + \left(\frac{\lambda^2}{\lambda^2 - 1}\right)(a^2 - 2ax) = 0$$

which is the equation of a circle.

ILLUSTRATION 4.21

Find the image of the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ in the line $2x - 3y + 5 = 0$.

Sol. The image of the circle in the line will be circle having same radius.

For given circle, centre is $C(1, -2)$ and radius is

$$\sqrt{(-1)^2 + (2)^2} = \sqrt{5}$$

Let image of C in the line be $C_1(h, k)$.

$$\therefore \frac{h-1}{2} = \frac{k+2}{-3} = \frac{-2(2(1) - 3(-2) + 5)}{(2)^2 + (-3)^2} = -2$$

$$\therefore h = -3 \text{ and } k = 4$$

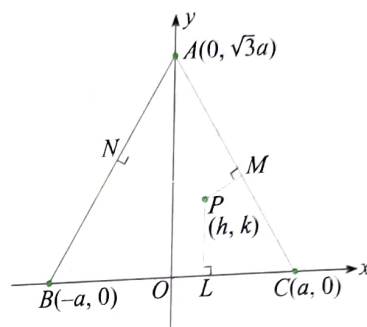
Thus, centre of the image circle is $C_1(-3, 4)$.

Hence, equation of the image circle is $(x + 3)^2 + (y - 4)^2 = 9$.

ILLUSTRATION 4.22

A point is moving in such a way that sum of the squares of perpendiculars drawn from it to the sides of an equilateral triangle is constant. Prove that its locus is a circle.

Sol.



Let vertices B and C of equilateral triangle be $(-a, 0)$ and $(a, 0)$.

$$OA = OB \tan 60^\circ = a\sqrt{3}$$

$$\therefore A \equiv (0, a\sqrt{3})$$

Let point P be (h, k) .

Equations of AC and BC are $x\sqrt{3} + y - a\sqrt{3} = 0$ and $x\sqrt{3} - y + a\sqrt{3} = 0$, respectively.

According to the question,

$$PL^2 + PM^2 + PN^2 = \text{constant (say } \lambda)$$

$$\therefore k^2 + \left(\frac{\sqrt{3}h + k - \sqrt{3}a}{2}\right)^2 + \left(\frac{\sqrt{3}h - k + \sqrt{3}a}{2}\right)^2 = \lambda$$

$$\Rightarrow 6h^2 + 6k^2 - 4\sqrt{3}ak + 6a^2 - 4\lambda = 0$$

Hence, the required locus is $6x^2 + 6y^2 - 4\sqrt{3}ay + 6a^2 - 4\lambda = 0$, which is a circle.

ILLUSTRATION 4.23

If $(m_i, 1/m_i)$, $m_i > 0$, $i = 1, 2, 3, 4$, are four distinct points on a circle, then show that $m_1 m_2 m_3 m_4 = 1$.

Sol. Given that $(m_i, 1/m_i)$, $m_i > 0$, $i = 1, 2, 3, 4$, are four distinct points on a circle.

Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

As the point $(m, 1/m)$ lies on it, we have

$$m^2 + \frac{1}{m^2} + 2gm + \frac{2f}{m} + c = 0$$

$$\text{or } m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$$

Since m_1, m_2, m_3 , and m_4 are the roots of this equation, the product of roots is 1, i.e.,

$$m_1 m_2 m_3 m_4 = 1$$

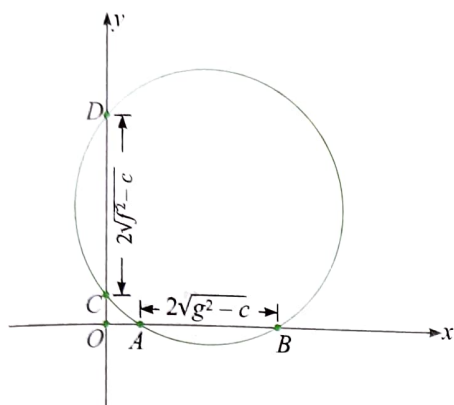
Intercept of Circle on Axes

One of the methods to find the intercepts of circle on axes is solving equation with the axes.

Another method uses the standard formula for intercepts.

Consider the following circle:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$



To solve this circle with x -axis, we put $y = 0$.

$$\therefore x^2 + 2gx + c = 0$$

$$\therefore x = \frac{-2g \pm \sqrt{4g^2 - 4c}}{2} = -g \pm \sqrt{g^2 - c}$$

Thus, circle intersects x -axis at $A(-g - \sqrt{g^2 - c}, 0)$ and $B(-g + \sqrt{g^2 - c}, 0)$.

$$\begin{aligned} \text{Intercept on } x\text{-axis, } AB &= (-g + \sqrt{g^2 - c}) - (-g - \sqrt{g^2 - c}) \\ &= 2\sqrt{g^2 - c} \end{aligned}$$

Similarly, points of intersection of circle with y -axis are $C(0, -f - \sqrt{f^2 - c})$ and $D(0, -f + \sqrt{f^2 - c})$ and intercept on

$$y\text{-axis is } CD = 2\sqrt{f^2 - c}.$$

ILLUSTRATION 4.24

Find the length of intercept which the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the axes.

Sol. Comparing the given equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$g = 5, f = -3 \text{ and } c = 9$$

$$\begin{aligned} \therefore \text{Length of intercept on } x\text{-axis} &= 2\sqrt{g^2 - c} \\ &= 2\sqrt{(5)^2 - 9} = 8 \end{aligned}$$

$$\begin{aligned} \text{Length of intercept on } y\text{-axis} &= 2\sqrt{f^2 - c} \\ &= 2\sqrt{(-3)^2 - 9} = 0 \end{aligned}$$

Thus, circle touches the y -axis.

Alternatively, putting $y = 0$ in the equation of circle, we get

$$x^2 + 10x + 9 = 0$$

$$\text{or } (x + 1)(x + 9) = 0$$

Thus, points of intersection with x -axis are $A(-1, 0)$ and $B(-9, 0)$.

Thus, $AB = 8$.

Putting $x = 0$ in the equation of circle, we get

$$y^2 - 6y + 9 = 0$$

$$\text{or } (y - 3)^2 = 0$$

Thus, circle touches the y -axis.

So, length of intercept on y -axis is 0.

ILLUSTRATION 4.25

If the intercepts of the variable circle on the x - and y -axes are 2 units and 4 units, respectively, then find the locus of the centre of the variable circle.

Sol. Let the equation of variable circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

We have to find the locus of the centre $C(-g, -f)$.

According to the question, we have

$$2\sqrt{g^2 - c} = 2 \text{ and } 2\sqrt{f^2 - c} = 4$$

$$\therefore g^2 - c = 1 \text{ and } f^2 - c = 4$$

Eliminating c , we get

$$f^2 - g^2 = 3 \text{ or } (-f)^2 - (-g)^2 = 3$$

Therefore, required locus is $y^2 - x^2 = 3$.

EQUATION OF CIRCLE PASSING THROUGH THREE POINTS

The general equation of circle, i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$ contains three independent constants g, f and c .

Hence, for determining the equation of a circle, three points or three conditions are required to form three equations in g, f and c .

If circle passes through three points (x_i, y_i) , $i = 1, 2, 3$, then we have three equations

$$x_i^2 + y_i^2 + 2gx_i + 2fy_i + c = 0, i = 1, 2, 3$$

Solving these three equations, we get values of g, f and c .

ILLUSTRATION 4.26

Find the equation of the circle which passes through the points $(1, -2)$, $(4, -3)$ and whose center lies on the line $3x + 4y = 7$.

Sol. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

If (1) passes through the points $(1, -2)$ and $(4, -3)$, then

$$5 + 2g - 4f + c = 0 \quad (2)$$

$$\text{and } 25 + 8g - 6f + c = 0 \quad (3)$$

Since the center $(-g, -f)$ lies on the line $3x + 4y = 7$, we have

$$-3g - 4f = 7 \quad (4)$$

Solving (2), (3), and (4), we get

$$g = -\frac{47}{15}, f = \frac{3}{5}, \text{ and } c = \frac{11}{3}$$

Substituting in (1), the equation of the circle is

$$15x^2 + 15y^2 - 94x + 18y + 55 = 0$$

ILLUSTRATION 4.27

Show that a cyclic quadrilateral is formed by the lines $5x + 3y = 9$, $x = 3y$, $2x = y$, and $x + 4y + 2 = 0$ taken in order. Find the equation of the circumcircle.

Sol. Solving the given equation in pairs taken in order, the coordinates of the vertices of the quadrilateral $ABCD$ are

$$A\left(\frac{3}{2}, \frac{1}{2}\right), B(0, 0), C\left(-\frac{2}{9}, -\frac{4}{9}\right), \text{ and } D\left(\frac{42}{17}, -\frac{19}{17}\right)$$

First, we will find the equation of the circle passing through A , B , and C . Let the equation of this circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

The coordinates of the points A , B , and C must satisfy (1). So, substituting in (1), we have

$$3g + f + c + \frac{5}{2} = 0 \quad (2)$$

$$c = 0 \quad (3)$$

$$\text{and } 4g + 8f - 9c = \frac{20}{9} \quad (4)$$

Solving (2), (3), and (4), we get $g = -10/9$, $f = 5/6$, and $c = 0$.

Substituting in (1), the equation of the circle through the vertices A , B , and C is

$$9x^2 + 9y^2 - 20x + 15y = 0 \quad (5)$$

Since the coordinates of the vertex $D(42/17, -19/17)$ also satisfy (5), a cyclic quadrilateral $ABCD$ is formed by the given lines, and (5) is the equation of the circumcircle of the quadrilateral.

CONCEPT APPLICATION EXERCISE 4.1

1. If a circle whose center is $(1, -3)$ touches the line $3x - 4y - 5 = 0$, then find its radius.
2. Find the equation of the circle which touches the x -axis and whose center is $(1, 2)$.
3. Find the equation of circle which touches both the axes and the line $x = 2$.
4. If $2x + y = 0$ is the equation of a diameter of the circle which touches the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$, then find the centre and radius of the circle.
5. Find the equation of the circle with centre at $(3, -1)$ and which cuts off an intercept of length 6 units from the line $2x - 5y + 18 = 0$.
6. If one end of the diameter is $(1, 1)$ and the other end lies on the line $x + y = 3$, then find the locus of the centre of the circle.
7. Tangent drawn from the point $P(4, 0)$ to the circle $x^2 + y^2 = 8$ touches it at the point A in the first quadrant. Find the coordinates of another point B on the circle such that $AB = 4$.
8. If the line $x + 2by + 7 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$, then find the value of b .
9. Find the length of the intercept that the circle $x^2 + y^2 + 10x - 6y + 9 = 0$ makes on the x -axis.
10. If one end of a diameter of the circle $2x^2 + 2y^2 - 4x - 8y + 2 = 0$ is $(3, 2)$, then find the other end of the diameter.
11. Prove that the locus of the point that moves such that the sum of the squares of its distances from the three vertices of a triangle is constant is a circle.

12. Find the number of integral values of λ for which $x^2 + y^2 + \lambda x + (1 - \lambda)y + 5 = 0$ is the equation of a circle whose radius cannot exceed 5.

13. Prove that the locus of the centroid of the triangle whose vertices are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$, and $(1, 0)$, where t is a parameter, is circle.
14. Find the locus of center of circle of radius 2 units, if intercept cut on the x -axis is twice of intercept cut on the y -axis by the circle.

ANSWERS

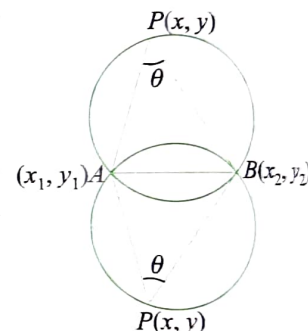
1. 2
2. $x^2 + y^2 - 2x - 4y + 1 = 0$
3. $(x - 1)^2 + (y \pm 1)^2 = 1$
4. Radius = 4 units; Centre $C(1, -2)$
5. $(x - 3)^2 + (y + 1)^2 = 38$
6. $2x + 2y - 5 = 0$
7. $(2, -2)$ or $(-2, 2)$
8. $b = 5$
9. 8
10. $(-1, 2)$
12. 10 values
14. $4x^2 - y^2 = 12$

SOME MORE FORMS OF EQUATION OF CIRCLE

EQUATION OF CIRCLE HAVING GIVEN END POINTS OF CHORD WHICH SUBTENDS GIVEN ANGLE AT THE CIRCUMFERENCE

Let the end points of given chord of circle be $A(x_1, y_1)$ and $B(x_2, y_2)$. Let AB subtends an angle θ at variable point P on the circumference.

The equation of circle is the equation of the locus of point P such that $\angle APB = \theta$.



$$\text{Slope of } AP, m_1 = \frac{y - y_1}{x - x_1}$$

$$\text{Slope of } BP, m_2 = \frac{y - y_2}{x - x_2}$$

$$\therefore \tan \theta = \left| \frac{\frac{y - y_1}{x - x_1} - \frac{y - y_2}{x - x_2}}{1 + \left(\frac{y - y_1}{x - x_1} \right) \left(\frac{y - y_2}{x - x_2} \right)} \right|$$

$$\Rightarrow \frac{(y - y_1)(x - x_2) - (x - x_1)(y - y_2)}{(x - x_1)(x - x_2) + (y - y_1)(y - y_2)} = \pm \tan \theta$$

Hence, equations of required circles are

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = \pm \cot \theta [(y - y_1)(x - x_2) - (x - x_1)(y - y_2)]$$

ILLUSTRATION 4.28

Find the equation of the circle if the chord of the circle joining $(1, 2)$ and $(-3, 1)$ subtends 90° at the center of the circle.

Sol. The chord joining points $A(1, 2)$ and $B(-3, 1)$ subtends 90° at the center of the circle. Then AB subtends an angle 45° at point P on the circumference.

$$(x-1)(x+3) + (y-2)(y-1) = \pm \cot 45^\circ [(y-2)(x+3) - (x-1)(y-1)]$$

$$\text{or } x^2 + y^2 + 2x - 3y - 1 = \pm [4y - x - 7]$$

$$\text{or } x^2 + y^2 + 3x - 7y + 6 = 0 \text{ and } x^2 + y^2 + x + y - 8 = 0$$

EQUATION OF CIRCLE HAVING GIVEN END POINTS OF DIAMETER

If chord AB joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is diameter of the circle, then for any point $P(x, y)$ on the circumference, $\angle APB = 90^\circ$.

In that case, equation of circle is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = \pm (\cot 90^\circ) [(y-y_1)(x-x_2) - (x-x_1)(y-y_2)]$$

$$\text{or } (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

This is the equation of the smallest circle which goes through points A and B .

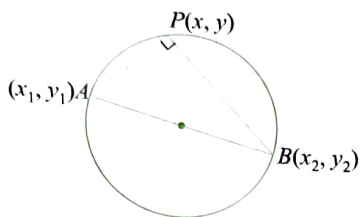


ILLUSTRATION 4.29

Find the equation of the circle which passes through $(1, 0)$ and $(0, 1)$ and has its radius as small as possible.

Sol. The radius will be minimum if the given points are the endpoints of a diameter.

Then, the equation of the circle is

$$(x-1)(x-0) + (y-0)(y-1) = 0$$

$$\text{or } x^2 + y^2 - x - y = 0$$

ILLUSTRATION 4.30

If the abscissa and ordinates of two points P and Q are the roots of the equations $x^2 + 2ax - b^2 = 0$ and $x^2 + 2px - q^2 = 0$, respectively, then find the equation of the circle with PQ as diameter.

Sol. Let x_1, x_2 and y_1, y_2 be the roots of $x^2 + 2ax - b^2 = 0$ and $x^2 + 2px - q^2 = 0$, respectively. Then,

$$x_1 + x_2 = -2a, x_1 x_2 = -b^2$$

$$\text{and } y_1 + y_2 = -2p, y_1 y_2 = -q^2$$

The equation of the circle with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as the endpoints of diameter is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

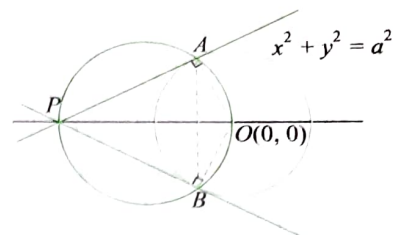
$$\text{or } x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1 x_2 + y_1 y_2 = 0$$

$$\text{or } x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$$

ILLUSTRATION 4.31

Tangents drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ touches the circle at A and B . Find the equation of the circumcircle of triangle PAB .

Sol.



From the figure, clearly the points O, A, P and B are concyclic and OP is diameter of the circle.

Thus, equation of circumcircle of triangle PAB is

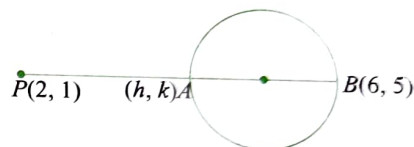
$$x(x-x_1) + y(y-y_1) = 0$$

$$\text{or } x^2 + y^2 - xx_1 - yy_1 = 0$$

ILLUSTRATION 4.32

The point on a circle nearest to the point $P(2, 1)$ is at distance 4 units and the farthest point is $(6, 5)$. Find the equation of the circle.

Sol.



In the figure, point nearest to $P(2, 1)$ on the circle is $A(h, k)$ such that $AP = 4$ units.

Also, point $B(6, 5)$ is on the circle which is farthest point from P . Clearly, AB is diameter of the circle.

$$\text{Now, } PB = \sqrt{(6-2)^2 + (5-1)^2} = 4\sqrt{2}$$

$$\therefore AB = PB - PA = 4(\sqrt{2} - 1)$$

$$\text{Thus, } \frac{AB}{AP} = \frac{\sqrt{2} - 1}{1}$$

$$\therefore A \equiv \left(\frac{6 + 2(\sqrt{2} - 1)}{1 + (\sqrt{2} - 1)}, \frac{5 + 1(\sqrt{2} - 1)}{1 + (\sqrt{2} - 1)} \right)$$

$$\equiv (2 + 2\sqrt{2}, 1 + 2\sqrt{2})$$

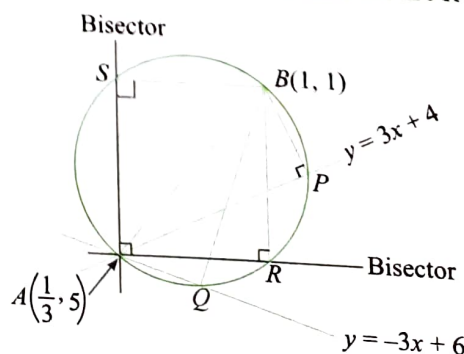
Hence, equation of circle having AB as diameter is

$$(x-6)(x-2-2\sqrt{2}) + (y-5)(y-1-2\sqrt{2}) = 0$$

ILLUSTRATION 4.33

Let P, Q, R and S be the feet of perpendiculars drawn from the point $(1, 1)$ upon the lines $y = 3x + 4$ and $y = -3x + 6$ and their angle bisectors, respectively. Find the equation of the circle extremities of one of whose diameters are R and S .

Sol.



Given lines intersect at point $A(1/3, 5)$.

Angle bisectors are always perpendicular.

So, if RS is diameter, then AB is also a diameter.

∴ Equation of the circle whose diameter is RS

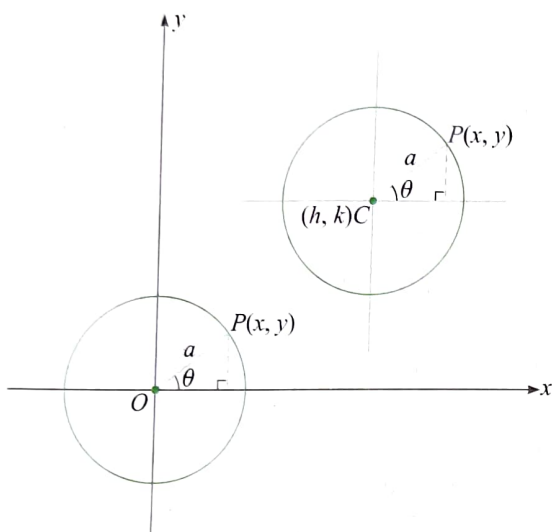
$$\equiv \text{Equation of circle whose diameter is } AB$$

$$\equiv (x-1)(x-1/3) + (y-5)(y-1) = 0$$

$$\equiv 3x^2 + 3y^2 - 4x - 18y + 16 = 0$$

EQUATION OF CIRCLE IN PARAMETRIC FORM

Since circle is bounded curve, its parametric equation has sine and cosine functions which are bounded functions.



For the circle $x^2 + y^2 = a^2$, parametric equation is $x = a \cos \theta$, $y = a \sin \theta$, $\theta \in [0, 2\pi)$.

Here, θ is called parameter, which is the angle of line joining point $P(a \cos \theta, a \sin \theta)$ and centre with the horizontal axis.

Point $P(a \cos \theta, a \sin \theta)$ is also expressed as $P(\theta)$. Here, ' θ ' is called parameter or parametric angle of point P .

For circle $(x-h)^2 + (y-k)^2 = a^2$, parametric equation is $x = h + a \cos \theta$, $y = k + a \sin \theta$, $\theta \in [0, 2\pi)$.

ILLUSTRATION 4.34

Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$.

Sol. The equation of the circle can be rewritten in the form

$$\left(x + \frac{p}{2}\right)^2 + \left(y + \frac{p}{2}\right)^2 = \frac{p^2}{2}$$

Therefore, the parametric form of the equation of the given circle is

$$x = -\frac{p}{2} + \frac{p}{\sqrt{2}} \cos \theta$$

$$= \frac{p}{2} (-1 + \sqrt{2} \cos \theta)$$

$$\text{and } y = -\frac{p}{2} + \frac{p}{\sqrt{2}} \sin \theta$$

$$= \frac{p}{2} (-1 + \sqrt{2} \sin \theta)$$

where $0 \leq \theta < 2\pi$.

ILLUSTRATION 4.35

Find the centre and radius of the circle whose parametric equation is $x = -1 + 2 \cos \theta$, $y = 3 + 2 \sin \theta$.

Sol. We have $\frac{x+1}{2} = \cos \theta$ and $\frac{y-3}{2} = \sin \theta$

Squaring and adding, we get

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$$

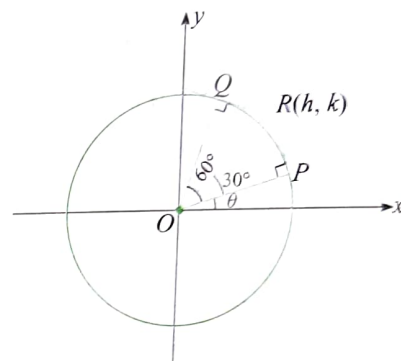
$$\text{or } (x+1)^2 + (y-3)^2 = 4$$

Centre of the circle is $(-1, 3)$ and radius is 2.

ILLUSTRATION 4.36

Find the locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ at points whose parametric angles differ by 60° .

Sol.



Let the points on the circle whose parametric angles differ by 60° be $P(a \cos \theta, a \sin \theta)$ and $Q(a \cos (\theta + 60^\circ), a \sin (\theta + 60^\circ))$.

Tangents at points P and Q intersect at $R(h, k)$.

In the figure, $\angle POQ = 60^\circ$ and $\angle POR = 30^\circ$.

In triangle OPR ,

$$OP = OR \cos 30^\circ$$

$$\therefore a = \sqrt{h^2 + k^2} \frac{\sqrt{3}}{2}$$

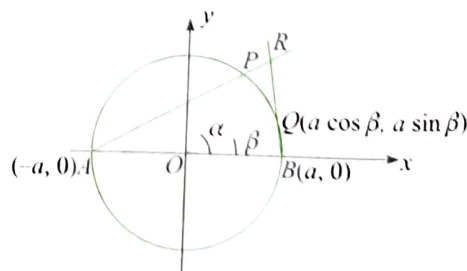
Hence, locus of R is $3(x^2 + y^2) = 4$.

ILLUSTRATION 4.37

A circle $x^2 + y^2 = a^2$ meets the x -axis at $A(-a, 0)$ and $B(a, 0)$.

$P(\alpha)$ and $Q(\beta)$ are two points on the circle so that $\alpha - \beta = 2\gamma$, where γ is a constant. Find the locus of the point of intersection of AP and BQ .

Sol. Let



The coordinates of A are $(-a, 0)$ and that of P are $(a \cos \alpha, a \sin \alpha)$. Therefore, the equation of AP is

$$y = \frac{a \sin \alpha}{a (\cos \alpha + 1)} (x + a)$$

$$\text{or } y = \tan \frac{\alpha}{2} (x + a) \quad (1)$$

Similarly, the equation of BQ is

$$y = \frac{a \sin \beta}{a (\cos \beta - 1)} (x - a)$$

$$\text{or } y = -\cot \frac{\beta}{2} (x - a) \quad (2)$$

We now eliminate α and β from (1) and (2). Therefore,

$$\tan \frac{\alpha}{2} = \frac{y}{a+x}, \quad \tan \frac{\beta}{2} = \frac{a-x}{y}$$

$$\text{Now, } \alpha - \beta = 2\gamma \Rightarrow \gamma = \frac{\alpha}{2} - \frac{\beta}{2}$$

$$\therefore \tan \gamma = \frac{\tan \frac{\alpha}{2} - \tan \frac{\beta}{2}}{1 + \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} = \frac{\frac{y}{a+x} - \frac{a-x}{y}}{1 + \frac{y}{a+x} \cdot \frac{a-x}{y}}$$

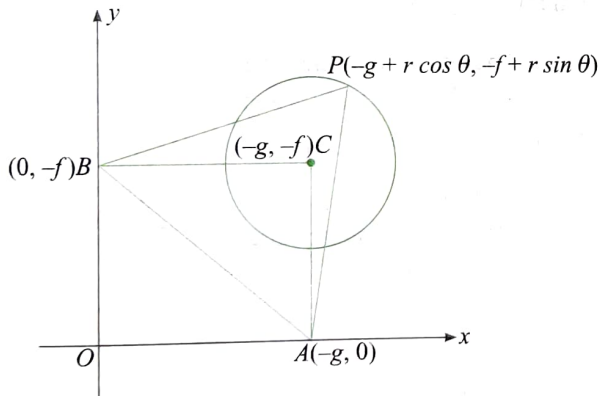
$$\text{or } \tan \gamma = \frac{y^2 - (a^2 - x^2)}{(a+x)y + (a-x)y} = \frac{x^2 + y^2 - a^2}{2ay}$$

$$\text{or } x^2 + y^2 - 2ay \tan \gamma = a^2$$

ILLUSTRATION 4.38

P is the variable point on the circle with centre at C . CA and CB are perpendiculars from C on x -axis and y -axis, respectively. Show that the locus of the centroid of triangle PAB is a circle with centre at the centroid of triangle CAB and radius equal to the one-third of the radius of the given circle.

Sol.



Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Let the centroid of triangle PAB be (h, k) .

$$\therefore 3h = -2g + r \cos \theta$$

$$\text{and } 3k = -2f + r \sin \theta$$

$$\Rightarrow (3h + 2g)^2 + (3k + 2f)^2 = r^2$$

$$\Rightarrow (x + 2g/3)^2 + (y + 2f/3)^2 = r^2/9$$

This is the required locus which is a circle.

Centre of the circle is $(-2g/3, -2f/3)$ which is centroid of the triangle CAB and radius $(r/3)$ which is one-third of the radius of the given circle.

EQUATION OF CIRCLE CIRCUMSCRIBING TRIANGLE AND QUADRILATERAL

Consider two second-degree equations in variables x and y .

$$S_1 \equiv a_1x^2 + b_1y^2 + 2h_1xy + 2g_1x + 2f_1y + c_1 = 0 \quad (1)$$

$$\text{and } S_2 \equiv a_2x^2 + b_2y^2 + 2h_2xy + 2g_2x + 2f_2y + c_2 = 0 \quad (2)$$

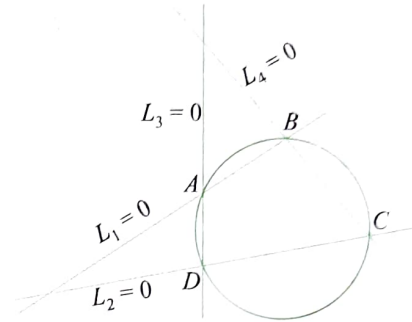
Second-degree equation satisfying the points of intersection of above two equations is given by

$$S_1 + \lambda S_2 = 0$$

If this equation represents equation of a circle, then the value of λ obtained by making coefficients of x^2 and y^2 equal and that by comparing coefficient of xy to zero must be equal.

Equation of circle circumscribing the quadrilateral formed by lines $L_1 = 0, L_2 = 0, L_3 = 0$ and $L_4 = 0$ is given by

$$L_1L_2 + \lambda L_3L_4 = 0$$



Here, $L_1L_2 = 0$ and $L_3L_4 = 0$ represent pairs of straight lines.

Equation of circle circumscribing the triangle formed by lines $L_1 = 0, L_2 = 0$ and $L_3 = 0$ is given by

$$L_1L_2 + \lambda L_2L_3 + \mu L_1L_3 = 0.$$

The values of λ and μ can be obtained by solving two equations formed by making coefficients of x^2 and y^2 equal and comparing coefficient of xy to zero.

ILLUSTRATION 4.39

Equations of sides AB, BC, CD and DA , respectively, are $x + y - 10, x - 7y + 50 = 0, 22x - 4y + 125 = 0$ and $2x - 4y - 5 = 0$. Prove that quadrilateral $ABCD$ is cyclic. Also, find the equation of circumcircle of $ABCD$.

Sol. If $ABCD$ is a cyclic quadrilateral, then the equation of circumcircle of $ABCD$ will be

$$(x + y - 10)(22x - 4y + 125) + \lambda(x - 7y + 50)(2x - 4y - 5) = 0$$

Since it represents a circle, we have

$$\text{Coefficient of } x^2 = \text{Coefficient of } y^2$$

$$\Rightarrow 22 + 2\lambda = -4 + 28\lambda$$

$$\Rightarrow \lambda = 1$$

Also, coefficient of xy is zero.

$$\therefore 22 - 4 + \lambda(-4 - 14) = 0$$

$$\therefore \lambda = 1$$

Thus, value of λ from two conditions is same.

Therefore, $ABCD$ is cyclic quadrilateral.

Also, equation of circle is

$$(x + y - 10)(22x - 4y + 125) + (x - 7y + 50)(2x - 4y - 5) = 0$$

$$\therefore 24x^2 + 24y^2 - 1500 = 0$$

$$\text{or } 2x^2 + 2y^2 = 125$$

CONCEPT APPLICATION EXERCISE 4.2

- Find the radius of the circle $(x - 5)(x - 1) + (y - 7)(y - 4) = 0$.
- Find the equations of the circles which pass through the origin and cut off chords of length a from each of the lines $y = x$ and $y = -x$.
- Find the equation of the circle passing through the origin and cutting intercepts of lengths 3 units and 4 units from the positive axes.
- Find the values of k for which the points $(2k, 3k)$, $(1, 0)$, $(0, 1)$, and $(0, 0)$ lie on a circle.
- If points A and B are $(1, 0)$ and $(0, 1)$, respectively, and point C is on the circle $x^2 + y^2 = 1$, then find the locus of the orthocenter of triangle ABC .

ANSWERS

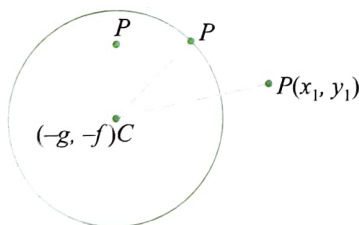
- 5/2
- $x^2 + y^2 \pm \sqrt{2}ax = 0$ and $x^2 + y^2 \pm \sqrt{2}ay = 0$
- $x^2 + y^2 - 3x - 4y = 0$
- 5/13
- $x^2 + y^2 - 2x - 2y + 1 = 0$

GEOMETRY WITH CIRCLE

POSITION OF POINT W.R.T. CIRCLE

Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Centre of the circle is $C(-g, -f)$.



Point $P(x_1, y_1)$ lies outside, on, or inside the circle accordingly as CP is greater than, equal to, or less than the radius, respectively. Mathematically,

$$\sqrt{(x_1 + g)^2 + (y_1 + f)^2} >, =, < \sqrt{g^2 + f^2 - c}$$

$$\text{or } x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$$

$$\text{or } S_1 >, =, < 0; \text{ where } S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

Thus, inequality $x^2 + y^2 + 2gx + 2fy + c < 0$ represents the region inside the circle and $x^2 + y^2 + 2gx + 2fy + c > 0$ represents the region outside the circle.

ILLUSTRATION 4.40

Find the values of α for which the point $(\alpha - 1, \alpha + 1)$ lies in the larger segment of the circle $x^2 + y^2 - x - y - 6 = 0$ made by the chord whose equation is $x + y - 2 = 0$.

Sol. The given circle

$$S(x, y) \equiv x^2 + y^2 - x - y - 6 = 0 \quad (1)$$

has center at $C \equiv (1/2, 1/2)$.

According to the required conditions, the given point $P(\alpha - 1, \alpha + 1)$ must lie inside the given circle, i.e.,

$$S(\alpha - 1, \alpha + 1) < 0$$

$$\text{or } (\alpha - 1)^2 + (\alpha + 1)^2 - (\alpha - 1) - (\alpha + 1) - 6 < 0$$

$$\text{or } \alpha^2 - \alpha - 2 < 0$$

$$\text{i.e., } (\alpha - 2)(\alpha + 1) < 0$$

$$\text{or } -1 < \alpha < 2 \quad (2)$$

Also, P and C must lie on the same side of the line

$$L(x, y) \equiv x + y - 2 = 0 \quad (3)$$

i.e., $L(1/2, 1/2)$ and $L(\alpha - 1, \alpha + 1)$ must have the same sign.

Now, since

$$L\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} - 2 < 0$$

$$\text{we have } L(\alpha - 1, \alpha + 1) = (\alpha - 1) + (\alpha + 1) - 2 < 0$$

$$\text{i.e., } \alpha < 1 \quad (4)$$

Inequalities (2) and (4) together give the possible values of α as $-1 < \alpha < 1$.

ILLUSTRATION 4.41

The circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect the coordinate axes, and the point $(1, 4)$ is inside the circle. Find the range of values of k .

Sol. The equation of the circle is

$$x^2 + y^2 - 6x - 10y + k = 0 \quad (1)$$

whose center is $C(3, 5)$ and radius

$$r = \sqrt{34 - k}$$

If the circle does not touch or intersect the x -axis, then radius

$$r < y\text{-coordinate of center } C$$

$$\text{or } \sqrt{34 - k} < 5$$

$$\text{or } 34 - k < 25$$

$$\text{or } k > 9 \quad (2)$$

Also, if the circle does not touch or intersect the y -axis, then the radius

$$r < x\text{-coordinate of center } C$$

$$\text{or } \sqrt{34 - k} < 3$$

$$\text{or } 34 - k < 9$$

$$\text{or } k > 25 \quad (3)$$

If the point $(1, 4)$ is inside the circle, then

Its distance from center $C < r$

$$\text{or } \sqrt{(3 - 1)^2 + (5 - 4)^2} < \sqrt{34 - k}$$

$$\text{or } 5 < 34 - k$$

$$\text{or } k < 29 \quad (4)$$

Now, all the conditions (2), (3), and (4) are satisfied if $25 < k < 29$ which is the required range of the values of k .

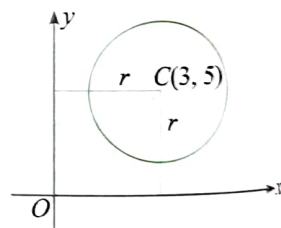
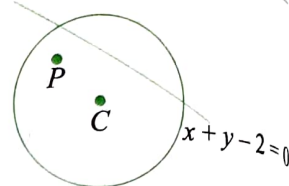
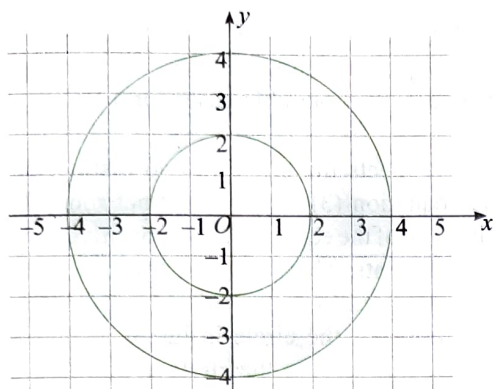


ILLUSTRATION 4.42

Find the area of the region in which the points satisfy the inequalities $4 < x^2 + y^2 < 16$ and $3x^2 - y^2 > 0$.

Sol. The points satisfying inequality $4 < x^2 + y^2 < 16$ lie in the region outside the circle $x^2 + y^2 = 4$ and inside the circle $x^2 + y^2 = 16$ as shown in the following figure.

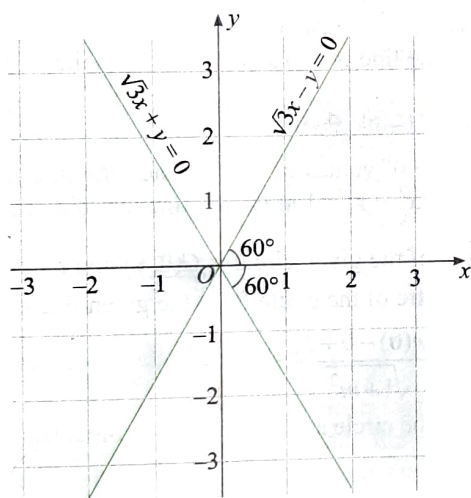


Now, $3x^2 - y^2 > 0$

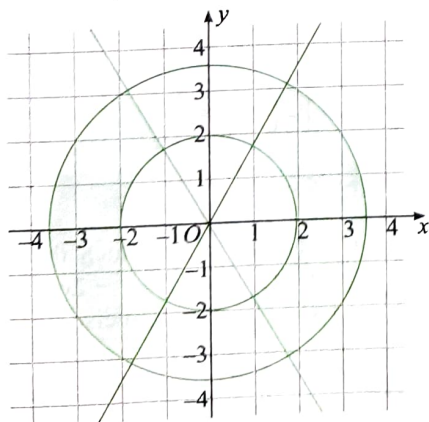
$$\therefore (\sqrt{3}x - y)(\sqrt{3}x + y) > 0$$

Now, draw the lines $\sqrt{3}x - y = 0$ and $\sqrt{3}x + y = 0$.

Therefore, $3x^2 - y^2 > 0$ represents the following region:



Hence, the region of the points satisfying both the inequalities is shown in the following figure.

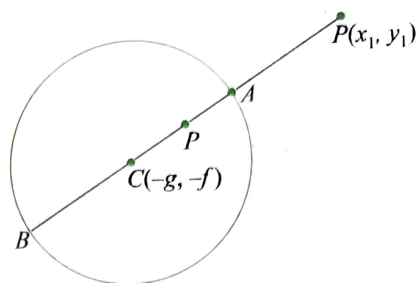


Hence, the required area is

$$\frac{4}{6} [\pi(4)^2 - \pi(2)^2] = 8\pi$$

MAXIMUM AND MINIMUM DISTANCE OF A POINT FROM THE CIRCLE

Consider point $P(x_1, y_1)$ and circle $x^2 + y^2 + 2gx + 2fy + c = 0$.



The centre of the circle is $C(-g, -f)$ and radius (r) is $\sqrt{g^2 + f^2 - c}$.

The maximum and minimum distance from $P(x_1, y_1)$ to the circle, respectively are

$$PB = CB + PC = r + PC$$

$$\text{and } PA = |CP - CA| = |CP - r|$$

ILLUSTRATION 4.43

Find the greatest and the least distances of point $P(10, 7)$ from circle $x^2 + y^2 - 4x - 2y - 20 = 0$.

Sol. Centre of the given circle is $C(2, 1)$ and radius is $r = \sqrt{(-2)^2 + (-1)^2 - (-20)} = 5$.

$$\text{Now, } PC = \sqrt{(10-2)^2 + (7-1)^2} = 10$$

Therefore,

$$\text{Greatest distance of point } P \text{ from circle} = PC + r = 10 + 5 = 15$$

$$\text{Least distance of point } P \text{ from circle} = |PC - r| = |10 - 5| = 5$$

ILLUSTRATION 4.44

Find the points on the circle $x^2 + y^2 - 2x + 4y - 20 = 0$ which are farthest and nearest to the point $(-5, 6)$.

Sol. The given circle is

$$x^2 + y^2 - 2x + 4y - 20 = 0$$

The center of the circle is $C(1, -2)$ and the radius is $r = 5$.

Here, point $P(-5, 6)$ lies outside the circle. Now,

$$\text{Slope of } CP = \frac{6 - (-2)}{-5 - 1} = -\frac{4}{3} = \tan \theta \text{ (say)}$$

Points A and B lie at distance of 5 units from the center C .

Now, points at distance 5 units from C on the line CP are

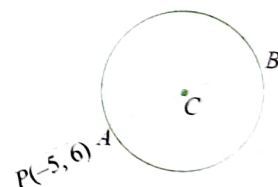
$$(1 \pm 5 \cos \theta, -2 \pm 5 \sin \theta)$$

$$\text{or } \left(1 \pm 5 \left(-\frac{3}{5} \right), -2 \pm 5 \left(\frac{4}{5} \right) \right)$$

$$\text{or } (1 \mp 3, -2 \pm 4)$$

$$\text{or } (-2, 2), (4, -6)$$

Clearly, point $A(-2, 2)$ is nearest to P and $B(4, -6)$ is farthest from P .



Alternative method:

$$CP = 10$$

Now, point A divides CP in ratio

$$\frac{AP}{AC} = \frac{CP - AC}{AC} = \frac{10 - 5}{5} = 1$$

Therefore, the coordinates of A are

$$\left(\frac{-5+1}{2}, \frac{6-2}{2} \right) \equiv (-2, 2)$$

Also, C is the midpoint of AB . Hence, the coordinates of B are $(4, -6)$.

ILLUSTRATION 4.45

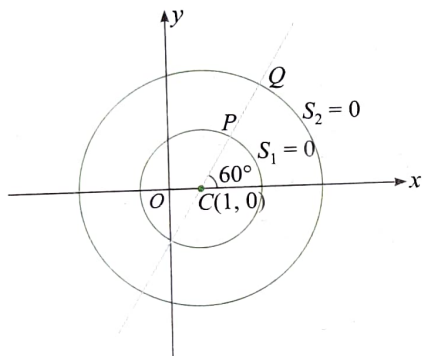
Find the positive values of a such that point $(a+1, \sqrt{3}a)$ lies inside the region bounded by the circles

$$x^2 + y^2 - 2x - 3 = 0 \text{ and } x^2 + y^2 - 2x - 15 = 0.$$

Sol. We have circles $(x-1)^2 + y^2 = 4$ (1)

and $(x-1)^2 + y^2 = 16$ (2)

Circles are concentric having centre at $C(1, 0)$ and radii 2 and 4.



Variable point $(a+1, \sqrt{3}a)$ lies on the line $y = \sqrt{3}(x-1)$, which passes through centre of the circles having inclination 60° .

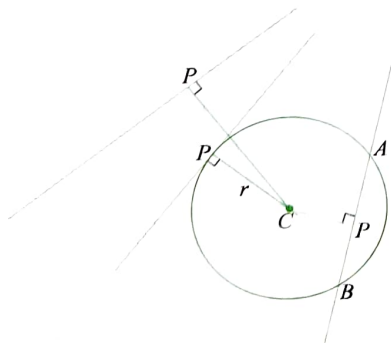
Line meets circles at $P(1+2\cos 60^\circ, 2\sin 60^\circ)$ and $Q(1+4\cos 60^\circ, 4\sin 60^\circ)$.

Thus, $P \equiv (2, \sqrt{3})$ and $Q \equiv (3, 2\sqrt{3})$

$$\therefore a \in (2, 3)$$

INTERSECTION OF LINE AND CIRCLE

A line may not meet the circle at all or meet the circle in one or two points.



Consider the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

and the line

$$y = mx + \alpha \quad (2)$$

Centre of the circle is $C(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$.

Solving circle and line, we have

$$x^2 + (mx + \alpha)^2 + 2gx + 2f(mx + \alpha) + c = 0 \quad (3)$$

This equation is quadratic in x .

Now, there are three possible cases for a line and a circle in a plane.

- If the line meets the circle in two distinct points A and B , then equation (3) has two distinct roots. In that case, discriminant of the equation will be positive. Geometrically, distance of centre C of the circle from the line is less than the radius, i.e., $CP < r$.
- If the line meets the circle in just one point P , then we say that it is tangent to the circle at point P . In that case, equation (3) has two identical roots and so, discriminant of the equation will be zero. Geometrically, distance of centre C of the circle from the line is equal to the radius, i.e., $CP = r$.
- If the line doesn't meet the circle, then equation (3) has non-real roots. So, discriminant of the equation will be negative. Geometrically, distance of centre of the circle C from the line is more than the radius, i.e., $CP > r$.

ILLUSTRATION 4.46

Find the range of values of m for which the line $y = mx + 2$ cuts the circle $x^2 + y^2 = 1$ at two distinct or coincident points.

Sol. Centre of the given circle is $C(0, 0)$ and radius is 1. Distance of centre of the circle from the given line is

$$CP = \frac{|m(0) - 0 + 2|}{\sqrt{1+m^2}} = \frac{2}{\sqrt{1+m^2}}$$

If the line cuts the circle at two distinct or coincident points, then

$$CP \leq 1$$

$$\therefore \frac{2}{\sqrt{1+m^2}} \leq 1$$

$$\Rightarrow 1 + m^2 \geq 4$$

$$\Rightarrow m^2 \geq 3$$

$$\Rightarrow m \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$$

ILLUSTRATION 4.47

Find the range of parameter a for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.

Sol. The given circles are

$$C_1: (x-1)^2 + (y-1)^2 = 1, r_1 = 1$$

$$\text{and } C_2: (x-8)^2 + (y-1)^2 = 4, r_2 = 2$$

The line $y - 2x - a = 0$ will lie between these circles if the centers of the circles lie on opposite sides of the line, i.e.,

$$(1 - 2 - a)(1 - 16 - a) < 0$$

$$\text{or } a \in (-15, -1)$$

The line will not touch or intersect the circles if

$$\frac{|1 - 2 - a|}{\sqrt{5}} > 1, \frac{|1 - 16 - a|}{\sqrt{5}} > 2$$

$$\text{or } |1 + a| > \sqrt{5}, |15 + a| > 2\sqrt{5}$$

$$\text{i.e., } a > \sqrt{5} - 1 \text{ or } a < \sqrt{5} - 1$$

$$\text{and } a > 2\sqrt{5} - 15 \text{ or } a < -2\sqrt{5} - 15$$

Hence, the common values of a are $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

ILLUSTRATION 4.48

Let $A \equiv (-1, 0)$, $B \equiv (3, 0)$, and PQ be any line passing through $(4, 1)$ having slope m . Find the range of m for which there exist two points on PQ at which AB subtends a right angle.

Sol. The equation of line PQ is

$$(y - 1) = m(x - 4)$$

$$\text{or } y - mx + 4m - 1 = 0$$

For the required m , we have to make sure that the line PQ meets the circle, with diameter AB , at real and distinct points.

The equation of the circle having AB as diameter is

$$x^2 + y^2 - 2x - 3 = 0$$

$$\therefore \frac{|0 - m + 4m - 1|}{\sqrt{1 + m^2}} < 2$$

$$\text{or } 5m^2 - 6m - 3 < 0$$

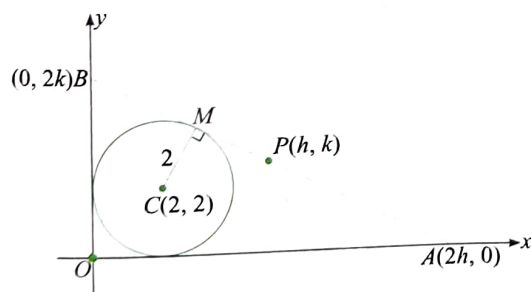
$$\text{or } m \in \left(\frac{3 - 2\sqrt{6}}{5}, \frac{3 + 2\sqrt{6}}{5} \right)$$

ILLUSTRATION 4.49

The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a variable triangle OAB . Sides OA and OB lie along the x - and y - axes respectively, where O is the origin. Find the locus of the midpoint of side AB .

Sol. Given equation of circle is $(x - 2)^2 + (y - 2)^2 = 2^2$.

It touches both the axes.



Let the midpoint of AB be $P(h, k)$.

$$\therefore A \equiv (2h, 0) \text{ and } B \equiv (0, 2k)$$

So, the equation of AB is $\frac{x}{2h} + \frac{y}{2k} = 1$.

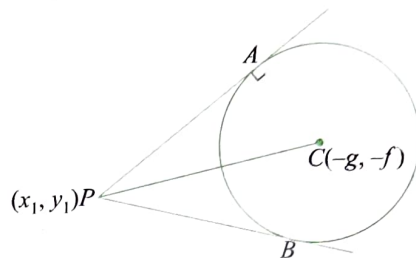
Since this line touches the given circle, we have

$$\frac{\left| \frac{2}{2h} + \frac{2}{2k} - 1 \right|}{\sqrt{\frac{1}{4h^2} + \frac{1}{4k^2}}} = 2$$

On simplifying, we get locus of point P as $x + y - xy + \sqrt{x^2 + y^2} = 0$.

LENGTH OF TANGENT TO CIRCLE

Let the tangents drawn to circle from point $P(x_1, y_1)$ touch the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at points A and B .



We know that lengths of tangent PA and PB are equal.

Centre of the circle is $C(-g, -f)$.

$$\text{Now, } PA^2 = CP^2 - CA^2$$

$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Therefore, length of tangent to circle from point $P(x_1, y_1)$ is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$

$$\text{where } S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

ILLUSTRATION 4.50

The lengths of the tangents from $P(1, -1)$ and $Q(3, 3)$ to a circle are $\sqrt{2}$ and $\sqrt{6}$, respectively. Then, find the length of the tangent from $R(-1, -5)$ to the same circle.

Sol. Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Let the tangents from points P, Q, R touch the circle at A, B, C , respectively. According to question,

$$PA^2 = 2$$

$$\text{or } 1 + 1 + 2g - 2f + c = 2$$

$$\text{or } 2g - 2f + c = 0 \quad (1)$$

$$\text{and } PB^2 = 6$$

$$\text{or } 9 + 9 + 6g + 6f + c = 6$$

$$\text{or } 6g + 6f + c = -12 \quad (2)$$

From (1) and (2),

$$g = -1 - \frac{c}{3}, f = -1 + \frac{c}{6}$$

$$\text{Now, } RC^2 = 1 + 25 - 2g - 10f + c$$

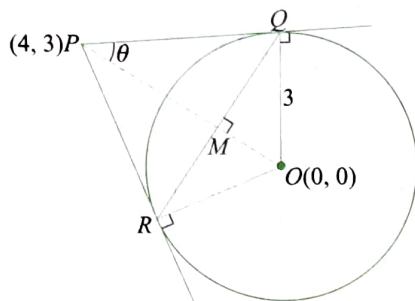
$$= 26 + 2 + \frac{2c}{3} + 10 - \frac{5c}{3} + c$$

$$= 38$$

$$\therefore RC = \sqrt{38}$$

ILLUSTRATION 4.51

Find the area of the triangle formed by the tangents from the point $(4, 3)$ to the circle $x^2 + y^2 = 9$ and the line joining their points of contact.

Sol.

In the figure, $OP = 5$.

$$\therefore PQ = 4.$$

In triangle OQP, $\tan \theta = \frac{3}{4}$

$$\begin{aligned} \therefore \text{Area of triangle } PQR &= \frac{1}{2} PQ \times PR \sin 2\theta \\ &= \frac{1}{2} \times 4 \times 4 \times \frac{2 \tan \theta}{1 + \tan^2 \theta} \\ &= 8 \frac{2 \left(\frac{3}{4}\right)}{1 + \left(\frac{3}{4}\right)^2} = \frac{192}{25} \text{ sq. unit} \end{aligned}$$

ILLUSTRATION 4.52

C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1 . From a point P on C_2 , tangents PA and PB are drawn to C_1 . Prove that the centroid of triangle PAB lies on C_1 .

Sol. Let r be the radius of C_1 and $2r$ be the radius of C_2 . Then, $OA = OB = r$

and $OP = 2r$

Since PA and PB are tangents to C_1 ,

$$\angle OAP = \angle OBP = 90^\circ$$

Let OP meet C_1 at G .

Let $\angle OPA = \theta$. Then,

$$\sin \theta = \frac{OA}{OP} = \frac{r}{2r} = \frac{1}{2}$$

or $\theta = 30^\circ$

or $\angle AOP = 60^\circ$

In $\triangle OAC$,

$$\cos 60^\circ = \frac{OC}{OA}$$

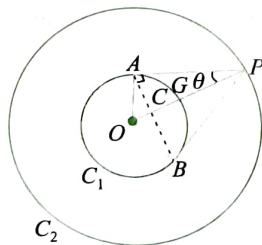
$$\text{or } \frac{1}{2} = \frac{OC}{r} \text{ or } OC = \frac{r}{2}$$

$$\therefore PC = 2r - \frac{r}{2} = \frac{3r}{2}$$

$$\text{Also, } CG = r - OC = r - \frac{r}{2} = \frac{r}{2}$$

$$\text{Clearly, } CG = \frac{1}{3} PC$$

Therefore, G is the centroid of $\triangle ABP$.

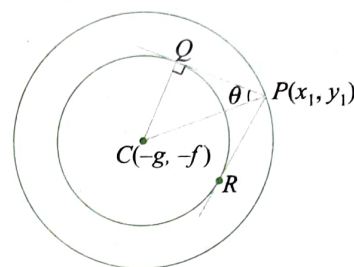
**ILLUSTRATION 4.53**

If from any point P on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, tangents are drawn to the circle

$x^2 + y^2 + 2gx + 2fy + c \sin^2 \alpha + (g^2 + f^2) \cos^2 \alpha = 0$, then find the angle between the tangents ($0 < \alpha < \pi/2$).

Sol. Given circles are $x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

$$\text{and } x^2 + y^2 + 2gx + 2fy + c \sin^2 \alpha + (g^2 + f^2) \cos^2 \alpha = 0 \quad (2)$$



Let $P(x_1, y_1)$ be a point on the circle (1).

$$\therefore x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad (3)$$

The length of the tangent drawn from $P(x_1, y_1)$ to the circle (2) is

$$\begin{aligned} PQ &= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \sin^2 \alpha + (g^2 + f^2) \cos^2 \alpha} \\ &= \sqrt{-c + c \sin^2 \alpha + (g^2 + f^2) \cos^2 \alpha} \quad (\text{Using (1)}) \\ &= (\sqrt{g^2 + f^2 - c}) \cos \alpha \end{aligned}$$

The radius of the circle (2) is

$$\begin{aligned} CQ &= \sqrt{g^2 + f^2 - c \sin^2 \alpha - (g^2 + f^2) \cos^2 \alpha} \\ &= (\sqrt{g^2 + f^2 - c}) \sin \alpha \end{aligned}$$

In $\triangle CPQ$,

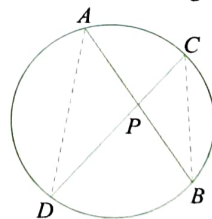
$$\tan \theta = \frac{CQ}{PQ} = \frac{\sqrt{g^2 + f^2 - c} \sin \alpha}{\sqrt{g^2 + f^2 - c} \cos \alpha} = \tan \alpha$$

$$\therefore \theta = \alpha$$

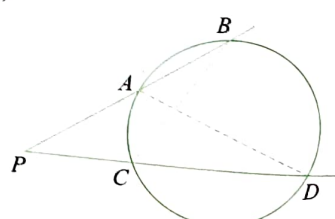
Therefore, angle between tangents is 2α .

SECANT OF CIRCLE

Secants AB and CD intersect inside the circle in Fig. (a) and outside the circle in Fig. (b).



(a)



(b)

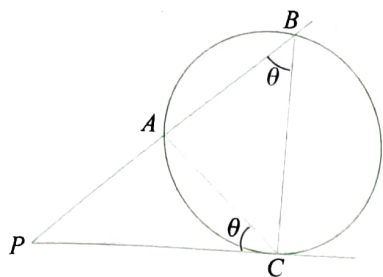
From the figure, we have

$$\angle DCB = \angle DAB \text{ and } \angle ADC = \angle ABC$$

Hence, $\triangle ADP \sim \triangle CBP$

$$\therefore \frac{AP}{CP} = \frac{DP}{BP}$$

$$\text{or } AP \times BP = CP \times DP$$



In the above figure, secant through P cuts the circle at A and B .
Tangent through P touches the circle at C .

Now, $\triangle PCB \sim \triangle PAC$

$$\therefore \frac{PC}{PA} = \frac{PB}{PC}$$

$$\text{or } PC^2 = PA \times PB$$

Alternative method:

Let the slope of variable line through $P(x_1, y_1)$ be $\tan \theta$.

Also, let the line cut the circle at A and B .

Equation of line is

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

$$\therefore x = x_1 + r \cos \theta, y = y_1 + r \sin \theta$$

Here, r is algebraic distance of the point (x, y) from the point $P(x_1, y_1)$ on the line.

Solving line with the circle, we have

$$(x_1 + r \cos \theta)^2 + (y_1 + r \sin \theta)^2 + 2gx_1 + 2fy_1 + c = 0$$

$$\Rightarrow r^2 + 2r(x_1 \cos \theta + y_1 \sin \theta + g \cos \theta + f \sin \theta) + (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = 0 \quad (1)$$

This is quadratic equation in r . The roots of the equation are lengths PA and PB .

$\therefore$ Product of roots,

$$PA \times PB = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = \text{constant.}$$

Also, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = PC^2$, where C is the point of contact in above figure.

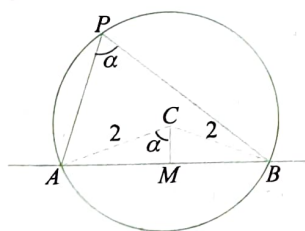
$$\therefore PA \times PB = PC^2$$

ILLUSTRATION 4.54

Find the length of the chord $x^2 + y^2 - 4y = 0$ along the line $x + y = 1$. Also find the angle that the chord subtends at the circumference of the larger segment.

Sol. Given circle is $x^2 + y^2 - 4y = 0$

Given chord is $x + y = 1$



Centre of the circle is $C(0, 2)$ and radius is $CA = 2$.

AB is chord of the circle.

M is foot of perpendicular from centre on the chord AB .

Clearly, M is midpoint of AB .

$$\text{Now, } CM = \frac{|0 + 2 - 1|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Length of chord, $AB = 2AM$

$$= 2\sqrt{CA^2 - CM^2} = 2\sqrt{4 - \frac{1}{2}} = \sqrt{14}$$

$$\text{Also, } \sin \alpha = \frac{\sqrt{7}}{2\sqrt{2}} \text{ or } \alpha = \sin^{-1} \frac{\sqrt{7}}{2\sqrt{2}}$$

ILLUSTRATION 4.55

If the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points, then prove that $|a_1a_2| = |b_1b_2|$.

Sol. Line $a_1x + b_1y + c_1 = 0$ meets the axes at points $P\left(-\frac{c_1}{a_1}, 0\right)$ and $Q\left(0, -\frac{c_1}{b_1}\right)$.

Line $a_2x + b_2y + c_2 = 0$ meets the axes at points $R\left(-\frac{c_2}{a_2}, 0\right)$ and $S\left(0, -\frac{c_2}{b_2}\right)$.

Since points P, Q, R and S are concyclic, we have

$OP \times OR = OQ \times OS$, where O is origin.

$$\Rightarrow \left| \left(-\frac{c_1}{a_1}\right) \left(-\frac{c_2}{a_2}\right) \right| = \left| \left(-\frac{c_1}{b_1}\right) \left(-\frac{c_2}{b_2}\right) \right|$$

$$\Rightarrow |a_1a_2| = |b_1b_2|$$

ILLUSTRATION 4.56

If a line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2 + y^2 = a^2$ at A and B , then find the value of $PA \cdot PB$.

Sol. From the figure,

$$PA \cdot PB = \text{Constant}$$

$$\text{Also, } PA \cdot PB = PC^2$$

$$\text{But } PC^2 = OP^2 - OC^2$$

$$= \alpha^2 + \beta^2 - a^2$$

$$\text{Hence, } PA \cdot PB = \alpha^2 + \beta^2 - a^2$$

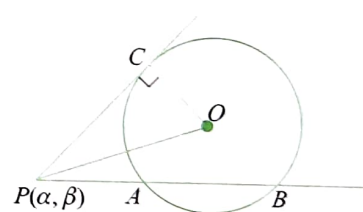


ILLUSTRATION 4.57

Two circles C_1 and C_2 intersect at two distinct points P and Q in a plane. Let a line passing through P meets circles C_1 and C_2 at A and B , respectively. Let Y be the midpoint of AB , and QY meets circles C_1 and C_2 at X and Z , respectively. Then prove that Y is the midpoint of XZ .

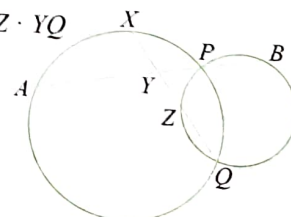
Sol. From the figure, $YP \cdot YB = YZ \cdot YQ$

$$\text{and } YA \cdot YP = YX \cdot YQ$$

$$\text{But } YA = YB$$

$$\text{Hence, } XY = YZ$$

So, Y is midpoint of XZ .



EQUATION OF CHORD OF CIRCLE BISECTED AT GIVEN POINT

Let chord AB of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is bisected at $M(x_1, y_1)$.

Also, foot of perpendicular from centre on the chord AB is M .

$$\text{Slope of } CM = \frac{y_1 + f}{x_1 + g}$$

$$\therefore \text{Slope of chord } AB = -\frac{x_1 + g}{y_1 + f}$$

Therefore, equation of chord AB is

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

$$\text{or } yy_1 + fy - y_1^2 - fy_1 = -xx_1 + x_1^2 - gx + gx_1$$

$$\Rightarrow xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

$$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

(Adding $gx_1 + fy_1 + c$ on both sides)

$$\Rightarrow T = S_1; \text{ where } T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

$$\text{and } S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

This chord is the farthest chord from the centre of the circle among all the chords passing through the point $M(x_1, y_1)$.

Since $AM^2 = AC^2 - CM^2$ (when CM is maximum, AM is minimum), this chord is of minimum length among all the chords passing through the point M .

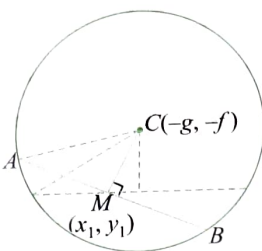


ILLUSTRATION 4.58

Find the equation of chord of the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ passing through the point $(2, 3)$ which has shortest length.

Sol. Clearly required chord of the circle is that one which is bisected at point $(2, 3)$.

So, using $T = S_1$, equation of chord is

$$2x + 3y - (x + 2) - 2(y + 3) - 4 = 2^2 + 3^2 - 2(2) - 4(3) - 4$$

$$\text{or } x + y - 5 = 0$$

ILLUSTRATION 4.59

A variable chord of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through the point $P(x_1, y_1)$. Find the locus of the midpoint of the chord.

Sol. Let $M(h, k)$ be the midpoint of the variable chord AB .

$\therefore$ Equation of the chord AB is

$$hx + ky + g(x + h) + f(y + k) + c = h^2 + k^2 + 2gh + 2fk + c$$

(Using $T = S_1$)

Since chord AB passes through $P(x_1, y_1)$, we have

$$hx_1 + ky_1 + g(x_1 + h) + f(y_1 + k) + c = h^2 + k^2 + 2gh + 2fk + c$$

$$\text{or } hx_1 + ky_1 + gx_1 + fy_1 = h^2 + k^2 + gh + fk$$

$\therefore$ Equation of locus of point M is

$$xx_1 + yy_1 + gx_1 + fy_1 = x^2 + y^2 + gx + fy$$

Alternative method:

Since $CM \perp AB$, slope of $CM \times$ Slope of $AB = -1$

$$\therefore \frac{h + g}{k + f} \times \frac{h - x_1}{k - y_1} = -1$$

$$\text{or } (x + g)(x - x_1) + (y + f)(y - y_1) = 0$$

or $x^2 + y^2 + gx + fy - gx_1 - fy_1 - xx_1 - yy_1 = 0$, which is equation of required locus.

CONCEPT APPLICATION EXERCISE 4.3

- Find the angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$.
- If the join of (x_1, y_1) and (x_2, y_2) makes an obtuse angle at (x_3, y_3) , then prove that $(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) < 0$.
- Triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates $(3, 4)$ and $(-4, 3)$, respectively, then find $\angle QPR$.
- Find the locus of centre of a circle which passes through the origin and cuts off a length of 4 units on the line $x = 3$.
- Find the least distance of the line $8x - 4y + 73 = 0$ from the circle $16x^2 + 16y^2 + 48x - 8y - 43 = 0$.
- If the length of tangent drawn from the point $(5, 3)$ to the circle $x^2 + y^2 + 2x + ky + 17 = 0$ is 7, then find the value of k .
- If the length of the tangent from any point on the circle $(x - 3)^2 + (y + 2)^2 = 5r^2$ to the circle $(x - 3)^2 + (y + 2)^2 = r^2$ is 4 units, then find the area between the circles.
- Find the locus of a point which moves so that the ratio of the length of the tangents to the circles $x^2 + y^2 + 4x + 3 = 0$ and $x^2 + y^2 - 6x + 5 = 0$ is $2 : 3$.
- Find the length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_2 = 0$.
- A tangent is drawn to each of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$. Show that if the two tangents are mutually perpendicular, the locus of their point of intersection is a circle concentric with the given circles.
- Find the equation of chord AB of the circle $x^2 + y^2 = r^2$ passing through the point $P(1, 1)$ such that $\frac{PB}{PA} = \frac{\sqrt{2} + r}{\sqrt{2} - r}$, $(0, r < \sqrt{2})$.
- If a circle passes through points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then find the value of λ .
- Two variable chords AB and BC of a circle $x^2 + y^2 = r^2$ are such that $AB = BC = r$. Find the locus of the point of intersection of tangents at A and C .
- If the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ intersects the line $3x - 4y = m$ at two distinct points, then find the values of m .
- Three concentric circles, of which the biggest is $x^2 + y^2 = 1$, have their radii in AP. If the line $y = x + 1$ cuts all the circles at real and distinct points, then find the interval in which the common difference of AP will lie.
- Find the middle point of the chord of the circle $x^2 + y^2 = 25$ intercepted on the line $x - 2y = 2$.
- Find the locus of the midpoint of the chord of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$, which makes an angle of 120° at the centre.

18. Through a fixed point (h, k) , secants are drawn to the circle $x^2 + y^2 = r^2$. Show that the locus of the midpoints of the secants by the circle is $x^2 + y^2 = hx + ky$.

ANSWERS

- | | | |
|--------------------------------|---|------------------------|
| 1. 90° | 3. $\pi/4$ | 4. $y^2 + 6x = 13$ |
| 5. $2\sqrt{5}$ | 6. -4 | 7. 16π |
| 8. $5x^2 + 5y^2 + 60x + 7 = 0$ | 9. $\sqrt{c_2 - c_1}$ | |
| 11. $y = x$ | 12. $\lambda = 2$ | 13. $x^2 + y^2 = 4r^2$ |
| 14. $-35 < m < 15$ | 15. $d \in \left(0, \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right)\right)$ | |
| 16. $(2/5, -4/5)$ | 17. $(x-1)^2 + (y-1)^2 = 1$ | |

TANGENT AND NORMAL TO CIRCLE

EQUATION OF TANGENT TO CIRCLE AT A GIVEN POINT ON THE CIRCLE

Consider circle having equation $x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

We want to find the equation of tangent at point $P(x_1, y_1)$ on the circle.

Centre of the circle is $C(-g, -f)$.

CP is perpendicular to tangent at point P .

$$\text{Slope of } CP = \frac{y_1 + f}{x_1 + g}$$

$$\therefore \text{Slope of tangent at point } P = -\frac{x_1 + g}{y_1 + f}$$

Slope of tangent can also be obtained using differentiation.

Differentiating equation (1), w.r.t. x , we get

$$2x + 2y \frac{dy}{dx} + 2g + 2f \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x+g}{y+f}$$

This is the slope of tangent at point (x, y) on the circle.

So, the slope of tangent at point $P(x_1, y_1)$ on the circle is

$$-\frac{x_1 + g}{y_1 + f}$$

Then equation of tangent at point P is

$$y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)$$

$$\text{or } xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$$

$$\text{or } xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

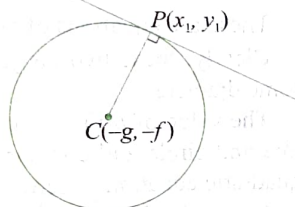
[Adding $(gx_1 + fy_1 + c)$ on both sides]

$$\text{or } xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

[As point (x_1, y_1) lies on the circle]

$$\text{or } T = 0, \text{ where } T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

Expression T is obtained by replacing x^2 by xx_1 , y^2 by yy_1 , $2x$ by $(x + x_1)$ and $2y$ by $(y + y_1)$ in the equation of circle.



Equation of tangent to circle $(x - h)^2 + (y - k)^2 = r^2$ at point $P(x_1, y_1)$ on it is given by $(x - h)(x_1 - h) + (y - k)(y_1 - k) = r^2$.

Equation of tangent to circle $x^2 + y^2 = a^2$ at point $P(\theta)$ i.e., $P(a \cos \theta, a \sin \theta)$ is $(a \cos \theta)x + (a \sin \theta)y = a^2$ or $(\cos \theta)x + (\sin \theta)y = a$.

ILLUSTRATION 4.60

Tangent to circle $x^2 + y^2 = 5$ at $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$. Find the coordinate of the corresponding point of contact.

Sol. Equation of tangent to $x^2 + y^2 = 5$ at $(1, -2)$ is $x - 2y - 5 = 0$.

Solving this with the second circle, we get

$$(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$$

$$\Rightarrow 5y^2 + 10y + 5 = 0$$

$$\Rightarrow (y + 1)^2 = 0$$

$$\Rightarrow y = -1$$

$$\Rightarrow x = -2 + 5 = 3$$

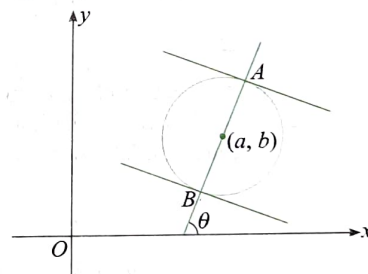
Thus, point of contact on second circle is $(3, -1)$.

ILLUSTRATION 4.61

Find the equation of tangent at the end points of diameter of circle $(x - a)^2 + (y - b)^2 = r^2$ which is inclined at an angle θ with positive x -axis.

Sol. Diameter makes an angle θ with x -axis.

So, the slope of diameter is $\tan \theta$.



Let this diameter meet the circle at points A and B .

Thus, the coordinates of A and B are given by $(a \pm r \cos \theta, b \pm r \sin \theta)$.

Therefore, equations of tangents at points A and B are given by

$$(x - a)(a \pm r \cos \theta - a) + (y - b)(b \pm r \sin \theta - b) = r^2$$

$$\text{or } \pm(x - a) \cos \theta \pm r(y - b) \sin \theta = r$$

$$\text{or } (x - a) \cos \theta + (y - b) \sin \theta = \pm r$$

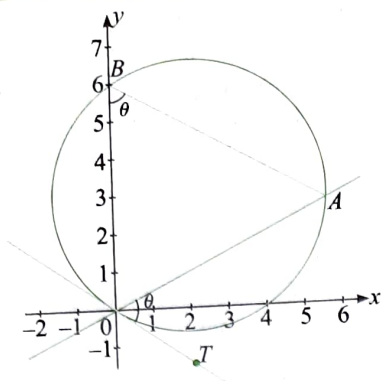
ILLUSTRATION 4.62

A chord of the circle $x^2 + y^2 - 4x - 6y = 0$ passing through the origin subtends an angle $\tan^{-1}\left(\frac{7}{4}\right)$ at the point where the circle meets positive y -axis. Find the equation of the chord.

Sol. Chord OA subtends an angle $\theta = \tan^{-1}\left(\frac{7}{4}\right)$ at point B on y -axis.

From the geometrical properties of the circle, we know that

$$\angle AOB = \angle AOT, \text{ where } OT \text{ is tangent at origin.}$$



Equation of tangent at (0, 0) is

$$-2(x + 0) - 3(y + 0) = 0$$

or $2x + 3y = 0$

Let the slope of OA be m .

$$\therefore \frac{7}{4} = \left| \frac{m + \frac{2}{3}}{1 - \frac{2}{3}m} \right|$$

$$\Rightarrow \frac{7}{4} = \left| \frac{3m + 2}{3 - 2m} \right|$$

$$\Rightarrow 21 - 14m = 12m + 8$$

$$\Rightarrow 26m = 13$$

$$\Rightarrow m = 1/2$$

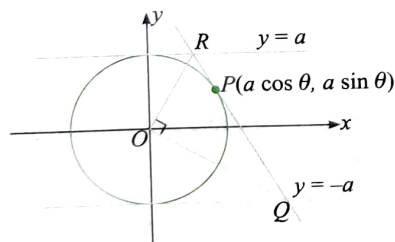
(Taking positive sign)

Therefore, the equation of chord OA is $x - 2y = 0$.

ILLUSTRATION 4.63

Two parallel tangents to a given circle are cut by a third tangent in the points R and Q. Show that the lines from R and Q to the centre of the circle are mutually perpendicular.

Sol.



Consider the circle $x^2 + y^2 = a^2$.

Let the two parallel tangents be $y = a$ and $y = -a$.

Third tangent at point $P(a \cos \theta, a \sin \theta)$ is $\cos \theta x + \sin \theta y = a$.

Solving these with $y = \pm a$, we get $x = \frac{a \mp a \sin \theta}{\cos \theta}$.

$$\therefore R \equiv \left(\frac{a - a \sin \theta}{\cos \theta}, a \right) \text{ and } Q \equiv \left(\frac{a + a \sin \theta}{\cos \theta}, -a \right)$$

$$\text{Slope of } OR, m_1 = \frac{a}{\frac{a - a \sin \theta}{\cos \theta}} = \frac{\cos \theta}{1 - \sin \theta}$$

$$\text{Slope of } OQ, m_2 = \frac{-a}{\frac{a + a \sin \theta}{\cos \theta}} = -\frac{\cos \theta}{1 + \sin \theta}$$

$$m_1 \times m_2 = \frac{\cos \theta}{1 - \sin \theta} \times \left(-\frac{\cos \theta}{1 + \sin \theta} \right) = -\frac{\cos^2 \theta}{1 - \sin^2 \theta} = -1$$

Thus, QR subtends right angle at centre.

EQUATION OF TANGENT TO CIRCLE HAVING GIVEN SLOPE

Consider a circle having the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

Centre of the circle is $C(-g, -f)$.

Let line $y = mx + \alpha$ touches the circle at point P.

$\therefore CP = \text{radius}$

$$\Rightarrow \frac{|m(-g) + f + \alpha|}{\sqrt{m^2 + 1}} = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow -mg + f + \alpha = \pm \sqrt{m^2 + 1} \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \alpha = mg - f \pm \sqrt{m^2 + 1} \sqrt{g^2 + f^2 - c}$$

Putting the value of α in $y = mx + \alpha$, we get

$$y = mx + mg - f \pm \sqrt{m^2 + 1} \sqrt{g^2 + f^2 - c}$$

or $y + f = m(x + g) \pm \sqrt{m^2 + 1} \sqrt{g^2 + f^2 - c}, m \in R$

These are equations of tangents to circle (1) having slope m .

Clearly, these two tangents are parallel at the end points of some diameter.

The value of α in $y = mx + \alpha$ can also be obtained by solving line and circle and comparing the discriminant of the resulting quadratic equation to zero.

For circle $x^2 + y^2 = a^2$, equation of tangent having slope m is

$$y = mx \pm a\sqrt{m^2 + 1}$$

Thus, if line $y = mx + \alpha$ touches the circle $x^2 + y^2 = a^2$, then $\alpha^2 = a^2 m^2 + a^2$.

To find the point of contact, we solve the line and circle.

$$\therefore x^2 + (mx + \alpha)^2 = a^2$$

$$\Rightarrow (1 + m^2)x^2 + 2\alpha mx + \alpha^2 - a^2 = 0$$

Since discriminant of the above quadratic is zero, we have

$$x = \frac{-\alpha m}{m^2 + 1}$$

$$\therefore y = \frac{-\alpha m^2}{m^2 + 1} + \alpha = \frac{\alpha}{m^2 + 1}$$

So, point of contact is $\left(\frac{-\alpha m}{m^2 + 1}, \frac{\alpha}{m^2 + 1} \right)$.

Putting the value of α , we get the points of contact as

$$\left(\frac{\pm a\sqrt{m^2 + 1} \cdot m}{m^2 + 1}, \frac{\mp a\sqrt{m^2 + 1}}{m^2 + 1} \right)$$

or $\left(\frac{\pm am}{\sqrt{m^2 + 1}}, \frac{\mp a}{\sqrt{m^2 + 1}} \right)$.

For tangent $y = mx + a\sqrt{m^2 + 1}$, the point of contact is

$$\left(\frac{-am}{\sqrt{m^2 + 1}}, \frac{a}{\sqrt{m^2 + 1}} \right).$$

For tangent $y = mx - a\sqrt{m^2 + 1}$, the point of contact is $\left(\frac{am}{\sqrt{m^2 + 1}}, \frac{-a}{\sqrt{m^2 + 1}}\right)$.

ILLUSTRATION 4.64

Find the equations of tangents to the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the straight line $4x + 3y + 5 = 0$.

Sol. Given circle is $x^2 + y^2 - 6x + 4y - 12 = 0$.

Centre of the circle is $(3, -2)$ and radius is

$$\sqrt{(-3)^2 + (2)^2 - (-12)} = 5.$$

Given that tangents are parallel to the line $4x + 3y + 5 = 0$.

Therefore, the equations of tangents take the form $4x + 3y + c = 0$.

Distance of centre of the circle from the tangent line is radius.

$$\therefore \left| \frac{4(3) + 3(-2) + c}{\sqrt{16 + 9}} \right| = 5$$

$$\Rightarrow 6 + c = \pm 25$$

$$\Rightarrow c = 19 \text{ and } -31$$

Therefore, the equations of required tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$.

Alternative method:

Given circle in center-radius form is $(x - 3)^2 + (y + 2)^2 = 5^2$

Tangents are parallel to the line $4x + 3y + 5 = 0$.

$$\therefore \text{Slope of tangents} = -\frac{4}{3}$$

Hence, using the equation of tangents

$$y + f = m(x + g) \pm \sqrt{m^2 + 1} \sqrt{g^2 + f^2 - c},$$

required equations of tangents are obtained as

$$y + 2 = -\frac{4}{3}(x - 3) \pm 5 \sqrt{\left(-\frac{4}{3}\right)^2 + 1}$$

$$\text{or } 4x + 3y + 19 = 0 \text{ and } 4x + 3y - 31 = 0$$

ILLUSTRATION 4.65

Prove that the line $y = m(x - 1) + 3\sqrt{1 + m^2} - 2$ touches the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ for all real values of m .

Sol. The equation of given circle is $(x - 1)^2 + (y + 2)^2 = 3^2$.

Therefore, equations of tangents to the circle having slope m are

$$y + 2 = m(x - 1) \pm 3\sqrt{1 + m^2}$$

Alternatively,

As any tangent to $x^2 + y^2 = r^2$ is given by $y = mx + 3\sqrt{1 + m^2}$, any tangent to the given circle will be

$$y + 2 = m(x - 1) + 3\sqrt{1 + m^2}$$

ILLUSTRATION 4.66

Find the equation of tangent at the end points of diameter of circle $(x - a)^2 + (y - b)^2 = r^2$ which is inclined at an angle θ with positive x -axis using slope method.

Sol. Diameter makes an angle θ with x -axis.

So, slope of diameter is $\tan \theta$.

Therefore, slope of tangent is $-\cot \theta$.

Hence, equations of tangents having slope $-\cot \theta$ are given by

$$y - b = -\cot \theta (x - a) \pm r \sqrt{1 + \cot^2 \theta}$$

$$\text{or } \cos \theta (x - a) + \sin \theta (y - b) = \pm r$$

ILLUSTRATION 4.67

If $a > 2b > 0$, then find the positive value of m for which

$y = mx - b\sqrt{1 + m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$.

Sol. $y = mx - b\sqrt{1 + m^2}$ is a tangent to the circle $x^2 + y^2 = b^2$ for all values of m .

If it also touches the circle $(x - a)^2 + y^2 = b^2$, then the length of the perpendicular from its center $(a, 0)$ on this line is equal to the radius b of the circle, which gives

$$\frac{ma - b\sqrt{1 + m^2}}{\sqrt{1 + m^2}} = \pm b$$

Taking the negative value of the RHS, we get $m = 0$. So, we neglect it.

Taking the positive value of the RHS, we get

$$ma = 2b\sqrt{1 + m^2}$$

$$\text{or } m^2(a^2 - 4b^2) = 4b^2$$

$$\text{or } m = \frac{2b}{\sqrt{a^2 - 4b^2}}$$

EQUATION OF TANGENTS TO CIRCLE DRAWN FROM EXTERNAL POINT

We know that from external point, two tangents can be drawn to circle.

Let tangents be drawn to circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from external point $P(x_1, y_1)$. To find the value of slope of the tangents, we take the equation of tangent as $y - y_1 = m(x - x_1)$. The distance from the centre of the circle to this tangent line will be radius of the circle.

$$\text{So, we get } \left| \frac{m(-g) + f - mx_1 + y_1}{\sqrt{m^2 + 1}} \right| = \sqrt{g^2 + f^2 - c}$$

Solving the resulting quadratic equation in m , we get two values of slopes of tangents.

For circle $x^2 + y^2 = \alpha^2$, equation of one of the tangents having slope m is $y = mx + \alpha\sqrt{1 + m^2}$.

If this tangent passes through the external point $P(x_1, y_1)$, then we have

$$y_1 = mx_1 + \alpha\sqrt{1 + m^2}$$

$$\text{or } (y_1 - mx_1)^2 = \alpha^2(1 + m^2)$$

Solving this quadratic equation in m , we get two values of slopes of tangents drawn from the point P .

ILLUSTRATION 4.68

Find the equation of tangents to circle $x^2 + y^2 - 2x + 4y - 4 = 0$ drawn from point $P(2, 3)$.

Sol. Equation of the circle is $x^2 + y^2 - 2x + 4y - 4 = 0$.

Centre of the circle is $C(1, -2)$ and radius is $\sqrt{(-1)^2 + 2^2 - (-4)} = 3$.

Tangents are drawn to the circle from point $P(2, 3)$.

Equation of the line through point P having slope m is

$$y - 3 = m(x - 2)$$

$$\text{or } mx - y - 2m + 3 = 0$$

If the line touches the circle, then the distance from the centre of the circle to this line will be radius of the circle.

$$\therefore \left| \frac{m(1) + 2 - 2m + 3}{\sqrt{m^2 + 1}} \right| = 3$$

$$\Rightarrow (5 - m)^2 = 9(m^2 + 1)$$

$$\Rightarrow 8m^2 + 10m - 16 = 0$$

$$\Rightarrow 4m^2 + 5m - 8 = 0$$

$$\Rightarrow m = \frac{-5 \pm \sqrt{153}}{8}$$

Therefore, required equations of tangents are

$$y - 3 = \left(\frac{-5 \pm \sqrt{153}}{8} \right) (x - 2).$$

ILLUSTRATION 4.69

Tangents drawn from point P to the circle $x^2 + y^2 = 16$ make the angles θ_1 and θ_2 with positive x -axis. Find the locus of point P such that $(\tan \theta_1 - \tan \theta_2) = c$ (constant).

Sol. Let point P be (h, k) .

Equation of tangent to circle $x^2 + y^2 = 16$ having slope m is

$$y = mx + 4\sqrt{1 + m^2}$$

If tangent is passing through P , then

$$k = mh + 4\sqrt{1 + m^2}$$

$$\Rightarrow (k - mh)^2 = 16 + 16m^2$$

$$\Rightarrow (h^2 - 16)m^2 - 2hkm + k^2 - 16 = 0$$

Roots of this equation are m_1 and m_2 , where $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$.

$$\text{Sum of roots, } m_1 + m_2 = \frac{2hk}{h^2 - 16}$$

$$\text{Product of roots, } m_1 m_2 = \frac{k^2 - 16}{h^2 - 16}$$

Given that $(\tan \theta_1 - \tan \theta_2) = c$

$$\Rightarrow m_1 - m_2 = c$$

$$\Rightarrow (m_1 + m_2)^2 - 4m_1 m_2 = c^2$$

$$\Rightarrow \left(\frac{2hk}{h^2 - 16} \right)^2 - 4 \frac{k^2 - 16}{h^2 - 16} = c^2$$

$$\Rightarrow 4h^2 k^2 - 4(k^2 - 16)(h^2 - 16) = c^2(h^2 - 16)^2$$

$$\Rightarrow 4x^2 y^2 - 4(y^2 - 16)(x^2 - 16) = c^2(x^2 - 16)^2$$

This is the required locus.

EQUATION OF PAIR OF TANGENTS

Let the circle be $x^2 + y^2 = a^2$.

Its center and radius are $C(0, 0)$

and a , respectively. Let the given

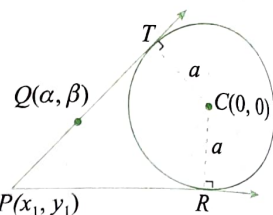
external point be $P(x_1, y_1)$.

From point $P(x_1, y_1)$, two tangents

PT and PR can be drawn to the

circle, touching the circle at T and

R , respectively.



Let $Q(\alpha, \beta)$ be on PT . Then the equation of PQ is

$$y - y_1 = \frac{\beta - y_1}{\alpha - x_1} (x - x_1)$$

$$\text{or } y(\alpha - x_1) - (\beta - y_1)x - \alpha y_1 + \beta x_1 = 0$$

Length of $\perp$ from $C(0, 0)$ on $PT = a$ (radius)

$$\text{or } \frac{|\beta x_1 - \alpha y_1|}{\sqrt{(\alpha - x_1)^2 + (\beta - y_1)^2}} = a$$

$$\text{or } (\beta x_1 - \alpha y_1)^2 = a^2 \{(\alpha - x_1)^2 + (\beta - y_1)^2\}$$

Therefore, the locus of $Q(\alpha, \beta)$ is

$$(yx_1 - xy_1)^2 = a^2 \{(x - x_1)^2 + (y - y_1)^2\}$$

$$\text{or } y^2 x_1^2 + x^2 y_1^2 - 2xyx_1 y_1 = a^2 \{x^2 + x_1^2 - 2xx_1 + y^2 + y_1^2 - 2yy_1\}$$

$$\text{or } y^2(x_1^2 + y_1^2 - a^2) + x^2(x_1^2 + y_1^2 - a^2) - a^2(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

$$\text{or } (x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$$

$$\text{or } SS_1 = T^2$$

This is the required equation of the pair of tangents drawn from (x_1, y_1) to the circle

$$x^2 + y^2 = a^2$$

where $S = x^2 + y^2 - a^2$

$$S_1 = x_1^2 + y_1^2 - a^2$$

and $T = xx_1 + yy_1 - a^2$

Similarly, for circle $x^2 + y^2 + 2gx + 2fy + c = 0$, equation of pair of tangents drawn from external point $P(x_1, y_1)$ is

$$T^2 = SS_1, \text{ where } S = x^2 + y^2 + 2gx + 2fy + c, S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \text{ and } T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c.$$

ILLUSTRATION 7.70

Find the equation of pair of tangents drawn to circle $x^2 + y^2 - 2x + 4y - 4 = 0$ from point $P(-2, 3)$. Also, find the angle between tangents.

Sol. We have circle $x^2 + y^2 - 2x + 4y - 4 = 0$ or $S = 0$.

Tangents are drawn from point $P(-2, 3)$ to the circle.

Equation of pair of tangents is

$$T^2 = SS_1$$

$$\text{or } (-2x + 3y - (x - 2) + 2(y + 3) - 4)^2$$

$$= (x^2 + y^2 - 2x + 4y - 4)((-2)^2 + 3^2 - 2(-2) + 4(3) - 4)$$

$$(-3x + 5y + 4)^2 = 25(x^2 + y^2 - 2x + 4y - 4)$$

$$16x^2 + 30xy - 26x + 60y - 116 = 0$$

$$8x^2 + 15xy - 13x + 30y - 58 = 0$$

Here, $a = 8$, $b = 0$ and $h = 15/2$.

If angle between tangents is θ , then

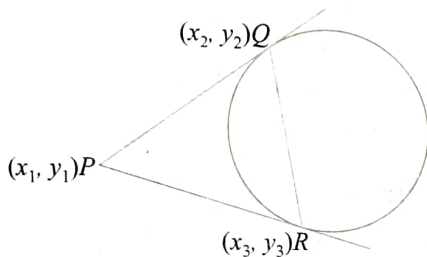
$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{|a + b|} = \frac{2\sqrt{\frac{225}{4} - 0}}{|8 + 0|} = \frac{15}{8}$$

$$\theta = \tan^{-1} \frac{15}{8}$$

CHORD OF CONTACT

If tangents drawn from point $P(x_1, y_1)$ to the circle touch the circle at points $Q(x_2, y_2)$ and $R(x_3, y_3)$, then line QR is called chord of contact.

Consider the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.



Equation of the tangent PQ is

$$xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0 \quad (1)$$

and equation of the tangent PR is

$$xx_3 + yy_3 + g(x + x_3) + f(y + y_3) + c = 0 \quad (2)$$

Since (1) and (2) pass through $P(x_1, y_1)$,

$$x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \quad (3)$$

$$x_1x_3 + y_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0 \quad (4)$$

Thus, points Q and R lie on the line

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \quad (5)$$

or $T = 0$, where $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Equation (5) is the equation of chord of contact w.r.t. point P .

If point P lies on the curve, then equation (5) represents the tangent at point P . In that case, points P , Q and R merge.

ILLUSTRATION 4.71

If the chord of contact of the tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then prove that a , b , and c are in GP.

Sol. Let (h, k) be a point on $x^2 + y^2 = a^2$. Then,

$$h^2 + k^2 = a^2 \quad (1)$$

The equation of the chord of contact of tangents drawn from (h, k) to $x^2 + y^2 = b^2$ is

$$hx + ky = b^2 \quad (2)$$

This touches the circle $x^2 + y^2 = c^2$. Therefore,

$$\left| \frac{-b^2}{\sqrt{h^2 + k^2}} \right| = c$$

$$\text{or } \left| \frac{-b^2}{\sqrt{a^2}} \right| = c$$

[Using (1)]

$$\text{or } b^2 = ac$$

Therefore, a , b , and c are in GP.

ILLUSTRATION 4.72

If the straight line $x - 2y + 1 = 0$ intersects the circle $x^2 + y^2 = 25$ at points P and Q , then find the coordinates of the point of intersection of the tangents drawn at P and Q to the circle $x^2 + y^2 = 25$.

Sol. Let $R(h, k)$ be the point of intersection of the tangents drawn at P and Q to the given circle. Then PQ is the chord of the contact of tangents drawn from R to $x^2 + y^2 = 25$.

So, its equation is

$$hx + ky - 25 = 0 \quad (1)$$

It is given that the equation of PQ is

$$x - 2y + 1 = 0 \quad (2)$$

Since (1) and (2) represent the same line, we have

$$\frac{h}{1} = \frac{k}{-2} = \frac{-25}{1}$$

$$\text{or } h = -25, k = 50$$

Hence, the required point is $(-25, 50)$.

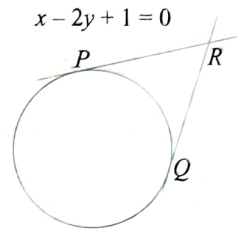
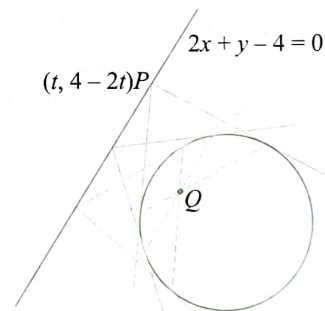


ILLUSTRATION 4.73

Tangents are drawn to $x^2 + y^2 = 1$ from any arbitrary point P on the line $2x + y - 4 = 0$. Prove that corresponding chords of contact pass through a fixed point and find that point.

Sol.



Variable point on the line $2x + y - 4 = 0$ is $P(t, 4 - 2t)$, $t \in R$.

Equation of chord of contact of the circle $x^2 + y^2 = 1$ w.r.t. point P is

$$tx + (4 - 2t)y = 1$$

$$\text{or } (4y - 1) + t(x - 2y) = 0$$

This is the equation of family of straight lines which are concurrent at point of intersection of lines $4y - 1 = 0$ and $x - 2y = 0$.

Therefore, lines are concurrent at $Q(1/2, 1/4)$.

DIRECTOR CIRCLE

Locus of the point of intersection of perpendicular tangents to given circle is circle which is called director circle.

Consider the circle $x^2 + y^2 = a^2$.

One of the equations of tangent to the circle having slope m is given by

$$y = mx + a\sqrt{m^2 + 1}, m \in R$$

If it passes through the point $P(h, k)$, then

$$k = mh + a\sqrt{m^2 + 1}$$

$$\Rightarrow [k - mh]^2 = a^2(m^2 + 1)$$

$$\Rightarrow (h^2 - a^2)m^2 - 2hkm + k^2 - a^2 = 0$$

This equation gives two values of slope (m_1 and m_2) of tangents drawn from point $P(h, k)$.

If tangents are perpendicular, then

$$m_1 m_2 = -1$$

i.e., product of roots is -1 .

$$\therefore \frac{k^2 - a^2}{h^2 - a^2} = -1$$

$$\Rightarrow k^2 - a^2 = a^2 - h^2$$

$$\Rightarrow h^2 + k^2 = 2a^2$$

Therefore, required equation of locus is $x^2 + y^2 = 2a^2$.

This equation represents a circle which is called director circle.

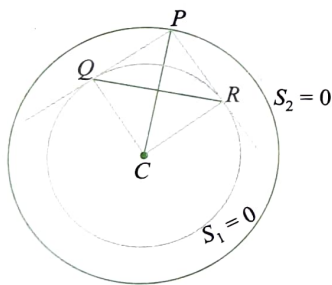
We observe that director circle is concentric with the given circle and has radius $\sqrt{2}$ times the radius of the given circle.

Thus, director circle of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $(x + g)^2 + (y + f)^2 = 2(g^2 + f^2 - c)$.

ILLUSTRATION 4.74

Find the length of the chord of contact with respect to the point on the director circle of circle $x^2 + y^2 + 2ax - 2by + a^2 - b^2 = 0$.

Sol. We have circle $S_1 \equiv (x + a)^2 + (y - b)^2 = 2b^2$.
Equation of director circle $S_2 \equiv (x + a)^2 + (y - b)^2 = 4b^2$.



Tangents PQ and PR are drawn to circle $S_1 = 0$ from a point P on the director circle $S_2 = 0$.

So, QR is the chord of contact.

In the figure, $PQCR$ is a square.

$\therefore QR = CP = \text{Radius of director circle} = 2b$

ILLUSTRATION 4.75

Find the locus of the centers of the circles $x^2 + y^2 - 2ax - 2by + 2 = 0$, where a and b are parameters, if the tangents from the origin to each of the circles are orthogonal.

Sol. The given circle is $x^2 + y^2 - 2ax - 2by + 2 = 0$
or $(x - a)^2 + (y - b)^2 = a^2 + b^2 - 2$

Its director circle is

$$(x - a)^2 + (y - b)^2 = 2(a^2 + b^2 - 2)$$

Given that tangents drawn from the origin to the circle are orthogonal. It implies that the director circle of the circle must pass through the origin, i.e.,

$$a^2 + b^2 = 2(a^2 + b^2 - 2)$$

$$\text{or } a^2 + b^2 = 4$$

Thus, the locus of the center of the given circle is $x^2 + y^2 = 4$.

NORMAL TO CIRCLE

Since the line perpendicular to the tangent to circle at the point of contact passes through the centre of the circle, normal to the circle is the line passing through the centre of the circle. In fact, all the diameters of the circle are normal to the circle.

Consider the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Equation of normal to the circle at point $P(x_1, y_1)$ on it is the line passing through the point P and centre $C(-g, -f)$ of the circle.

Thus, equation of normal is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

Equation of normal to the circle having slope m is

$$y - y_1 = m(x - x_1)$$

ILLUSTRATION 4.76

Find the equations of the normals to the circle $x^2 + y^2 - 8x - 2y + 12 = 0$ at the points whose ordinate is -1 .

Sol. We have circle $x^2 + y^2 - 8x - 2y + 12 = 0$.

Centre of the circle is $C(4, 1)$.

Putting $y = -1$ in the equation of circle, we get

$$x^2 - 8x + 15 = 0$$

$$\Rightarrow (x - 3)(x - 5) = 0$$

$$\Rightarrow x = 5 \text{ or } 3$$

Thus, the points on the circle are $P(5, -1)$ and $Q(3, -1)$.

Equation of normal at P is

$$y + 1 = \frac{-1 - 1}{5 - 4}(x - 5)$$

$$\text{or } 2x + y - 9 = 0$$

Equation of normal at Q is

$$y + 1 = \frac{-1 - 1}{3 - 4}(x - 5)$$

$$\text{or } 2x - y - 7 = 0$$

ILLUSTRATION 4.77

Find the equation of the normal to the circle $x^2 + y^2 - 2x = 0$ parallel to the line $x + 2y = 3$.

Sol. Slope of given line is $-\frac{1}{2}$.

So, slope of normal is $-\frac{1}{2}$.

Normal passes through the centre of the circle.

Here, centre is $(1, 0)$.

Therefore, equation of normal is

$$y = 0 = -\frac{1}{2}(x-1)$$

$$x + 2y - 1 = 0$$

CONCEPT APPLICATION EXERCISE 4.4

- Find the equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which makes equal intercepts on the positive coordinates axes.
- Find the equations of tangents to the circle $x^2 + y^2 - 22x - 4y + 25 = 0$ which are perpendicular to the line $5x + 12y + 8 = 0$.
- If the line $lx + my + n = 0$ is tangent to the circle $x^2 + y^2 = a^2$, then find the condition.
- Find the equation of pair of tangents drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$.
- Find the area of the triangle formed by the positive x -axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.
- If the tangent at $(3, -4)$ to the circle $x^2 + y^2 - 4x + 2y - 5 = 0$ cuts the circle $x^2 + y^2 + 16x + 2y + 10 = 0$ in A and B , then find the midpoint of AB .
- If $3x + y = 0$ is a tangent to a circle whose center is $(2, -1)$, then find the equation of the other tangent to the circle from the origin.
- Let A be the center of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $B(1, 7)$ and $D(4, -2)$ on the circle meet at the point C . Find the area of quadrilateral $ABCD$.
- An infinite number of tangents can be drawn from $(1, 2)$ to the circle $x^2 + y^2 - 2x - 4y + \lambda = 0$. Then find the value of λ .
- Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3, with center in the first quadrant. If A is one of the points of contact, find the length of OA .
- From the variable point A on a circle $x^2 + y^2 = 2a^2$, two tangents are drawn to the circle $x^2 + y^2 = a^2$ which meet the curve at B and C . Find the locus of the circumcenter of $\triangle ABC$.
- Find the distance between the chords of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) .
- Find the point P for which the line $9x + y - 28 = 0$ is the chord of contact of the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$ w.r.t. point P .
- Find the equation of the normal to the circle $x^2 + y^2 = 9$ at the point $(3/\sqrt{2}, 3/\sqrt{2})$.

ANSWERS

- $x + y = \pm 2\sqrt{2}$
- $12x - 5y + 8 = 0$ and $12x - 5y = 252$
- $a^2(l^2 + m^2) = n^2$
- $2x^2 + 2y^2 + 5xy = 0$
- $2\sqrt{3}$ sq. unit
- $(-6, -7)$

$$7. x - 3y = 0$$

$$9. 5$$

$$11. x^2 + y^2 = \frac{a^2}{2}$$

$$13. (3, -1)$$

$$8. 75 \text{ sq. units}$$

$$10. 3(3 + \sqrt{10})$$

$$12. \frac{|g^2 + f^2 - c|}{2\sqrt{g^2 + f^2}}$$

$$14. x - y = 0$$

DIFFERENT CASES OF TWO CIRCLES

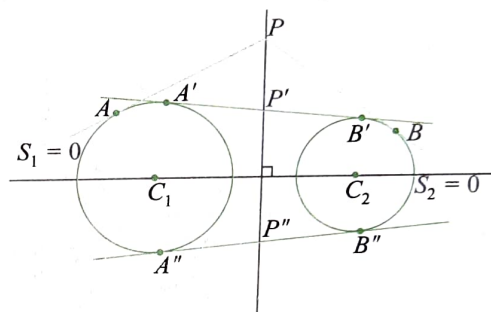
RADICAL AXIS OF TWO CIRCLES

The radical axis of two circles is the locus of a point which moves such that the lengths of the tangents drawn from it to the two circles are equal.

Consider two circles,

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad (2)$$



Let the tangents drawn to above two circles from variable point $P(h, k)$ touch the circles at A and B .

If $PA = PB$, then

$$\Rightarrow \sqrt{h^2 + k^2 + 2g_1h + 2f_1k + c_1} = \sqrt{h^2 + k^2 + 2g_2h + 2f_2k + c_2}$$

$$\Rightarrow h^2 + k^2 + 2g_1h + 2f_1k + c_1 = h^2 + k^2 + 2g_2h + 2f_2k + c_2$$

$$\Rightarrow 2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$$

Therefore, locus of point P is

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0.$$

This is equation of straight line which is called radical axis of two circles.

We observe that equation of radical axis is $S_1 - S_2 = 0$.

Notes:

- Slope of radical axis is $-\frac{g_1 - g_2}{f_1 - f_2}$ and slope of line joining centres C_1 and C_2 of the circles is $\frac{f_1 - f_2}{g_1 - g_2}$.

Thus, radical axis is perpendicular to the line joining the centres of the circles.

- Radical axis bisects the length of common tangents between the points of contact, i.e., points P' and P'' bisect the length of common tangents $A'B'$ and $A''B''$, respectively.

ILLUSTRATION 4.78

Find the equation of radical axis of the circles $x^2 + y^2 - 3x + 5y - 7 = 0$ and $2x^2 + 2y^2 - 4x + 8y - 13 = 0$.

Sol. We have circles

$$x^2 + y^2 - 3x + 5y - 7 = 0$$

$$\text{and } 2x^2 + 2y^2 - 4x + 8y - 13 = 0$$

(1)

(2)

Equation of radical axis of the two circles can be obtained by subtracting the equation of one circle from that of the other in such a way that terms x^2 and y^2 gets cancelled out.

So, multiplying equation (1) by 2 and then subtracting equation (2) from it, we get

$$-2x + 2y - 1 = 0$$

$$\text{or } 2x - 2y + 1 = 0$$

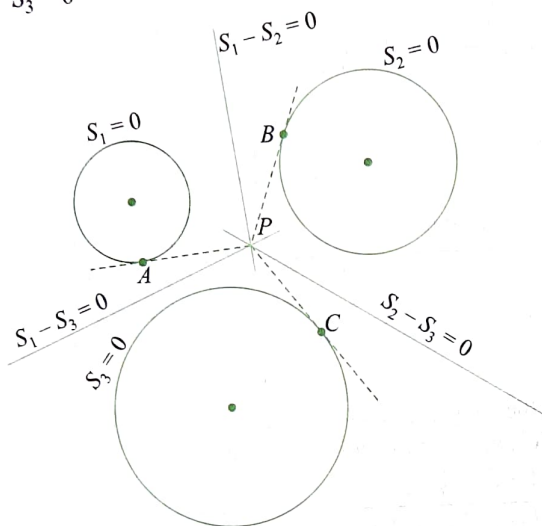
RADICAL AXES OF THREE CIRCLES TAKEN TWO AT A TIME AND RADICAL CENTRE

Consider three circles,

$$S_1 = 0 \quad (1)$$

$$S_2 = 0 \quad (2)$$

$$\text{and } S_3 = 0 \quad (3)$$



Radical axis of circles (1) and (2) is $S_1 - S_2 = 0$.

Radical axis of circles (2) and (3) is $S_2 - S_3 = 0$.

Now, equation of family of lines through point of intersection of above two radical axes is

$$S_1 - S_2 + \lambda(S_2 - S_3) = 0, \lambda \in R.$$

For $\lambda = 1$, we have line $S_1 - S_3 = 0$, which is radical axis of circles (1) and (3). Thus, three radical axes of three circles, taken two at a time, are concurrent.

The point of concurrency of three radical axes is called radical centre.

Radical centre is the only point in the plane of three circles from which length of tangents to three circles is same.

In the figure, $PA = PB = PC$.

ILLUSTRATION 4.79

The equation of three circles are given $x^2 + y^2 = 1$, $x^2 + y^2 - 8x + 15 = 0$, $x^2 + y^2 + 10y + 24 = 0$. Determine the coordinates of the point P such that the tangents drawn from it to the circles are equal in length.

Sol. We know that the point from which the lengths of tangents are equal in length is the radical center of the given three circles. Now, the radical axis of the first two circles is

$$(x^2 + y^2 - 1) - (x^2 + y^2 - 8x + 15) = 0$$

$$\text{i.e., } x - 2 = 0$$

and the radical axis of the second and third circle is

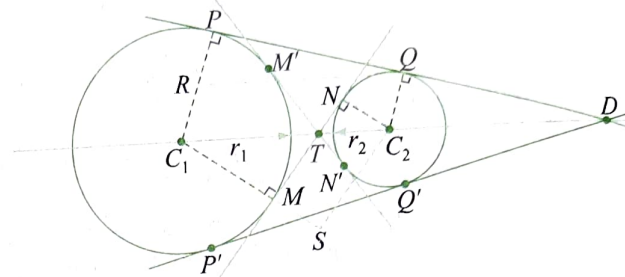
$$(x^2 + y^2 - 8x + 15) - (x^2 + y^2 + 10y + 24) = 0$$

$$\text{i.e., } 8x + 10y + 9 = 0 \quad (2)$$

Solving (1) and (2), the coordinates of the radical center, i.e., of point P , are $P(2, -5/2)$.

TWO DISJOINT CIRCLES

Two disjoint circles neither have common region nor touch each other as shown in the following figure.



Let the two circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad (2)$$

with centres $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$ and radii $r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$ and $r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$, respectively.

We observe that distance between centres of the circles is more than the sum of radii.

Thus, $C_1C_2 > r_1 + r_2$.

In this case, four common tangents can be drawn to two circles.

Two direct common tangents PQ and $P'Q'$ intersect at point D .

Two indirect or transverse common tangents MN and $M'N'$ intersect at point T .

Also, points C_1, C_2, T and D are collinear.

Now, triangles C_1MT and C_2NT are similar.

$$\therefore \frac{C_1T}{C_2T} = \frac{C_1M}{C_2N} = \frac{r_1}{r_2}$$

Thus, point T divides C_1C_2 internally in the ratio $r_1 : r_2$.

Similarly, triangles C_1PD and C_2QD are similar.

$$\therefore \frac{C_1D}{C_2D} = \frac{C_1P}{C_2Q} = \frac{r_1}{r_2}$$

Thus, point D divides C_1C_2 externally in the ratio $r_1 : r_2$.

If two circles have same radius, point D does not exist.

Length of Common Tangents between Points of Contact

In the figure, line QR is drawn parallel to C_1C_2 .

In right angled triangle QPR ,

$$RQ = C_1C_2 \text{ and } RP = C_1P - C_1R = r_1 - r_2$$

$$\therefore PQ^2 = RQ^2 - PR^2 = (C_1C_2)^2 - (r_1 - r_2)^2 = (P'Q')^2$$

This gives length of direct common tangents PQ and $P'Q'$.

Now, extend C_1M to point S such that $MS = r_2$.

Thus, C_2S is parallel to MN .

In right angled triangle C_1SC_2 ,

$$C_2S = MN \text{ and } C_1S = C_1M + MS = C_1M + NC_2 = r_1 + r_2$$

$$\therefore MN^2 = (C_1C_2)^2 - (C_1S)^2 = (C_1C_2)^2 - (r_1 + r_2)^2 = (MN')^2$$

This gives length of indirect common tangent MN and $M'N'$.

Equations of Common Tangents

Let the coordinates of point T be (x_3, y_3) .

Equation of line passing through point T having slope m is

$$y - y_3 = m(x - x_3)$$

If this line touches the first circle, then distance from the centre of the circle to the line is radius.

From this condition, we will get two values of slope m for two indirect common tangents.

With the similar algorithm, we can get two values of slope of direct common tangents using point D .

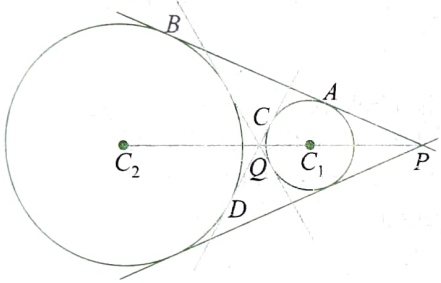
ILLUSTRATION 4.80

Find all the common tangents to the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$. Also find the lengths of the direct and indirect common tangents.

Sol. Here, for the first circle, the center is $C_1(1, 3)$ and radius r_1 is 1.

For the second circle, the center is $C_2(-3, 1)$ and radius r_2 is 3.

Thus, $C_1C_2 = 2\sqrt{5}$ and $r_1 + r_2 = 4$ or $C_1C_2 > r_1 + r_2$, $r_1 \neq r_2$. Thus, we have four common tangents.



To find direct common tangents:

The coordinates of the point P dividing line C_1C_2 in the ratio $r_1 : r_2$, i.e., 1 : 3, externally, are

$$\left(\frac{1 \times (-3) - 3 \times 1}{1 - 3}, \frac{1 \times 1 - 3 \times 3}{1 - 3} \right) \text{ or } (3, 4)$$

Therefore, the equation of any line through point $P(3, 4)$ is

$$y - 4 = m_1(x - 3)$$

$$\text{or } m_1x - y + 4 - 3m_1 = 0 \quad (1)$$

If (1) is tangent to the first circle, then the length of perpendicular from the center $C_1(1, 3)$ of (1) is equal to r_1 (radius). Therefore,

$$\frac{|m_1 - 3 + 4 - 3m_1|}{\sqrt{(m_1^2 + 1)}} = 1 \text{ or } (1 - 2m_1)^2 = m_1^2 + 1$$

$$\text{or } 3m_1^2 - 4m_1 = 0$$

$$\text{or } m_1 = 0, \frac{4}{3}$$

Substituting $m_1 = 0$ and $4/3$ in (1), the equations of direct common tangents are $y = 4$ and $4x - 3y = 0$.

To find transverse common tangents:

The coordinates of the point Q dividing the line C_1C_2 in the ratio $r_1 : r_2$, i.e., 1 : 3, internally, are $(0, 5/2)$.

Therefore, the equation of any line through $Q(0, 5/2)$ is

$$y - \frac{5}{2} = m_2(x - 0) \text{ or } m_2x - y + \frac{5}{2} = 0 \quad (2)$$

If (2) is tangent to the first circle, then the length of perpendicular from the center $C_1(1, 3)$ on (2) is equal to r_1 (radius). Therefore,

$$\frac{|m_2 - 3 + 5/2|}{\sqrt{(m_2^2 + 1)}} = 1$$

$$\text{or } (2m_2 - 1)^2 = 4(m_2^2 + 1)$$

$$\text{or } -4m_2 - 3 = 0$$

$$\therefore m_2 = -\frac{3}{4} \text{ and } \infty \quad (\text{As coefficient of } m_2^2 \text{ is zero})$$

Substituting $m_2 = -3/4$ and ∞ in (2), the equations of transverse common tangents are

$$3x + 4y - 10 = 0 \text{ and } x = 0$$

$$AB^2 = C_1C_2^2 - (r_1 - r_2)^2 \\ = 20 - (3 - 1)^2 = 16$$

$$\therefore AB = 4$$

$$CD^2 = C_1C_2^2 - (r_1 + r_2)^2 \\ = 20 - (3 + 1)^2 = 4$$

$$\therefore CD = 2$$

TWO CIRCLES TOUCHING EXTERNALLY

Let the two circles be

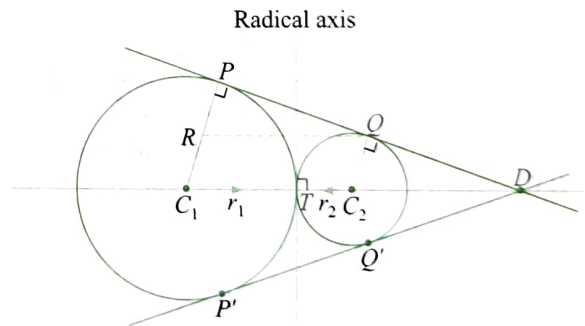
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad (2)$$

with centres $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$ and radii

$$r_1 = \sqrt{g_1^2 + f_1^2 - c_1} \text{ and } r_2 = \sqrt{g_2^2 + f_2^2 - c_2}, \text{ respectively.}$$

Let two circles touch externally as shown in the following figure.



We observe that distance between centres of the circles is equal to the sum of radii.

$$\text{Thus, } C_1C_2 = C_1T + C_2T = r_1 + r_2.$$

In this case, three common tangents can be drawn to two circles.

Two direct common tangents PQ and $P'Q'$ intersect at point D .

Point D divides C_1C_2 externally in the ratio $r_1 : r_2$.

If two circles have same radius, point D does not exist.

One common tangent is at point of contact T , which is radical axis of two circles whose equation is given by $S_1 - S_2 = 0$.

Point T divides C_1C_2 internally in the ratio $r_1 : r_2$.

Length of direct common tangent is given by

$$PQ^2 = (C_1C_2)^2 - (r_1 - r_2)^2 = P'Q'^2$$

ILLUSTRATION 4.81

Show that the circles $x^2 + y^2 - 10x + 4y - 20 = 0$ and $x^2 + y^2 + 14x - 6y + 22 = 0$ touch each other. Find the coordinates of the point of contact and the equation of the common tangent at the point of contact.

Sol. Two circles are

$$S_1 \equiv x^2 + y^2 - 10x + 4y - 20 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 14x - 6y + 22 = 0 \quad (2)$$

For the circle (1), centre is $C_1(5, -2)$ and radius,

$$r_1 = \sqrt{(-5)^2 + 2^2 - (-20)} = 7.$$

For the circle (2), centre is $C_2(-7, 3)$ and radius,

$$r_2 = \sqrt{7^2 + (-3)^2 - 22} = 6.$$

$$\text{Now, } C_1C_2 = \sqrt{(5+7)^2 + (-2-3)^2} = 13 = r_1 + r_2$$

Thus, two circles are touching each other externally.

Point of contact T divides C_1C_2 internally in the ratio $r_1 : r_2 = 7 : 6$.

$$\therefore T \equiv \left(\frac{7(-7) + 6(5)}{7+6}, \frac{7(3) + 6(-2)}{7+6} \right) \equiv (-19/13, 9/13)$$

Common tangent at point of contact is radical axis, whose equation is given by $S_1 - S_2 = 0$.

$$\text{Therefore, equation of common tangent is } -24x + 10y - 42 = 0 \\ \text{or } 12x - 5y + 21 = 0.$$

ILLUSTRATION 4.82

If two circles $x^2 + y^2 + c^2 = 2ax$ and $x^2 + y^2 + c^2 - 2by = 0$ touch each other externally, then prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Sol. The two circles are $x^2 + y^2 - 2ax + c^2 = 0$ and $x^2 + y^2 - 2by + c^2 = 0$.

Respective centres are $C_1(a, 0)$ and $C_2(0, b)$.

Respective radii are $r_1 = \sqrt{a^2 - c^2}$ and $r_2 = \sqrt{b^2 - c^2}$.

Since the two circles touch each other externally, we have

$$\begin{aligned} C_1C_2 &= r_1 + r_2 \\ \Rightarrow \sqrt{a^2 + b^2} &= \sqrt{a^2 - c^2} + \sqrt{b^2 - c^2} \\ \Rightarrow a^2 + b^2 &= a^2 - c^2 + b^2 - c^2 + 2\sqrt{a^2 - c^2}\sqrt{b^2 - c^2} \\ \Rightarrow c^2 &= \sqrt{a^2 - c^2}\sqrt{b^2 - c^2} \\ \Rightarrow c^4 &= a^2b^2 - c^2(a^2 + b^2) + c^4 \\ \Rightarrow a^2b^2 &= c^2(a^2 + b^2) \\ \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} &= \frac{1}{c^2} \end{aligned}$$

TWO CIRCLES TOUCHING INTERNALLY

Let the two circles be

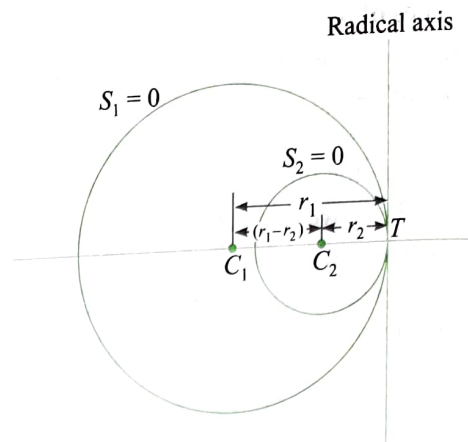
$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad (2)$$

with centres $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$ and radii $r_1 =$

$$\sqrt{g_1^2 + f_1^2 - c_1} \text{ and } r_2 = \sqrt{g_2^2 + f_2^2 - c_2}, \text{ respectively.}$$

Let the two circles touch internally as shown in the following figure.



We observe that distance between centres of the circles is equal to the difference of radii.

$$\text{Thus, } C_1C_2 = C_1T - C_2T = |r_1 - r_2|$$

There is only one common tangent at point of contact T .

This common tangent is radical axis whose equation is $S_1 - S_2 = 0$.

Also, point T divides C_1C_2 externally in the ratio $r_1 : r_2$.

ILLUSTRATION 4.83

Find the equation of a circle with center $(4, 3)$ touching the circle $x^2 + y^2 = 1$.

Sol. Let the circle be $x^2 + y^2 - 8x - 6y + k = 0$ touching the circle $x^2 + y^2 = 1$. Then the equation of the common tangent at point of contact is $S_1 - S_2 = 0$ or $8x + 6y - 1 - k = 0$.

This is a tangent to the circle $x^2 + y^2 = 1$. Therefore,

$$\pm 1 = \frac{k+1}{\sqrt{8^2 + 6^2}}$$

$$\text{or } k+1 = \pm 10$$

$$\text{i.e., } k = -11 \text{ or } 9$$

Hence, the circles are $x^2 + y^2 - 8x - 6y + 9 = 0$ and $x^2 + y^2 - 8x - 6y - 11 = 0$.

Alternative method:

The given circle is $x^2 + y^2 = 1$, which has center $C_1(0, 0)$ and radius $r_1 = 1$. The required circle has center $C_2(4, 3)$ and radius r_2 .

If the circles are touching externally, then

$$r_1 + r_2 = C_1C_2$$

$$\text{or } r_2 = 5 - 1 = 4$$

If the circles are touching internally, then

$$r_2 - r_1 = C_1C_2 \quad (\because (4, 3) \text{ lies outside circle } x^2 + y^2 = 1)$$

$$\text{or } r_2 = 6$$

Thus, the required circles are

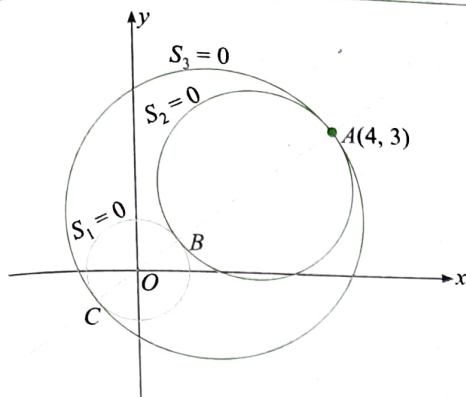
$$(x-4)^2 + (y-3)^2 = 16$$

$$\text{or } (x-4)^2 + (y-3)^2 = 36$$

ILLUSTRATION 4.84

Find the equation of the smaller circle that touches the circle $x^2 + y^2 = 1$ and passes through the point $(4, 3)$.

Sol. Required circle through point $A(4, 3)$ touches the circle $S_1 \equiv x^2 + y^2 = 1$.



This circle is smallest if it touches the given circle externally and point of contact B lies on the common normal.

Equation of line OA is $y = \frac{3}{4}x$. (1)

Putting this value of y from (1) in $x^2 + y^2 = 1$, we get

$$x = \pm \frac{4}{5} \text{ and } y = \pm \frac{3}{5}$$

$$\therefore B \equiv \left(\frac{4}{5}, \frac{3}{5}\right)$$

$$\text{and } C \equiv \left(\frac{-4}{5}, \frac{-3}{5}\right)$$

Therefore, equation of circle with AB as diameter is

$$S_2 \equiv \left(x - \frac{4}{5}\right)(x - 4) + (y - 3)\left(y - \frac{3}{5}\right) = 0$$

$$\text{or } x^2 + y^2 - \frac{24}{5}x - \frac{18}{5}y + 5 = 0$$

Required circle is the largest if given circle touches given circle internally and point of contact C lies on the common normal.

Therefore, equation of circle with AC as diameter is

$$S_3 \equiv \left(x + \frac{4}{5}\right)(x - 4) + \left(y + \frac{3}{5}\right)(y - 3) = 0$$

$$\text{or } x^2 + y^2 - \frac{16}{5}x - \frac{12}{5}y - 5 = 0$$

ILLUSTRATION 4.85

Find the locus of centre of variable circle C , that touches the circle $x^2 + y^2 = 4$ internally and passes through the point $(1, 0)$.

Sol. Given circle is $x^2 + y^2 = 4$.

Centre of the circle is $C_1(0, 0)$ and radius $r_1 = 2$.

Let the centre of the variable circle touching the given circle be $C_2(h, k)$ and radius be r_2 .

We have

$$C_1C_2 = r_1 - r_2 \quad (1)$$

Since variable circle passes through point $P(1, 0)$, its radius

$$r_2 = C_2P = \sqrt{(h-1)^2 + k^2}.$$

From (1), we have

$$\sqrt{h^2 + k^2} = 2 - \sqrt{(h-1)^2 + k^2}$$

($\because$ point $(1, 0)$ lies inside the given circle)

$$\Rightarrow \sqrt{(h-1)^2 + k^2} = 2 - \sqrt{h^2 + k^2}$$

$$\Rightarrow h^2 + k^2 - 2h + 1 = 4 + h^2 + k^2 - 4\sqrt{h^2 + k^2}$$

$$\Rightarrow 3 + 2h = 4\sqrt{h^2 + k^2}$$

$$\Rightarrow 9 + 4h^2 + 12h = 16h^2 + 16k^2$$

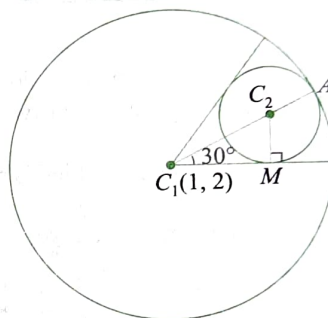
$$\Rightarrow 12x^2 + 16y^2 - 12x = 9$$

This is the required locus.

ILLUSTRATION 4.86

Find the locus of the centre of a circle touching the circle $x^2 + y^2 - 2x - 4y = 4$ internally and tangents on which from $(1, 2)$ are making an angle of 60° with each other.

Sol.



Given circle is $x^2 + y^2 - 2x - 4y = 4$.

Centre is $C_1(1, 2)$ and radius is $r_1 = \sqrt{(-2)^2 + (-4)^2 - (-4)} = 3$.

Let the centre of the variable circles touching given circle be $C_2(h, k)$.

Angle between tangents drawn from $C_1(1, 2)$ to variable circle is 60° .

In triangle C_1MC_2 ,

$$\sin 30^\circ = \frac{C_2M}{C_1C_2}$$

$$\Rightarrow \sin 30^\circ = \frac{C_2A}{C_1C_2}$$

$$\Rightarrow \frac{1}{2} = \frac{C_1A - C_1C_2}{C_1C_2} = \frac{3 - C_1C_2}{C_1C_2}$$

$$\Rightarrow C_1C_2 = 2$$

$$\Rightarrow \sqrt{(h-1)^2 + (k-2)^2} = 2$$

Therefore, required locus is $(x-1)^2 + (y-2)^2 = 4$.

TWO INTERSECTING CIRCLES

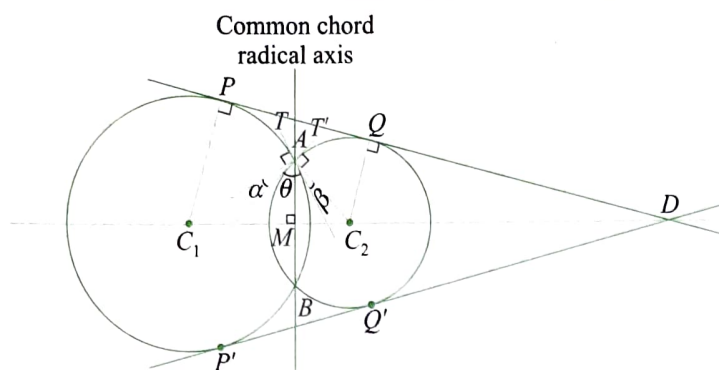
Let the two circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad (2)$$

with centres $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$ and

radii $r_1 = \sqrt{g_1^2 + f_1^2 - c_1}$ and $r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$, respectively.



We observe that distance between centres of the circles is less than the sum of radii.

$$\therefore C_1C_2 < r_1 + r_2.$$

Also, we must have $C_1C_2 > |r_1 - r_2|$, otherwise one circle may lie inside the other.

$$\therefore |r_1 - r_2| < C_1C_2 < r_1 + r_2$$

In this case, only two direct common tangents can be drawn.

These tangents intersect at point D , which divides C_1C_2 externally in the ratio $r_1 : r_2$.

If two circles have same radius, point D does not exist.

In the above figure, circles intersect at two distinct points A and B .

AB is common chord of the circle, which is radical axis whose equation is $S_1 - S_2 = 0$.

Length of direct common tangent is given by

$$PQ^2 = (C_1C_2)^2 - (r_1 - r_2)^2 = P'Q'^2$$

Points of Intersection of Circles and Length of Common Chord

To find the points of intersection of circles A and B , we solve one of the circles and the common chord i.e., radical axis. Using distance formula, we can find the length of common chord.

Also, in right angled triangle AMC_1 , $AM^2 = AC_1^2 - C_1M^2$; where $AC_1 = r_1$ and C_1M is distance of centre of the first circle from the common chord.

$$\text{So, length of common chord is } 2AM = \sqrt{AC_1^2 - C_1M^2}.$$

Angle between Two Circles

We know that angle between two curves is angle between tangents at their point of intersection.

In the figure, tangent to circle S_1 at A is line T and C_1A is perpendicular to this tangent.

Tangent to circle S_2 at A is line T' and C_2A is perpendicular to this tangent.

If the angle between tangents is θ , then angle between radius C_1A and tangent T is $\alpha = 90^\circ - \theta$.

Similarly, angle between radius C_2A and tangent T is $\beta = 90^\circ - \theta$.

$$\text{Thus, } \angle C_1AC_2 = 90^\circ - \theta + \theta + 90^\circ - \theta = 180^\circ - \theta.$$

In triangle C_1AC_2 , using cosine rule, we have

$$\cos(180^\circ - \theta) = \frac{C_1A^2 + C_2A^2 - (C_1C_2)^2}{2C_1A \cdot C_2A}$$

$$\therefore \cos \theta = \frac{(C_1C_2)^2 - r_1^2 - r_2^2}{2r_1 \cdot r_2}$$

Orthogonal Intersection of Two Circles

If two circles intersect at right angle, then we say that circles are intersecting orthogonally.

Clearly, here tangent to one circle goes through the centre of the other circle.

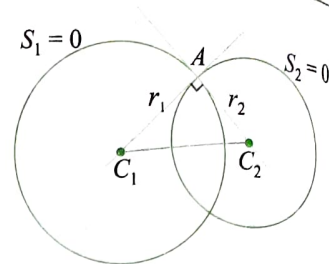
$$\text{In this case, } (C_1C_2)^2 = r_1^2 + r_2^2$$

For above equations of circles, we have

$$(g_1 - g_2)^2 + (f_1 - f_2)^2 = g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2$$

$$\therefore 2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

This is the condition for two circles given by equations (1) and (2) to intersect orthogonally.

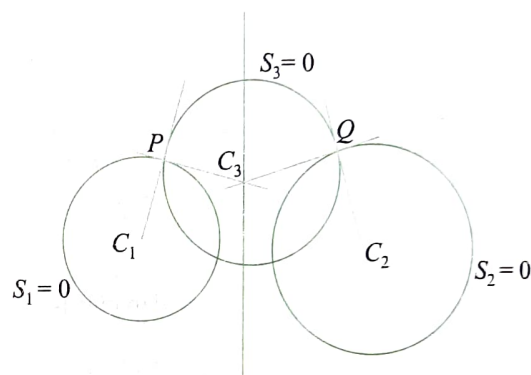


Radical Axis and Orthogonal Intersection of Circles

If two circles cut a third circle orthogonally, then the radical axis of the two circles will pass through the centre of the third circle.

OR

The centre of the variable circle cutting two given circles orthogonally lies on the radical axis of the given two circles.



In the figure, circle $S_3 = 0$ intersects the circles $S_1 = 0$ and $S_2 = 0$ orthogonally.

$$C_1P \perp C_3P \text{ and } C_2Q \perp C_3Q.$$

But $C_3P = C_3Q = \text{Radius of the circle } S_3 = 0$.

Hence, C_3 lies on the radical axis of the circles $S_1 = 0$ and $S_2 = 0$, as C_3P and C_3Q are also lengths of tangents from C_3 to the given circles.

From the above discussion, we understand that radical centre is the centre of the circle which intersects the given three circles orthogonally. The radius of this circle is the length of tangent from radical centre to any of the three circles.

ILLUSTRATION 4.87

If two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then find the values of r .

Sol. We have circle $(x - 1)^2 + (y - 3)^2 = r^2$ having centre $C_1(1, 3)$ and radius $r_1 = r$ and circle $x^2 + y^2 - 8x + 2y + 8 = 0$ having centre

$$C_2(4, -1) \text{ and radius } r_2 = \sqrt{4^2 + (-1)^2 - 8} = 3.$$

Circles intersect in two distinct points if

$$|r_1 - r_2| < C_1C_2 < r_1 + r_2$$

$$\therefore |r - 3| < 5 < r + 3$$

$$\Rightarrow |r - 3| < 5 \text{ and } r > 2$$

$$\Rightarrow -5 < r - 3 < 5 \text{ and } r > 2$$

$$\Rightarrow -2 < r < 8 \text{ and } r > 2$$

$$\Rightarrow 2 < r < 8$$

ILLUSTRATION 4.88

Find the angle at which the circles $(x-1)^2 + y^2 = 10$ and $x^2 + (y-2)^2 = 5$ intersect.

Sol. Here, $C_1 \equiv (1, 0)$ and $r_1 = \sqrt{10}$.

And $C_2 \equiv (0, 2)$ and $r_2 = \sqrt{5}$.

Also, $C_1C_2 = \sqrt{1^2 + 2^2} = \sqrt{5}$.

If circles intersect at an angle θ , then

$$\cos \theta = \frac{(C_1C_2)^2 - r_1^2 - r_2^2}{2r_1 \cdot r_2} = \frac{5 - 10 - 5}{2 \cdot \sqrt{10} \cdot \sqrt{5}} = -\frac{1}{\sqrt{2}}$$

$$\therefore \theta = \frac{3\pi}{4}$$

ILLUSTRATION 4.89

If the two circles $2x^2 + 2y^2 - 3x + 6y + k = 0$ and $x^2 + y^2 - 4x + 10y + 16 = 0$ cut orthogonally, then find the value of k .

Sol. The given circles are

$$2x^2 + 2y^2 - 3x + 6y + k = 0$$

$$\text{or } x^2 + y^2 - \frac{3}{2}x + 3y + \frac{k}{2} = 0 \quad (1)$$

$$\text{and } x^2 + y^2 - 4x + 10y + 16 = 0 \quad (2)$$

Since circles cut orthogonally, then

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$\text{or } 2\left(-\frac{3}{4}\right)(-2) + 2\left(\frac{3}{2}\right)5 = \frac{k}{2} + 16$$

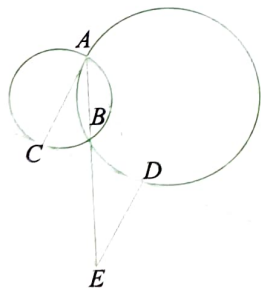
$$\text{or } 3 + 15 = \frac{k}{2} + 16 \text{ or } k = 4$$

ILLUSTRATION 4.90

Two circles passing through $A(1, 2)$, $B(2, 1)$ touch the line $4x + 8y - 7 = 0$ at C and D such that $ACED$ is a parallelogram. Then, find the coordinates of E .

Sol. The midpoint of AE must be the point of intersection of diagonals of parallelogram.

Let $E \equiv (h, k)$.



So, $\left(\frac{h+1}{2}, \frac{k+2}{2}\right)$ must lie on common tangent $4x + 8y - 7 = 0$.

$$\therefore 2h + 4k + 3 = 0 \quad (1)$$

Also, (h, k) lies on AB whose equation is $x + y = 3$.

$$\therefore h + k = 3 \quad (2)$$

Solving (1) and (2), we get

$$h = \frac{15}{2}, k = -\frac{9}{2}$$

ILLUSTRATION 4.91

Find the equation of the smallest circle which cuts circles $x^2 + y^2 = 1$ and $x^2 + y^2 + 8x + 8y - 33 = 0$ orthogonally.

Sol. Let the equation of the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

This circle cuts the given two circles orthogonally.

$$\therefore 2g(0) + 2f(0) = -1 + c,$$

$$\therefore c = 1$$

$$\text{and } 2(g)(4) + 2(f)(4) = -33 + c = -32$$

$$\therefore g + f = -4$$

$$\begin{aligned} \therefore \text{Radius of the circle} &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{g^2 + (g+4)^2 - 1} \\ &= \sqrt{2g^2 + 8g + 15} \\ &= \sqrt{2(g+2)^2 + 7} \end{aligned}$$

Radius is minimum if $g+2=0$ or $g=-2$.

$$\therefore f = -2$$

Hence, equation of the circle is $x^2 + y^2 - 4x - 4y + 1 = 0$.

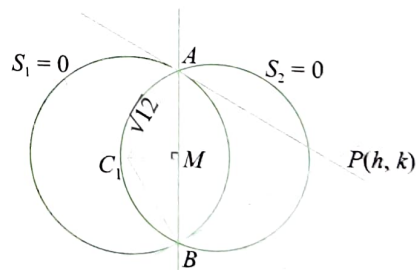
ILLUSTRATION 4.92

If the tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it meets the circle $x^2 + y^2 - 5x + 3y - 2 = 0$, then find the point of intersection of these tangents. Also, find the length of common chord.

Sol. We have circles

$$S_1 \equiv x^2 + y^2 - 12 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 - 5x + 3y - 2 = 0 \quad (2)$$



Circles are intersecting at points A and B .

Tangents are drawn to circle $S_1 = 0$ at A and B , which meet at point $P(h, k)$.

AB is common chord of the circles, whose equation is $S_1 - S_2 = 0$.

$$\text{or } 5x - 3y - 10 = 0 \quad (3)$$

Also, line AB is chord of contact of the circle $S_1 = 0$, w.r.t. point P .

$\therefore$ Equation of AB is

$$hx + ky - 12 = 0 \quad (4)$$

Equations (3) and (4) represent the same straight line.

$$\therefore \frac{h}{5} = \frac{k}{-3} = \frac{-12}{-10}$$

$$\Rightarrow h = 6, k = -18/5$$

Hence, the required point is $P(6, -18/5)$.

$$\text{In the figure, } C_1M = \frac{|5(0) - 3(0) - 10|}{\sqrt{5^2 + (-3)^2}} = \frac{10}{\sqrt{34}}$$

In triangle C_1MA , $AM^2 = C_1A^2 - C_1M^2 = 12 - \frac{100}{34} = \frac{154}{17}$

$$\therefore \text{Length of common chord, } AB = 2AM = 2\sqrt{\frac{154}{17}}$$

ILLUSTRATION 4.93

If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then prove that $2g'(g - g') + 2f'(f - f') = c - c'$.

Sol. Given circles are

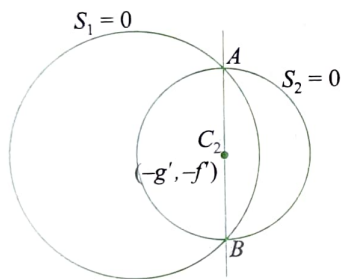
$$S_1 \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

$$S_2 \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad (2)$$

It is given that circle $S_1 = 0$ bisects the circumference of the circle $S_2 = 0$. So, the common chord of circles passes through the centre of the circle $S_2 = 0$.

Equation of common chord of the circles is $2x(g - g') + 2y(f - f') + c - c' = 0$.

This chord passes through the centre $(-g', -f')$ of second circle.



$$\therefore -2g'(g - g') - 2f'(f - f') + c - c' = 0$$

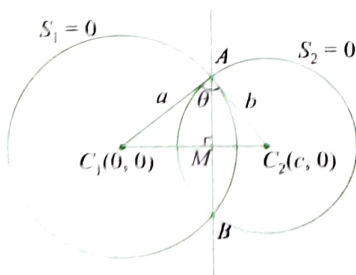
$$\Rightarrow 2g'(g - g') + 2f'(f - f') = c - c'$$

ILLUSTRATION 4.94

If θ is the angle between the two radii (one to each circle) drawn from one of the points of intersection of two circles $x^2 + y^2 = a^2$ and $(x - c)^2 + y^2 = b^2$, then prove that the length of

common chord of two circles is $\frac{2ab \sin \theta}{\sqrt{a^2 + b^2 - 2ab \cos \theta}}$.

Sol.



From the figure,

$$\cos \theta = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore c = \sqrt{a^2 + b^2 - 2ab \cos \theta} = C_1C_2$$

$$\text{Area of triangle } C_1AC_2 = \frac{1}{2} ab \sin \theta = \frac{1}{2} AM \times C_1C_2$$

$$\therefore AM = \frac{ab \sin \theta}{\sqrt{a^2 + b^2 - 2ab \cos \theta}}$$

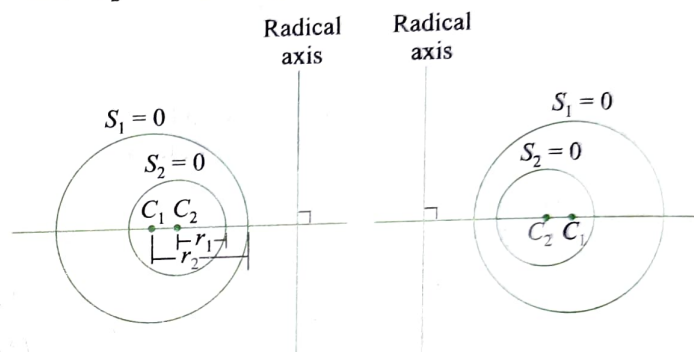
$$\therefore \text{Length of common chord, } AB = 2AM = \frac{2ab \sin \theta}{\sqrt{a^2 + b^2 - 2ab \cos \theta}}$$

ONE CIRCLE CONTAINED IN THE OTHER CIRCLE WITHOUT TOUCHING

Let the two circles be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad (1)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad (2)$$



As shown in the figure, circle $S_2 = 0$ is completely lying inside circle $S_1 = 0$ without touching it.

Clearly, $C_1C_2 < |r_2 - r_1|$

Radical axis doesn't meet the circles and circles have no common tangent.

ILLUSTRATION 4.95

If the circle $x^2 + y^2 = 1$ is completely contained in the circle $x^2 + y^2 + 4x + 3y + k = 0$, then find the values of k .

Sol. Given circles are

$$x^2 + y^2 = 1 \quad (1)$$

$$\text{and } x^2 + y^2 + 4x + 3y + k = 0 \quad (2)$$

$$C_1(0, 0), r_1 = 1.$$

$$C_2(-2, -3/2), r_2 = \sqrt{4 + \frac{9}{4} - k} = \sqrt{\frac{25}{4} - k}.$$

Circle (1) is completely contained by circle (2).

$$\Rightarrow C_1C_2 < r_2 - r_1$$

$$\Rightarrow \sqrt{4 + \frac{9}{4}} < \sqrt{\frac{25}{4} - k} - 1$$

$$\Rightarrow \frac{5}{2} + 1 < \sqrt{\frac{25}{4} - k}$$

$$\Rightarrow \frac{25}{4} - k > \frac{49}{4}$$

$$\Rightarrow k < -6$$

Also, for these values of k , $\frac{25}{4} - k > 0$.

CONCEPT APPLICATION EXERCISE 4.5

- How the following pair of circles are situated in the plane? Also, find the number of common tangents.
 - $x^2 + (y-1)^2 = 9$ and $(x-1)^2 + y^2 = 25$
 - $x^2 + y^2 - 12x - 12y = 0$ and $x^2 + y^2 + 6x + 6y = 0$
- If the two circles of same radius having centres at (2, 3) and (5, 6) cut orthogonally, then find the radius of the circles.
- Circle of radius 5 units intersects the circle $(x-1)^2 + (y-2)^2 = 9$ in such a way that the length of the common chord is of maximum length. If the slope of common chord is $\frac{3}{4}$, then find the centre of the circle.
- The equation of common chord of two circles is $x + y = 1$. One of the circles has the end points of diameter as (1, -3) and (4, 1) while the other passes through the point (1, 2). Then find the equation of these circles.
- Let two parallel lines L_1 and L_2 with positive slopes be tangent to the circle $S_1 \equiv x^2 + y^2 - 2x - 16y + 64 = 0$. If L_1 is also tangent to the circle $S_2 \equiv x^2 + y^2 - 2x + 2y - 2 = 0$ such that S_1 and S_2 lie on different sides of L_1 , then find the equation of L_2 .
- Find the coordinates of the point at which the circles $x^2 + y^2 - 4x - 2y = -4$ and $x^2 + y^2 - 12x - 8y = -36$ touch each other. Also find the equations of common tangents touching the circles at distinct points.
- The equation of a circle is $x^2 + y^2 = 4$. Find the center of the smallest circle touching the circle and the line $x + y = 5\sqrt{2}$.
- Consider four circles $(x \pm 1)^2 + (y \pm 1)^2 = 1$. Find the equation of the smaller circle touching these four circles.
- Find the equation of the circle whose radius is 3 and which touches internally the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ at the point (-1, -1).
- Two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents, ($a > b \geq 2$). Then prove that $\theta = 2 \sin^{-1} \left(\frac{a-b}{a+b} \right)$.
- If the radii of the circles $(x-1)^2 + (y-2)^2 = 1$ and $(x-7)^2 + (y-10)^2 = 4$ are increasing uniformly w.r.t. time as 0.3 unit/s and 0.4 unit/s, respectively, then at what value of t will they touch each other?
- Let T_1 and T_2 be two tangents drawn from (-2, 0) onto the circle $C: x^2 + y^2 = 1$. Determine the circles touching C and having T_1 and T_2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time.

ANSWERS

- (i) One circle completely lying inside other. No common tangent.
(ii) Touching externally. Three common tangents.
- 3 units
- $\left(\frac{17}{5}, \frac{-6}{5} \right)$ and $\left(\frac{-7}{5}, \frac{26}{5} \right)$
- $x^2 + y^2 - \frac{15}{2}x - \frac{y}{2} + \frac{7}{2} = 0$
- $2\sqrt{2}x - y + 11 - 2\sqrt{2} = 0$
- $\left(\frac{7}{2\sqrt{2}}, \frac{7}{2\sqrt{2}} \right)$
- $x^2 + y^2 = 3 - 2\sqrt{2}$
- $5x^2 + 5y^2 - 8x - 14y - 32 = 0$
- $t = 10, 90$

FAMILY OF CIRCLES

We know that three non-collinear points are required for a fixed circle. If two points are given, then we can draw innumerable circles of which the smallest circle is that for which given two points are end points of diameter. The set of these circles is called family of circles.

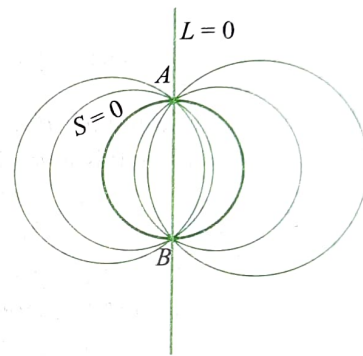
Now, two points may be given points or points of intersection of fixed circle and fixed line or points of intersection of two fixed circles. Let us find the general equation of family of circles. We have following possibilities:

1. Family of circles through points of intersection of fixed circle and fixed line.

Let given circle $S = 0$ and line $L = 0$ intersect at two points $A(x_1, y_1)$ and $B(x_2, y_2)$. General equation of circles through points A and B is given by

$$S + \lambda L = 0, \text{ where } \lambda \in R.$$

Here, λ is a parameter.

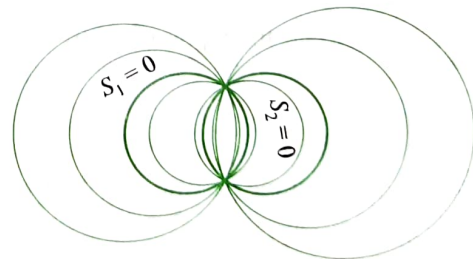


Since points A and B lie on given circle and line, these points also satisfy equation $S + \lambda L = 0$ for any real value of λ .

2. Family of circles through points of intersection of two circles.

Let given circles $S_1 = 0$ and $S_2 = 0$ intersect at two points $A(x_1, y_1)$ and $B(x_2, y_2)$. General equation of circles through points A and B is given by

$$S_1 + \lambda S_2 = 0, \text{ where } \lambda \in R.$$



Here, λ cannot take the value for which terms x^2 and y^2 get cancelled out, otherwise above equation reduces to radical axis of two circles.

3. Family of circles through given two points

Let given points be $A(x_1, y_1)$ and $B(x_2, y_2)$.

Equation of circle for which AB is diameter is

$$S \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Also, equation of line through points A and B is

$$L \equiv y - y_1 - \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = 0$$

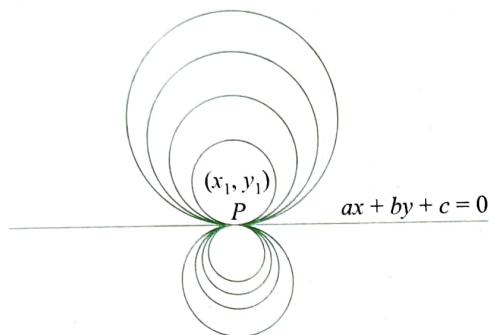
Therefore, equation of family of circles through A and B is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \left(y - y_1 - \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \right) = 0$$

4. Family of circles touching given line at a point

Let $P(x_1, y_1)$ is a point on the line $ax + by + c = 0$.

Now, innumerable circles can be drawn touching the line at point P as shown in the following figure.



Equation of above family of circles is given by

$$S + \lambda L = 0, \text{ where } \lambda \in R.$$

Here, $S = 0$ is the point circle at point P whose equation is $(x - x_1)^2 + (y - y_1)^2 = 0$ and $L = 0$ is given line.

ILLUSTRATION 4.96

Prove that the equation $x^2 + y^2 - 2x - 2ay - 8 = 0$, $a \in R$ represents the family of circles passing through two fixed points on x -axis.

Sol. We have $x^2 + y^2 - 2x - 2ay - 8 = 0$

$$\therefore (x^2 + y^2 - 2x - 8) - 2a(y) = 0, a \in R$$

This equation is of the form $S + \lambda L = 0$, which represents family of circles passing through points of intersection of circle $x^2 + y^2 - 2x - 8 = 0$ and line $y = 0$ (x -axis).

Solving circle and line, we have

$$x^2 - 2x - 8 = 0$$

$$\Rightarrow (x - 4)(x + 2) = 0$$

$$\Rightarrow x = 4, x = -2$$

So, circles pass through two fixed points $(4, 0)$ and $(-2, 0)$ on x -axis.

ILLUSTRATION 4.97

Find the equation of the circle passing through $(1, 1)$ and the points of intersection of the circles $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$.

Sol. Equation of the circle passing through the points of intersection of the given circles is

$$(x^2 + y^2 + 13x - 3y) + \lambda(2x^2 + 2y^2 + 4x - 7y - 25) = 0 \quad (1)$$

If this circle passes through the point $(1, 1)$, then

$$(1 + 1 + 13 - 3) + \lambda(2 + 2 + 4 - 7 - 25) = 0$$

$$\Rightarrow \lambda = 1/2$$

Substituting $\lambda = 1/2$ in (1), we get the equation of the required circle as $4x^2 + 4y^2 + 30x - 13y - 25 = 0$.

ILLUSTRATION 4.98

Find the equation of the smallest circle passing through the point of intersection of the line $x + y = 1$ and the circle $x^2 + y^2 = 9$.

Sol. Let the given circle and line intersect at points A and B . Equation of family of circles through A and B is

$$x^2 + y^2 - 9 + \lambda(x + y - 1) = 0, \lambda \in R$$

Variable centre of the circle is $\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$.

The smallest circle of this family is that for which A and B are end points of diameter.

So, centre $\left(-\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$ lies on the chord $x + y - 1 = 0$.

$$\Rightarrow -\frac{\lambda}{2} - \frac{\lambda}{2} - 1 = 0$$

$$\Rightarrow \lambda = -1$$

Using this value for λ , the equation of the smallest circle is

$$x^2 + y^2 - 9 - (x + y - 1) = 0$$

$$\text{or } x^2 + y^2 - x - y - 8 = 0$$

ILLUSTRATION 4.99

Find the equation of the circle which passes through the point $(1, 1)$ and which touches the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ at the point $(2, 3)$.

Sol. Given circle is

$$x^2 + y^2 + 4x - 6y - 3 = 0 \quad (1)$$

Required circle touches above circle at point $P(2, 3)$.

So, it also touches the tangent at point P to the given circle.

Equation of tangent at point $P(2, 3)$ to the given circle is

$$2x + 3y + 2(x + 2) - 3(y + 3) - 3 = 0$$

$$\text{or } x - 2 = 0$$

Equation of family of circles touching line $x - 2 = 0$ at $(2, 3)$ is given by

$$[(x - 2)^2 + (y - 3)^2] + \lambda(x - 2) = 0, \lambda \in R \quad (2)$$

If it passes through $(1, 1)$, then

$$1 + 4 + \lambda(-1) = 0 \text{ or } \lambda = 5$$

Putting the value of λ in equation (2), we get

$$(x - 2)^2 + (y - 3)^2 + 5(x - 2) = 0$$

$$\text{or } x^2 + y^2 + x - 6y + 3 = 0$$

This is the required circle.

ILLUSTRATION 4.100

Consider a family of circles passing through two fixed points $A(3, 7)$ and $B(6, 5)$. Show that the chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

Sol. The equation of the line passing through the points $A(3, 7)$ and $B(6, 5)$ is

$$y - 7 = -\frac{2}{3}(x - 3)$$

$$2x + 3y - 27 = 0$$

Also, the equation of the circle with A and B as the endpoints of diameter is

$$(x - 3)(x - 6) + (y - 7)(y - 5) = 0$$

Now, the equation of the family of circles through A and B is

$$(x - 3)(x - 6) + (y - 7)(y - 5) + \lambda(2x + 3y - 27) = 0 \quad (1)$$

The equation of the common chord of (1) and $x^2 + y^2 - 4x - 6y - 3 = 0$ is the radical axis, which is

$$[(x - 3)(x - 6) + (y - 7)(y - 5) + \lambda(2x + 3y - 27)] - [x^2 + y^2 - 4x - 6y - 3] = 0$$

$$\text{or } (2\lambda - 5)x + (3\lambda - 6)y + (-27\lambda + 56) = 0$$

$$\text{or } (-5x - 6y + 56) + \lambda(2x + 3y - 27) = 0$$

This is the family of lines which passes through the point of intersection of $-5x - 6y + 56 = 0$ and $2x + 3y - 27 = 0$, i.e., $(2, 23/3)$.

ILLUSTRATION 4.101

If C_1 , C_2 and C_3 belong to a family of circles through the points (x_1, y_1) and (x_2, y_2) , then prove that the ratio of the lengths of the tangents from any point on C_1 to the circles C_2 and C_3 is constant.

Sol. Equations of circles C_1 , C_2 and C_3 through $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = f(x, y)$$

$$+ \lambda_r \left(y - y_1 - \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \right) = 0; r = 1, 2, 3$$

$$\text{or } f(x, y) + \lambda_r g(x, y) = 0; r = 1, 2, 3$$

Consider point $P(h, k)$ on the circle C_1 .

$$\therefore f(h, k) + \lambda_1 g(h, k) = 0 \quad (1)$$

Let T_2 and T_3 be the lengths of the tangents from P to C_2 and C_3 , respectively.

$$\therefore \frac{T_2}{T_3} = \frac{\sqrt{f(h, k) + \lambda_2 g(h, k)}}{\sqrt{f(h, k) + \lambda_3 g(h, k)}}$$

$$= \frac{\sqrt{-\lambda_1 g(h, k) + \lambda_2 g(h, k)}}{\sqrt{-\lambda_1 g(h, k) + \lambda_3 g(h, k)}} = \sqrt{\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}}$$

Clearly, this ratio is independent of the choice of (h, k) and hence, a constant.

CONCEPT APPLICATION EXERCISE 4.6

- If the circle $x^2 + y^2 + 2x + 3y + 1 = 0$ cuts $x^2 + y^2 + 4x + 3y + 2 = 0$ at A and B , then find the equation of the circle on AB as diameter.
- Find the radius of the smallest circle which touches the straight line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$. Compute up to one place of decimal only.
- Let S_1 be a circle passing through $A(0, 1)$ and $B(-2, 2)$ and S_2 be a circle of radius $\sqrt{10}$ units such that AB is the common chord of S_1 and S_2 . Find the equation of S_2 .

4. Find the equation of the circle touching the line $2x + 3y + 1 = 0$ at $(1, -1)$ and cutting orthogonally the circle having line segment joining $(0, 3)$ and $(-2, -1)$ as diameter.

5. A variable circle which always touches the line $x + y - 2 = 0$ at $(1, 1)$ cuts the circle $x^2 + y^2 + 4x + 5y - 6 = 0$. Prove that all the common chords of intersection pass through a fixed point. Find that point.

ANSWERS

- $2x^2 + 2y^2 + 2x + 6y + 1 = 0$
- 1.5 unit
- $x^2 + y^2 + 2x - 3y + 2 \pm \sqrt{7}(x + 2y - 2) = 0$
- $2x^2 + 2y^2 - 10x - 5y + 1 = 0$
- $(6, -4)$

Solved Examples

EXAMPLE 4.1

The line $Ax + By + C = 0$ cuts the circle $x^2 + y^2 + ax + by + c = 0$ at P and Q . The line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ at R and S . If P , Q , R , and S are concyclic, then show that

$$\begin{vmatrix} a - a' & b - b' & c - c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Sol. P and Q are the points of intersection of the line

$$L_1 \equiv Ax + By + C = 0 \quad (1)$$

and the circle

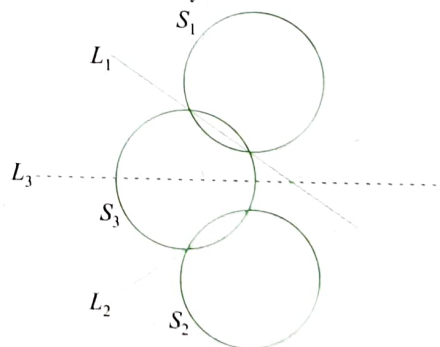
$$S_1 \equiv x^2 + y^2 + ax + by + c = 0 \quad (2)$$

R and S are the points of intersection of the line

$$L_2 \equiv A'x + B'y + C' = 0 \quad (3)$$

and the circle

$$S_2 \equiv x^2 + y^2 + a'x + b'y + c' = 0 \quad (4)$$



The radical axis of the circles $S_1 = 0$ and $S_2 = 0$ is

$$S_1 - S_2 = 0$$

$$\text{i.e., } L_3 \equiv (a - a')x + (b - b')y + (c - c') = 0 \quad (5)$$

If P , Q , R , and S are concyclic and $S_3 = 0$ is the equation of this circle through P , Q , R , and S , line (1) is the radical axis of the circles $S_1 = 0$ and $S_3 = 0$ and line (2) is the radical axis of the circles $S_2 = 0$ and $S_3 = 0$.

Thus, the straight lines given by (5), (1), and (3) are the radical axes of the circles $S_1 = 0$, $S_2 = 0$, and $S_3 = 0$ taken in pairs.

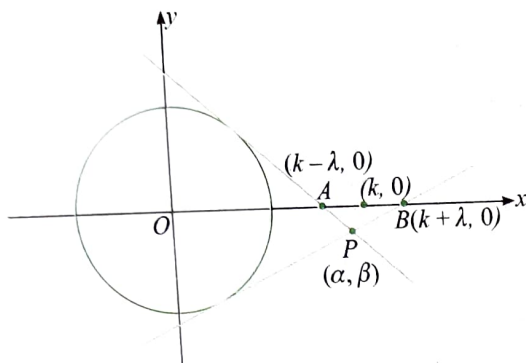
Since the radical axes of the three circles taken in pairs are concurrent or parallel, we have

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

EXAMPLE 4.2

Tangents are drawn to the circle $x^2 + y^2 = a^2$ from two points on the x -axis, equidistant from the point $(k, 0)$. Show that the locus of their point of intersection is $ky^2 = a^2(k - x)$.

Sol.



As shown in the figure, points A and B are equidistant from point $(k, 0)$.

Tangents drawn from points A and B intersect at point $P(\alpha, \beta)$.

Equation of pair of tangents from point P is

$$(x^2 + y^2 - a^2)(\alpha^2 + \beta^2 - a^2) = (\alpha x + \beta y - a^2)^2 \quad (\text{Using } T^2 = SS_1)$$

This equation is satisfied by points A and B .

$$\therefore ((k - \lambda)^2 - a^2)(\alpha^2 + \beta^2 - a^2) = (\alpha(k - \lambda) - a^2)^2 \quad (1)$$

$$\text{and } ((k + \lambda)^2 - a^2)(\alpha^2 + \beta^2 - a^2) = (\alpha(k + \lambda) - a^2)^2 \quad (2)$$

Subtracting (1) from (2), we get

$$(\alpha^2 + \beta^2 - a^2)(4k\lambda) = [(k + \lambda)\alpha - a^2 + (k - \lambda)\alpha - a^2] [(k + \lambda)\alpha - (k - \lambda)\alpha]$$

$$= [2(k\alpha - a^2)][2\lambda\alpha] = 4\lambda\alpha[k\alpha - a^2]$$

$$\Rightarrow k(\alpha^2 + \beta^2 - a^2) = k\alpha^2 - a^2\alpha$$

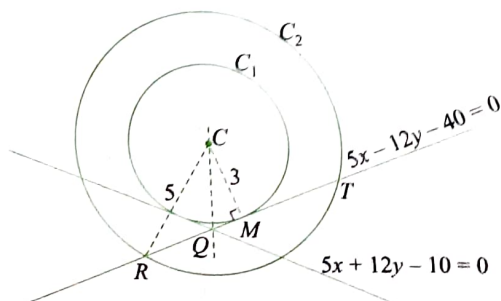
$$\Rightarrow k\beta^2 = a^2(k - \alpha)$$

Hence, the locus is $ky^2 = a^2(k - x)$.

EXAMPLE 4.3

Lines $5x + 12y - 10 = 0$ and $5x - 12y - 40 = 0$ touch a circle C_1 of diameter 6. If the centre of C_1 lies in the first quadrant, find the equation of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 units on these lines.

Sol. According to the question, circles are drawn as shown in the figure.



Centre of the circles lies on one of the angle bisectors of the given lines.

$$\frac{5x + 12y - 10}{13} = \pm \frac{5x - 12y - 40}{13}$$

$$\text{or } 24y = -30 \text{ and } 10x = 50$$

But centre lies in first quadrant.

So, for centre, $x = 5$.

Distance of centre $C(5, y)$ from the line $5x + 12y - 10 = 0$ is 3.

$$\therefore \frac{5(5) + 12y - 10}{13} = \pm 3$$

$$\therefore y = 2$$

(As centre lies in first quadrant)

So, centre is $(5, 2)$.

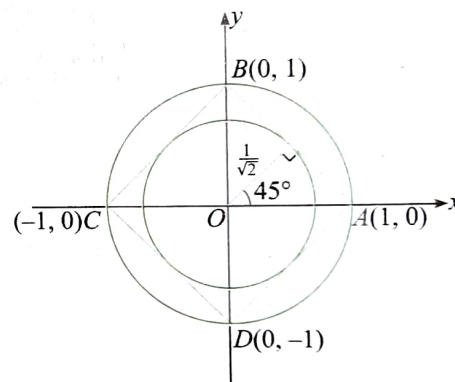
Also, in right angled triangle CMR , $CR = 5$ (as $RM = 4$).

Hence, equation of circle is $(x - 5)^2 + (y - 2)^2 = 5^2$.

EXAMPLE 4.4

If eight distinct points can be found on the curve $|x| + |y| = 1$ such that from each point, two mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$, then find the range of a .

Sol. $|x| + |y| = 1$ forms a square having vertices at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$.



Mutually perpendicular tangents meet on the director circle whose equation is $x^2 + y^2 = 2a^2$.

Equation of circle passing through vertices of square is $x^2 + y^2 = 1$ and its radius is 1.

Equation of circle inscribed in the circle is $x^2 + y^2 = \frac{1}{2}$ and its radius is $\frac{1}{\sqrt{2}}$.

For eight points on the square, radius of the director circle must lie between $\frac{1}{\sqrt{2}}$ and 1.

$$\text{Thus, } \frac{1}{\sqrt{2}} < \sqrt{2}a < 1.$$

$$\text{Hence, } \frac{1}{2} < a < \frac{1}{\sqrt{2}}$$

EXAMPLE 4.5

Let AB be the chord of contact of circle $x^2 + y^2 = 5$ w.r.t. the point $(5, -5)$. Then find the locus of the orthocenter of the triangle PAB , where P is variable point on the circle.

Sol. Equation of chord of contact AB of circle $x^2 + y^2 = 5$ w.r.t. point $(5, -5)$ is
 $5x - 5y = 5$ or $x - y = 1$.

Solving with the circle we get

$$\begin{aligned}x^2 + (x-1)^2 &= 5 \\x^2 - x - 2 &= 0 \\(x-2)(x+1) &= 0 \\x &= -1, 2 \\y &= -2, 1\end{aligned}$$

So, chord of contact meets the circle at $A(-1, -2)$ and $B(2, 1)$.

Let point P on the circle be $(\sqrt{5} \cos \theta, \sqrt{5} \sin \theta)$.

Now, as circumcentre of the triangle PAB is origin.

For orthocentre (h, k) , we have

$$h = -1 + 2 + \sqrt{5} \cos \theta \text{ and } k = -2 + 1 + \sqrt{5} \sin \theta$$

$$\text{or } h - 1 = \sqrt{5} \cos \theta \text{ and } k + 1 = \sqrt{5} \sin \theta$$

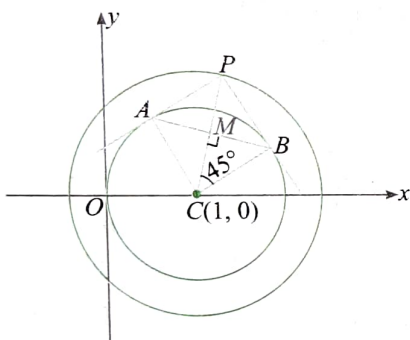
Squaring and adding, we get $(h-1)^2 + (k+1)^2 = 5$.

Hence, required locus is $(x-1)^2 + (y+1)^2 = 5$.

EXAMPLE 4.6

Let P be a variable point on the circle $x^2 + y^2 - 2x = 1$ and AB be the chord of contact of the circle $x^2 + y^2 - 2x = 0$ w.r.t. point P . Find the locus of the circumcentre of the triangle CAB , C being centre of the given circle.

Sol.



The two circles are

$$(x-1)^2 + y^2 = 1 \quad (1)$$

$$(x-1)^2 + y^2 = 2 \quad (2)$$

Clearly, the second circle is the director circle of the first circle.

$$\therefore \angle APB = 90^\circ$$

So, $APBC$ is square.

Now, circumcentre of the right-angled triangle CAB will be midpoint of hypotenuse.

Let midpoint of AB be $M(h, k)$.

$$\text{Now, } CM = CB \sin 45^\circ = \frac{1}{\sqrt{2}}$$

$$\text{So, } (h-1)^2 + k^2 = \frac{1}{2}$$

$$\text{Hence, locus of } M \text{ is } (x-1)^2 + y^2 = \frac{1}{2}.$$

EXAMPLE 4.7

A and B are two points in the xy -plane, which are $2\sqrt{2}$ unit distance apart and subtend an angle of 90° at the point $C(1, 2)$ on the line $x - y + 1 = 0$, which is larger than any angle subtended by the line segment AB at any other point on the line. Find the equation(s) of the circle through the points A, B , and C .

Sol. AB subtends the greatest angle at C . So, the line $x - y + 1 = 0$ touches the circle at C and, hence, AB is the diameter.

The family of circles touching the line

$$x - y + 1 = 0 \text{ at point } (1, 2) \text{ is } (x-1)^2 + (y-2)^2 + \lambda(x-y+1) = 0$$

Its radius is

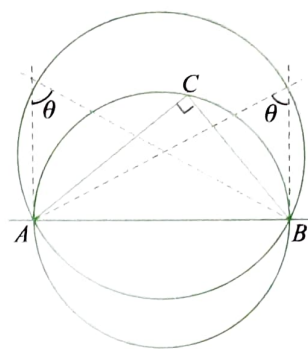
$$\sqrt{\left(\frac{\lambda-2}{2}\right)^2 + \left(\frac{\lambda+4}{2}\right)^2 - (5+\lambda)} = \sqrt{2}$$

$$\text{or } \lambda = \pm 2$$

Therefore, the equations of the circles are

$$x^2 + y^2 - 6y + 7 = 0$$

$$\text{and } x^2 + y^2 - 4x - 2y + 3 = 0$$



EXAMPLE 4.8

Let a given line L_1 intersect the x - and y -axes at P and Q , respectively. Let another line L_2 , perpendicular to L_1 , cut the x - and the y -axis at R and S , respectively. Show that the locus of the point of intersection of the lines PS and QR is a circle passing through the origin.

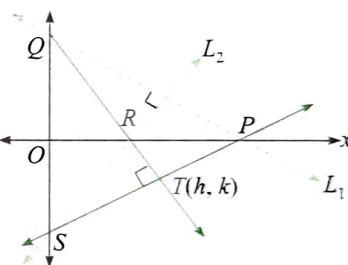
Sol. From the figure in ΔPQS ,

$$PQ \perp SR, PR \perp QS$$

Then we have $QT \perp PS$ (as the altitudes of a triangle are concurrent).

Therefore, points O, R, T , and S are concyclic.

Hence, the locus of T is a circle which passes through the origin.



EXAMPLE 4.9

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends a right angle at the origin.

Sol. The equation of the given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

Let $P(h, k)$ be the foot of perpendicular drawn from the origin to the chord LM of circle S .

$$\text{Slope of } OP = \frac{k}{h}$$

$$\therefore \text{Slope of } LM = -\frac{h}{k}$$

Therefore, the equation of LM is

$$y - k = -\frac{h}{k}(x - h)$$

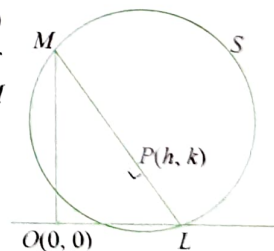
$$\text{or } ky - k^2 = -hx + h^2$$

$$\text{or } hx + ky = h^2 + k^2 \quad (2)$$

Now, the combined equations of lines joining the points of intersection of (1) and (2) to the origin can be obtained by making (1) homogeneous with the help of (2) as follows:

$$x^2 + y^2 + (2gx + 2fy) \left(\frac{hx + ky}{h^2 + k^2} \right) + c \left(\frac{hx + ky}{h^2 + k^2} \right)^2 = 0$$

$$\text{or } (h^2 + k^2)^2(x^2 + y^2) + (h^2 + k^2)(2gx + 2fy)(hx + ky) + c(hx + ky)^2 = 0 \quad (3)$$



Equation (3) represents the combined equation of OL and OM .

But $\angle LOM = 90^\circ$.

Therefore, from (3), we must have

Coeff. of x^2 + Coeff. of $y^2 = 0$

$$\text{or } (h^2 + k^2)^2 + 2gh(h^2 + k^2) + ch^2 + (h^2 + k^2)^2 + 2fk(h^2 + k^2) + ck^2 = 0$$

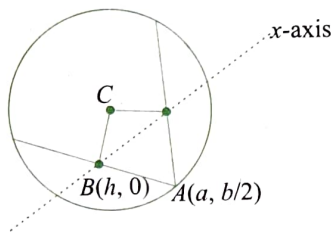
$$\text{or } h^2 + k^2 + gh + fk + \frac{c}{2} = 0$$

Therefore, the locus of (h, k) is $x^2 + y^2 + gx + fy + \frac{c}{2} = 0$.

EXAMPLE 4.10

Let a circle be given by $2x(x-a) + y(2y-b) = 0$, ($a \neq 0, b \neq 0$). Find the condition on a and b if two chords, each bisected by the x -axis, can be drawn to the circle from $(a, b/2)$.

Sol.



The given circle is

$$2x(x-a) + y(2y-b) = 0 \quad (a, b \neq 0)$$

$$\text{or } x^2 + y^2 - ax - \frac{b}{2}y = 0 \quad (1)$$

Its center is $C(a/2, b/4)$.

Also, point $A(a, b/2)$ lies on the circle.

Since chord is bisected at point $B(h, 0)$ on the x -axis, we have

$$(\text{Slope of } BC) \times (\text{Slope of } AB) = -1$$

$$\text{or } \left\{ \frac{(b/4) - 0}{(a/2) - h} \right\} \left\{ \frac{(b/2) - 0}{a - h} \right\} = -1$$

$$h^2 - 3\frac{a}{2}h + \frac{a^2}{2} + \frac{b^2}{8} = 0$$

Since two such chords exist, the above equation must have two distinct real roots, for which its discriminant must be positive, i.e.,

$$\text{or } \frac{9a^2}{4} - 4\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$$

$$\text{or } a^2 > 2b^2$$

EXAMPLE 4.11

Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line from point P intersects the curve at points Q and R . If the product $PQ \cdot PR$ is independent of the slope of the line, then show that the curve is a circle.

Sol. The given curve is

$$ax^2 + 2hxy + by^2 = 1 \quad (1)$$

Let the point P not lying on curve (1) be (x_1, y_1) .

Let θ be the inclination of line through P which intersects the given curve at Q and R .

The equation of line through P is

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

$$\text{or } x = r \cos \theta + x_1 \text{ and } y = r \sin \theta + y_1$$

For point Q or R , the above point must lie on curve (1). So,

$$a[x_1 + r \cos \theta]^2 + 2h[x_1 + r \cos \theta][y_1 + r \sin \theta] + b[y_1 + r \sin \theta]^2 = 1$$

$$\text{or } (a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta)r^2 + 2(ax_1 \cos \theta + hy_1 \sin \theta)r + (ax_1^2 + 2hx_1y_1 + by_1^2 - 1) = 0$$

It is a quadratic in r giving two values of r for PQ and PR . Therefore,

Product of roots = $PQ \cdot PR$

$$= \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta}$$

Here, $ax_1^2 + 2hx_1y_1 + by_1^2 - 1 \neq 0$ as (x_1, y_1) does not lie on curve (1). Also,

$$\text{Denominator} = a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta$$

$$= a + 2h \sin \theta \cos \theta + (b - a) \sin^2 \theta$$

$$= a + h \sin 2\theta + \frac{(b-a)}{2} (1 - \cos 2\theta)$$

$$= \left(\frac{a+b}{2}\right) + h \sin 2\theta + \left(\frac{a-b}{2}\right) \cos 2\theta$$

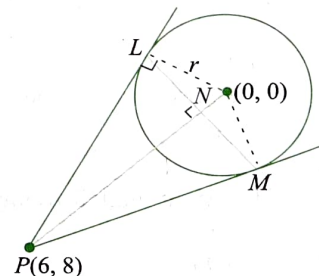
which is independent of θ if $h = 0$, and $(a-b)/2 = 0$, i.e., $h = 0$ and $a = b$.

Hence, the given equation is a circle.

EXAMPLE 4.12

For the circle $x^2 + y^2 = r^2$, find the value of r for which the area enclosed by the tangents drawn from the point $P(6, 8)$ to the circle and the chord of contact is maximum.

Sol. The given circle is $x^2 + y^2 = r^2$. From point $(6, 8)$, tangents are drawn to this circle.



Then, the length of tangent is

$$PL = \sqrt{6^2 + 8^2 - r^2} = \sqrt{100 - r^2}$$

Also, the equation of chord of contact LM is $6x + 8y - r^2 = 0$.

PN = Length of $\perp$ from P to LM

$$= \frac{36 + 64 - r^2}{\sqrt{36 + 64}} = \frac{100 - r^2}{10}$$

Now, in right-angled $\triangle PLN$,

$$LN^2 = PL^2 - PN^2 = (100 - r^2) - \frac{(100 - r^2)^2}{100} = \frac{(100 - r^2)r^2}{100}$$

$$\begin{aligned} \text{or } LN &= \frac{r\sqrt{100-r^2}}{10} \\ \therefore LM &= \frac{r\sqrt{100-r^2}}{5} \quad (\because LM = 2LN) \\ \therefore \text{Area of } \triangle PLM = \Delta &= \frac{1}{2} \times LM \times PN \\ &= \frac{1}{2} \times \frac{r\sqrt{100-r^2}}{5} \times \frac{100-r^2}{10} \\ &= \frac{1}{100} [r(100-r^2)^{3/2}] \end{aligned}$$

For the maximum value of Δ , we should have

$$\begin{aligned} \frac{d\Delta}{dr} &= 0 \\ \text{or } \frac{1}{100} \left[(100-r^2)^{3/2} + r \cdot \frac{3}{2} (100-r^2)^{1/2} (-2r) \right] &= 0 \\ \text{or } (100-r^2)^{1/2} [100-r^2-3r^2] &= 0 \\ \text{i.e., } r &= 10 \text{ or } r = 5 \end{aligned}$$

But $r = 10$ gives the length of tangent PL . Therefore, $r \neq 10$.

Hence, $r = 5$.

EXAMPLE 4.13

A circle of radius 1 unit touches positive x -axis and positive y -axis at A and B , respectively. A variable line passing through origin intersects the circle in two points D and E . The area of the triangle DEB is maximum when the slope of the line is m . Find the value of m .

Sol. The equation of the circle is

$$(x-1)^2 + (y-1)^2 = 1$$

$$\text{or } x^2 + y^2 - 2x - 2y + 1 = 0 \quad (1)$$

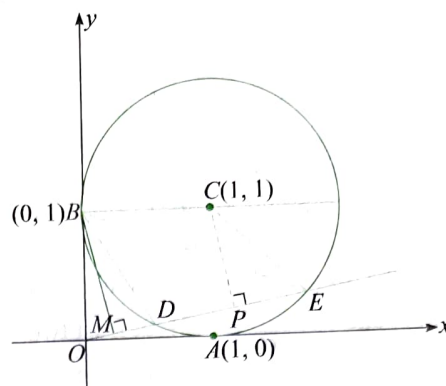
Let the equation of the variable straight line through origin be

$$y = mx \quad (2)$$

This line intersects the circle at two points D and E .

Distance of line from point $B(0, 1)$ is

$$BM = \frac{|m(0) - 1|}{\sqrt{1+m^2}} = \frac{1}{\sqrt{1+m^2}}$$



$$\text{Also, } CP = \frac{|m(1) - 1|}{\sqrt{1+m^2}} = \frac{|m-1|}{\sqrt{1+m^2}}$$

$$\therefore DE = 2PE = 2\sqrt{CE^2 - CP^2} = 2\sqrt{1 - \frac{(m-1)^2}{1+m^2}} = 2\sqrt{\frac{2m}{1+m^2}}$$

$\therefore$ Area of triangle DEB ,

$$\begin{aligned} \Delta &= \frac{1}{2} BM \times DE \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{1+m^2}} \times 2\sqrt{\frac{2m}{1+m^2}} = \frac{\sqrt{2m}}{1+m^2} \end{aligned}$$

Differentiating Δ w.r.t. m , we get

$$\frac{d\Delta}{dm} = \sqrt{2} \left[\frac{\frac{1}{2\sqrt{m}}(1+m^2) - 2m\sqrt{m}}{(1+m^2)^2} \right] = \frac{1-3m^2}{\sqrt{2}\sqrt{m}(1+m^2)^2}$$

If $\frac{d\Delta}{dm} = 0$, then $m = \frac{1}{\sqrt{3}}$, which is point of maxima.

Therefore, area is maximum if $m = \frac{1}{\sqrt{3}}$.

Exercises

Single Correct Answer Type

- The number of rational point(s) [a point (a, b) is called rational, if a and b both are rational numbers] on the circumference of a circle having center (π, e) is
 - at most one
 - at least two
 - exactly two
 - infinite
- The equation of the circle which has normals $(x-1) \times (y-2) = 0$ and a tangent $3x + 4y = 6$ is
 - $x^2 + y^2 - 2x - 4y + 4 = 0$
 - $x^2 + y^2 - 2x - 4y + 5 = 0$
 - $x^2 + y^2 = 5$
 - $(x-3)^2 + (y-4)^2 = 5$
- In triangle ABC , the equation of side BC is $x - y = 0$. The circumcenter and orthocenter of the triangle are $(2, 3)$ and $(5, 8)$, respectively. The equation of the circumcircle of the triangle is
 - $x^2 + y^2 - 4x - 6y - 27 = 0$
 - $x^2 + y^2 - 4x - 6y - 36 = 0$
 - $x^2 + y^2 - 4x - 6y - 24 = 0$
 - $x^2 + y^2 - 4x - 6y - 15 = 0$
- A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centers of the circles. The area of the rhombus is
 - $8\sqrt{3}$ sq. units
 - $4\sqrt{3}$ sq. units
 - $6\sqrt{3}$ sq. units
 - none of these
- The locus of the center of the circles such that the point $(2, 3)$ is the midpoint of the chord $5x + 2y = 16$ is
 - $2x - 5y + 11 = 0$
 - $2x + 5y - 11 = 0$
 - $2x + 5y + 11 = 0$
 - none of these
- Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to the x -axis. If (h, k) are the coordinates of the center of the circles, then the set of values of k is given by the interval
 - $k \geq \frac{1}{2}$
 - $-\frac{1}{2} \leq k \leq \frac{1}{2}$
 - $k \leq \frac{1}{2}$
 - $0 < k < \frac{1}{2}$
- The line $2x - y + 1 = 0$ is tangent to the circle at the point $(2, 5)$ and the center of the circle lies on $x - 2y = 4$. The radius of the circle is
 - $3\sqrt{5}$
 - $5\sqrt{3}$
 - $2\sqrt{5}$
 - $5\sqrt{2}$
- A right-angled isosceles triangle is inscribed in the circle $x^2 + y^2 - 4x - 2y - 4 = 0$ then length of its side is
 - $3\sqrt{2}$
 - $2\sqrt{2}$
 - $\sqrt{2}$
 - $4\sqrt{2}$
- $f(x, y) = x^2 + y^2 + 2ax + 2by + c = 0$ represents a circle. If $f(x, 0) = 0$ has equal roots, each being 2, and $f(0, y) = 0$ has 2 and 3 as its roots, then the center of the circle is
 - $(2, 5/2)$
 - Data are not sufficient.
 - $(-2, -5/2)$
 - Data are inconsistent.
- The equation of the circumcircle of an equilateral triangle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and one vertex of the triangle is $(1, 1)$. The equation of the incircle of the triangle is
 - $4(x^2 + y^2) = g^2 + f^2$
 - $4(x^2 + y^2) + 8gx + 8fy = (1-g)(1+3g) + (1-f)(1+3f)$
 - $4(x^2 + y^2) + 8gx + 8fy = g^2 + f^2$
 - none of these
- If it is possible to draw a triangle which circumscribes the circle $(x - (\alpha - 2\beta))^2 + (y - (\alpha + \beta))^2 = 1$ and is inscribed by $x^2 + y^2 - 2x - 4y + 1 = 0$, then
 - $\beta = -\frac{1}{3}$
 - $\beta = \frac{2}{3}$
 - $\alpha = \frac{5}{3}$
 - $\alpha = -\frac{5}{2}$
- The locus of centre of the circle $(x \cos \alpha + y \sin \alpha - a)^2 + (x \sin \alpha - y \cos \alpha - b)^2 = k^2$ if α varies, is
 - $x^2 - y^2 = a^2 + b^2$
 - $x^2 - y^2 = a^2 b^2$
 - $x^2 + y^2 = a^2 + b^2$
 - $x^2 + y^2 = a^2 b^2$
- A circle of radius unity is centered at the origin. Two particles start moving at the same time from the point $(1, 0)$ and move around the circle in opposite direction. One of the particle moves anticlockwise with constant speed v and the other moves clockwise with constant speed $3v$. After leaving $(1, 0)$, the two particles meet first at a point P , and continue until they meet next at point Q . The coordinates of the point Q are
 - $(1, 0)$
 - $(0, 1)$
 - $(0, -1)$
 - $(-1, 0)$
- $ABCD$ is a square of unit area. A circle is tangent to two sides of $ABCD$ and passes through exactly one of its vertices. The radius of the circle is
 - $2 - \sqrt{2}$
 - $\sqrt{2} - 1$
 - $1/2$
 - $1/\sqrt{2}$
- A circle of constant radius a passes through the origin O and cuts the axes of coordinates at points P and Q . Then the equation of the locus of the foot of perpendicular from O to PQ is
 - $(x^2 + y^2) \left(\frac{1}{x^2} + \frac{1}{y^2} \right) = 4a^2$
 - $(x^2 + y^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2} \right) = a^2$
 - $(x^2 + y^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2} \right) = 4a^2$
 - $(x^2 + y^2) \left(\frac{1}{x^2} + \frac{1}{y^2} \right) = a^2$

16. The circle $x^2 + y^2 = 4$ cuts the line joining the points $A(1, 0)$ and $B(3, 4)$ at two points, P and Q . Let $BP/PA = \alpha$ and $BQ/QA = \beta$. Then α and β are the roots of the quadratic equation
- (1) $3x^2 - 16x + 21 = 0$ (2) $x^2 - 8x + 7 = 0$
 (3) $x^2 - 9x + 8 = 0$ (4) none of these
17. If a circle of constant radius $3k$ passes through the origin O and meets the coordinate axes at A and B , then the locus of the centroid of triangle OAB is
- (1) $x^2 + y^2 = (2k)^2$ (2) $x^2 + y^2 = (3k)^2$
 (3) $x^2 + y^2 = (4k)^2$ (4) $x^2 + y^2 = (6k)^2$
18. $(-6, 0)$, $(0, 6)$, and $(-7, 7)$ are the vertices of a $\triangle ABC$. The incircle of the triangle has equation
- (1) $x^2 + y^2 - 9x - 9y + 36 = 0$
 (2) $x^2 + y^2 + 9x - 9y + 36 = 0$
 (3) $x^2 + y^2 + 9x + 9y - 36 = 0$
 (4) $x^2 + y^2 + 18x - 18y + 36 = 0$
19. If O is the origin and OP and OQ are the tangents from the origin to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$, then the circumcenter of triangle OPQ is
- (1) $(3, -2)$ (2) $(3/2, -1)$
 (3) $(3/4, -1/2)$ (4) $(-3/2, 1)$
20. The difference between the radii of the largest and the smallest circles which have their center on the circumference of the circle $x^2 + y^2 + 2x + 4y - 4 = 0$ and pass through the point (a, b) lying outside the given circle is
- (1) 6 (2) $\sqrt{(a+1)^2 + (b+2)^2}$
 (3) 3 (4) $\sqrt{(a+1)^2 + (b+2)^2} - 3$
21. If the conics whose equations are $S \equiv \sin^2 \theta x^2 + 2hxy + \cos^2 \theta y^2 + 32x + 16y + 19 = 0$, $S' \equiv \cos^2 \theta x^2 + 2h'xy + \sin^2 \theta y^2 + 16x + 32y + 19 = 0$ intersect at four concyclic points, then, (where $\theta \in R$)
- (1) $h + h' = 0$ (2) $h = h'$
 (3) $h + h' = 1$ (4) none of these
22. From a point $R(5, 8)$, two tangents RP and RQ are drawn to a given circle $S = 0$ whose radius is 5. If the circumcenter of triangle PQR is $(2, 3)$, then the equation of the circle $S = 0$ is
- (1) $x^2 + y^2 + 2x + 4y - 20 = 0$
 (2) $x^2 + y^2 + x + 2y - 10 = 0$
 (3) $x^2 + y^2 - x - 2y - 20 = 0$
 (4) $x^2 + y^2 - 4x - 6y - 12 = 0$
23. The ends of a quadrant of a circle have the coordinates $(1, 3)$ and $(3, 1)$. Then the center of such a circle is
- (1) $(2, 2)$ (2) $(1, 1)$ (3) $(4, 4)$ (4) $(2, 6)$
24. Let P be a point on the circle $x^2 + y^2 = 9$, Q a point on the line $7x + y + 3 = 0$, and the perpendicular bisector of PQ be the line $x - y + 1 = 0$. Then the coordinates of P are
- (1) $(0, -3)$ (2) $(0, 3)$
 (3) $(72/25, 21/25)$ (4) $(-72/25, 21/25)$
25. The locus of the center of the circle touching the line $2x - y = 1$ at $(1, 1)$ is
- (1) $x + 3y = 2$ (2) $x + 2y = 3$
 (3) $x + y = 2$ (4) none of these
26. A triangle is inscribed in a circle of radius 1. The distance between the orthocenter and the circumcenter of the triangle cannot be
- (1) 1 (2) 2 (3) $3/2$ (4) 4
27. The equation of the chord of the circle $x^2 + y^2 - 3x - 4y - 4 = 0$, which passes through the origin such that the origin divides it in the ratio $4 : 1$, is
- (1) $x = 0$ (2) $24x + 7y = 0$
 (3) $7x + 24y = 0$ (4) $7x - 24y = 0$
28. If OA and OB are equal perpendicular chords of the circles $x^2 + y^2 - 2x + 4y = 0$, then the equations of OA and OB are, where O is the origin,
- (1) $3x + y = 0$ and $3x - y = 0$
 (2) $3x + y = 0$ or $3y - x = 0$
 (3) $x + 3y = 0$ and $y - 3x = 0$
 (4) $x + y = 0$ or $x - y = 0$
29. A region in the x - y plane is bounded by the curve $y = \sqrt{25 - x^2}$ and the line $y = 0$. If the point $(a, a + 1)$ lies in the interior of the region, then
- (1) $a \in (-4, 3)$ (2) $a \in (-\infty, -1) \in (3, \infty)$
 (3) $a \in (-1, 3)$ (4) none of these
30. A circle is inscribed in a rhombus $ABCD$ with one angle 60° . The distance from the center of the circle to the nearest vertex is equal to 1. If P is any point on the circle, then $|PA|^2 + |PB|^2 + |PC|^2 + |PD|^2$ is equal to
- (1) 12 (2) 11
 (3) 9 (4) none of these
31. The equation of a line inclined at an angle of $\pi/4$ to the X -axis, such that the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal lengths on it, is
- (1) $2x - 2y - 3 = 0$ (2) $2x - 2y + 3 = 0$
 (3) $x - y + 6 = 0$ (4) $x - y - 6 = 0$
32. If the chord $y = mx + 1$ of the circles $x^2 + y^2 = 1$ subtends an angle of 45° at the major segment of the circle, then the value of m is
- (1) 2 (2) -2
 (3) -1 (4) none of these
33. A straight line l_1 with equation $x - 2y + 10 = 0$ meets the circle with equation $x^2 + y^2 = 100$ at B in the first quadrant. A line through B perpendicular to l_1 cuts the y -axis at $P(0, t)$. The value of t is
- (1) 12 (2) 15 (3) 20 (4) 25

34. A variable chord of the circle $x^2 + y^2 = 4$ is drawn from the point $P(3, 5)$ meeting the circle at the points A and B . A point Q is taken on this chord such that $2PQ = PA + PB$. The locus of Q is
- (1) $x^2 + y^2 + 3x + 4y = 0$ (2) $x^2 + y^2 = 36$
 (3) $x^2 + y^2 = 16$ (4) $x^2 + y^2 - 3x - 5y = 0$
35. The range of values of r for which the point $\left(-5 + \frac{r}{\sqrt{2}}, -3 + \frac{r}{\sqrt{2}}\right)$ is an interior point of the major segment of the circle $x^2 + y^2 = 16$, cut-off by the line $x + y = 2$, is
- (1) $(-\infty, 5\sqrt{2})$
 (2) $(4\sqrt{2} - \sqrt{14}, 5\sqrt{2})$
 (3) $(4\sqrt{2} - \sqrt{14}, 4\sqrt{2} + \sqrt{14})$
 (4) none of these
36. A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the coordinate axes. The coordinates of its vertices are
- (1) $(-6, -9), (-6, 5), (8, -9), (8, 5)$
 (2) $(-6, 9), (-6, -5), (8, -9), (8, 5)$
 (3) $(-6, -9), (-6, 5), (8, 9), (8, 5)$
 (4) $(-6, -9), (-6, 5), (8, -9), (8, -5)$
37. If a line passes through the point $P(1, -2)$ and cuts the circle $x^2 + y^2 - x - y = 0$ at A and B , then the maximum value of $PA + PB$ is
- (1) $\sqrt{26}$ (2) 8 (3) $\sqrt{8}$ (4) $2\sqrt{8}$
38. The area of the triangle formed by joining the origin to the points of intersection of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and the circle $x^2 + y^2 = 10$ is
- (1) 3 (2) 4 (3) 5 (4) 6
39. If (α, β) is a point on the circle whose center is on the x -axis and which touches the line $x + y = 0$ at $(2, -2)$, then the greatest value of α is
- (1) $4 - \sqrt{2}$ (2) 6 (3) $4 + 2\sqrt{2}$ (4) $4 + \sqrt{2}$
40. The area bounded by the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, and the pair of lines $\sqrt{3}(x^2 + y^2) = 4xy$ in first quadrant is equal to
- (1) $\pi/2$ (2) $5\pi/2$ (3) 3π (4) $\pi/4$
41. The number of integral values of y for which the chord of the circle $x^2 + y^2 = 125$ passing through the point $P(8, y)$ gets bisected at the point $P(8, y)$ and has integral slope is
- (1) 8 (2) 6 (3) 4 (4) 2
42. The straight line $x \cos \theta + y \sin \theta = 2$ will touch the circle $x^2 + y^2 - 2x = 0$ if
- (1) $\theta = n\pi, n \in I$ (2) $A = (2n + 1)\pi, n \in I$
 (3) $\theta = 2n\pi, n \in I$ (4) none of these
43. The range of values of λ , $\lambda > 0$, such that the angle θ between the pair of tangents drawn from $(\lambda, 0)$ to the circle $x^2 + y^2 = 4$ lies in $(\pi/2, 2\pi/3)$ is
- (1) $(4/\sqrt{3}, 2\sqrt{2})$ (2) $(0, \sqrt{2})$
 (3) $(1, 2)$ (4) none of these
44. The circles which can be drawn to pass through $(1, 0)$ and $(3, 0)$ and to touch the y -axis intersect at an angle θ . Then $\cos \theta$ is equal to
- (1) $1/2$ (2) $1/3$ (3) $1/4$ (4) $-1/4$
45. The locus of the midpoints of the chords of contact of $x^2 + y^2 = 2$ from the points on the line $3x + 4y = 10$ is a circle with center P . If O is the origin, then OP is equal to
- (1) 2 (2) 3 (3) $1/2$ (4) $1/3$
46. If a circle of radius r is touching the lines $x^2 - 4xy + y^2 = 0$ in the first quadrant at points A and B , then the area of triangle OAB (O being the origin) is
- (1) $3\sqrt{3} r^2/4$ (2) $\sqrt{3} r^2/4$
 (3) $3r^2/4$ (4) r^2
47. The locus of the midpoints of the chords of the circle $x^2 + y^2 - ax - by = 0$ which subtend a right angle at $(a/2, b/2)$ is
- (1) $ax + by = 0$
 (2) $ax + by = a^2 + b^2$
 (3) $x^2 + y^2 - ax - by + \frac{a^2 + b^2}{8} = 0$
 (4) $x^2 + y^2 - ax - by - \frac{a^2 + b^2}{8} = 0$
48. A circle through the point of intersection R of the lines $2x - y = 1$ and $3x + y = 2$ intersects these lines at points P and Q , respectively, such that $\angle PRQ$ is an acute angle. Then the angle subtended by the arc PQ at its centre is
- (1) 180°
 (2) 90°
 (3) 120°
 (4) Depends on centre and radius
49. If the pair of straight lines $xy\sqrt{3} - x^2 = 0$ is tangent to the circle at P and Q from the origin O such that the area of the smaller sector formed by CP and CQ is 3π sq. unit, where C is the center of the circle, then OP equals
- (1) $(3\sqrt{3})/2$ (2) $3\sqrt{3}$
 (3) 3 (4) $\sqrt{3}$
50. The condition that the chord $x \cos \alpha + y \sin \alpha - p = 0$ of $x^2 + y^2 - a^2 = 0$ may subtend a right angle at the center of the circle is
- (1) $a^2 = 2p^2$ (2) $p^2 = 2a^2$
 (3) $a = 2p$ (4) $p = 2a$
51. The centers of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is
- (1) $4 \leq x^2 + y^2 \leq 64$ (2) $x^2 + y^2 \leq 25$
 (3) $x^2 + y^2 \geq 25$ (4) $3 \leq x^2 + y^2 \leq 9$
52. The equation of the locus of the middle point of a chord of the circle $x^2 + y^2 = 2(x + y)$ such that the pair of lines joining the origin to the point of intersection of the chord and the circle are equally inclined to the x -axis is
- (1) $x + y = 2$ (2) $x - y = 2$
 (3) $2x - y = 1$ (4) none of these

53. The angle between the pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9 \sin^2 \alpha + 13 \cos^2 \alpha = 0$ is 2α . Then the equation of the locus of the point P is
- $x^2 + y^2 + 4x - 6y + 4 = 0$
 - $x^2 + y^2 + 4x - 6y - 9 = 0$
 - $x^2 + y^2 + 4x - 6y - 4 = 0$
 - $x^2 + y^2 + 4x - 6y + 9 = 0$
54. If two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 - px - qy = 0$ (where $pq \neq 0$) are bisected by the x -axis, then
- $p^2 = q^2$
 - $p^2 = 8q^2$
 - $p^2 < 8q^2$
 - $p^2 > 8q^2$
55. If one of the diameters of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with center $(2, 1)$, then the radius of the circle is
- $\sqrt{3}$
 - $\sqrt{2}$
 - 3
 - 2
56. Through the point $P(3, 4)$ a pair of perpendicular lines are drawn which meet x -axis at the points A and B . The locus of incentre of triangle PAB is
- $x^2 - y^2 - 6x - 8y + 25 = 0$
 - $x^2 + y^2 - 6x - 8y + 25 = 0$
 - $x^2 - y^2 + 6x + 8y + 25 = 0$
 - $x^2 + y^2 + 6x + 8y + 25 = 0$
57. A circle with center (a, b) passes through the origin. The equation of the tangent to the circle at the origin is
- $ax - by = 0$
 - $ax + by = 0$
 - $bx - ay = 0$
 - $bx + ay = 0$
58. A straight line with slope 2 and y -intercept 5 touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q . Then the coordinates of Q are
- $(-6, 11)$
 - $(-9, -13)$
 - $(-10, -15)$
 - $(-6, -7)$
59. The locus of the point from which the lengths of the tangents to the circles $x^2 + y^2 = 4$ and $2(x^2 + y^2) - 10x + 3y - 2 = 0$ are equal is
- a straight line inclined at $\pi/4$ with the line joining the centers of the circles
 - a circle
 - an ellipse
 - a straight line perpendicular to the line joining the centers of the circles
60. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets a straight line $5x - 2y + 6 = 0$ at a point Q on the y -axis, then the length of PQ is
- 4
 - $2\sqrt{5}$
 - 5
 - $3\sqrt{5}$
61. A line meets the coordinate axes at A and B . A circle is circumscribed about the triangle OAB . If d_1 and d_2 are the distances of the tangents to the circle at the origin O from the points A and B , respectively, then the diameter of the circle is
- $\frac{2d_1 + d_2}{2}$
 - $\frac{d_1 + 2d_2}{2}$
 - $d_1 + d_2$
 - $\frac{d_1 d_2}{d_1 + d_2}$
62. The range of values of α for which the line $2y = gx + \alpha$ is a normal to the circle $x^2 + y^2 + 2gx + 2gy - 2 = 0$ for all values of g is
- $[1, \infty)$
 - $[-1, \infty)$
 - $(0, 1)$
 - $(-\infty, 1]$
63. The equation of the tangent to the circle $x^2 + y^2 = a^2$, which makes a triangle of area a^2 with the coordinate axes, is
- $x \pm y = \pm a$
 - $x \pm y = \pm a\sqrt{2}$
 - $x \pm y = 3a$
 - $x \pm y = \pm 2a$
64. From an arbitrary point P on the circle $x^2 + y^2 = 9$, tangents are drawn to the circle $x^2 + y^2 = 1$, which meet $x^2 + y^2 = 9$ at A and B . The locus of the point of intersection of tangents at A and B to the circle $x^2 + y^2 = 9$ is
- $x^2 + y^2 = (27/7)^2$
 - $x^2 - y^2 = (27/7)^2$
 - $y^2 - x^2 = (27/7)^2$
 - none of these
65. If the radius of the circumcircle of triangle TPQ , where PQ is the chord of contact corresponding to the point T with respect to the circle $x^2 + y^2 - 2x + 4y - 11 = 0$, is 12 units, then the minimum distance of T from the director circle of the given circle is
- 6
 - 12
 - $6\sqrt{2}$
 - $12 - 4\sqrt{2}$
66. A straight line moves such that the algebraic sum of the perpendiculars drawn to it from two fixed points is equal to $2k$. Then, the straight line always touches a fixed circle of radius
- $2k$
 - $k/2$
 - k
 - none of these
67. If the line $ax + by = 2$ is a normal to the circle $x^2 + y^2 - 4x - 4y = 0$ and a tangent to the circle $x^2 + y^2 = 1$, then
- $a = \frac{1}{2}, b = \frac{1}{2}$
 - $a = \frac{1 + \sqrt{7}}{2}, b = \frac{1 - \sqrt{7}}{2}$
 - $a = \frac{1}{4}, b = \frac{3}{4}$
 - $a = 1, b = \sqrt{3}$
68. A light ray gets reflected from $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and the point of incident is $(-2, -4)$, then the equation of the incident ray is
- $4y + 3x + 22 = 0$
 - $3y + 4x + 20 = 0$
 - $4y + 2x + 20 = 0$
 - $y + x + 6 = 0$
69. A tangent at a point on the circle $x^2 + y^2 = a^2$ intersects a concentric circle C at two points P and Q . The tangents to the circle C at P and Q meet at a point on the circle $x^2 + y^2 = b^2$. Then the equation of the circle C is
- $x^2 + y^2 = ab$
 - $x^2 + y^2 = (a - b)^2$
 - $x^2 + y^2 = (a + b)^2$
 - $x^2 + y^2 = a^2 + b^2$
70. The greatest integral value of a such that $\sqrt{9 - a^2 + 2x - x^2} \geq \sqrt{16 - x^2}$ for at least one positive value of x
- 8
 - 7
 - 6
 - 4

71. The chords of contact of tangents from three points A , B , and C to the circle $x^2 + y^2 = a^2$ are concurrent. Then A , B , and C will
- be concyclic
 - be collinear
 - form the vertices of a triangle
 - none of these
72. The chord of contact of tangents from a point P to a circle passes through Q . If l_1 and l_2 are the lengths of the tangents from P and Q to the circle, then PQ is equal to
- $\frac{l_1 + l_2}{2}$
 - $\frac{l_1 - l_2}{2}$
 - $\sqrt{l_1^2 + l_2^2}$
 - $2\sqrt{l_1^2 + l_2^2}$
73. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is touched by $y = x$ at P such that $OP = 6\sqrt{2}$, then the value of c is
- 36
 - 144
 - 72
 - none of these
74. Tangents PA and PB are drawn to $x^2 + y^2 = 9$ from any arbitrary point P on the line $x + y = 25$. The locus of the midpoint of chord AB is
- $25(x^2 + y^2) = 9(x + y)$
 - $25(x^2 + y^2) = 3(x + y)$
 - $5(x^2 + y^2) = 3(x + y)$
 - none of these
75. A circle with radius $|a|$ and center on the y -axis slides along it and a variable line through $(a, 0)$ cuts the circle at points P and Q . The region in which the point of intersection of the tangents to the circle at points P and Q lies is represented by
- $y^2 \geq 4(ax - a^2)$
 - $y^2 \leq 4(ax - a^2)$
 - $y \geq 4(ax - a^2)$
 - $y \leq 4(ax - a^2)$
76. Consider a circle $x^2 + y^2 + ax + by + c = 0$ lying completely in the first quadrant. If m_1 and m_2 are the maximum and minimum values of y/x for all ordered pairs (x, y) on the circumference of the circle, then the value of $(m_1 + m_2)$ is
- $\frac{a^2 - 4c}{b^2 - 4c}$
 - $\frac{2ab}{b^2 - 4c}$
 - $\frac{2ab}{4c - b^2}$
 - $\frac{2ab}{b^2 - 4ac}$
77. The squared length of the intercept made by the line $x = h$ on the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is
- $\frac{4ch^2}{(g^2 - c)^2} (g^2 + f^2 - c)$
 - $\frac{4ch^2}{(f^2 - c)^2} (g^2 + f^2 - c)$
 - $\frac{4ch^2}{(g^2 - f^2)^2} (g^2 + f^2 - c)$
 - none of these
78. Let AB be the chord of contact of the point $(3, -3)$ w.r.t. the circle $x^2 + y^2 = 9$. Then the locus of the orthocentre of the $\triangle PAB$, where P is any point moving on the circle is
- $(x - 3)^2 + (y + 3)^2 = 9$
 - $(x - 3)^2 + (y + 3)^2 = 9/2$
 - $(x + 3)^2 + (y - 3)^2 = 9$
 - $(x + 3)^2 + (y - 3)^2 = 9/2$
79. Two congruent circles with centers at $(2, 3)$ and $(5, 6)$, which intersect at right angles, have radius equal to
- $2\sqrt{2}$
 - 3
 - 4
 - none of these
80. The distance from the center of the circle $x^2 + y^2 = 2x$ to the common chord of the circles $x^2 + y^2 + 5x - 8y + 1 = 0$ and $x^2 + y^2 - 3x + 7y - 25 = 0$ is
- 2
 - 4
 - $34/13$
 - $26/17$
81. C_1 is a circle of radius 1 unit touching x -axis and y -axis, C_2 is another circle of radius > 1 and touching the axes as well as the circle C_1 . Then the radius of C_2 is
- $3 - 2\sqrt{2}$
 - $3 + 2\sqrt{2}$
 - $3 + 2\sqrt{3}$
 - $6 + \sqrt{3}$
82. Suppose $ax + by + c = 0$, where a , b , and c are in AP be normal to a family of circles. The equation of the circle of the family intersecting the circle $x^2 + y^2 - 4x - 4y - 1 = 0$ orthogonally is
- $x^2 + y^2 - 2x + 4y - 3 = 0$
 - $x^2 + y^2 + 2x - 4y - 3 = 0$
 - $x^2 + y^2 - 2x + 4y - 5 = 0$
 - $x^2 + y^2 - 2x - 4y + 3 = 0$
83. Two circles of radii a and b touching each other externally, are inscribed in the area bounded by $y = \sqrt{1 - x^2}$ and the x -axis. If $b = 1/2$, then a is equal to
- $1/4$
 - $1/8$
 - $1/2$
 - $1/\sqrt{2}$
84. If the length of the common chord of two circles $x^2 + y^2 + 8x + 1 = 0$ and $x^2 + y^2 + 2\mu y - 1 = 0$ is $2\sqrt{6}$, then the values of μ are
- ± 2
 - ± 3
 - ± 4
 - none of these
85. If r_1 and r_2 are the radii of the smallest and the largest circles, respectively, which pass through $(5, 6)$ and touch the circle $(x - 2)^2 + y^2 = 4$, then $r_1 r_2$ is
- $31/4$
 - $41/4$
 - $41/3$
 - 17
86. If $C_1: x^2 + y^2 = (3 + 2\sqrt{2})^2$ is a circle and PA and PB are a pair of tangents on C_1 , where P is any point on the director circle of C_1 , then the radius of the smallest circle which touches C_1 externally and also the two tangents PA and PB is
- $2\sqrt{3} - 3$
 - $2\sqrt{2} - 1$
 - $2\sqrt{2} - 1$
 - 1
87. P is a point (a, b) in the first quadrant. If the two circles which pass through P and touch both the coordinate axes cut at right angles, then
- $a^2 - 6ab + b^2 = 0$
 - $a^2 + 2ab - b^2 = 0$
 - $a^2 - 4ab + b^2 = 0$
 - $a^2 - 8ab + b^2 = 0$
88. The number of common tangent(s) to the circles $x^2 + y^2 + 2x + 8y - 23 = 0$ and $x^2 + y^2 - 4x - 10y + 19 = 0$ is
- 1
 - 2
 - 3
 - 4
89. The locus of the centers of the circles which cut the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 5x + 4y - 2 = 0$ orthogonally is
- $9x + 10y - 7 = 0$
 - $x - y + 2 = 0$
 - $9x - 10y + 11 = 0$
 - $9x + 10y + 7 = 0$

90. Tangents are drawn to the circle $x^2 + y^2 = 1$ at the points where it is met by the circles $x^2 + y^2 - (\lambda + 6)x + (8 - 2\lambda)y - 3 = 0$, λ being the variable. The locus of the point of intersection of these tangents is
 (1) $2x - y + 10 = 0$ (2) $x + 2y - 10 = 0$
 (3) $x - 2y + 10 = 0$ (4) $2x + y - 10 = 0$
91. If the line $x \cos \theta + y \sin \theta = 2$ is the equation of a transverse common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6\sqrt{3}x - 6y + 20 = 0$, then the value of θ is
 (1) $5\pi/6$ (2) $2\pi/3$ (3) $\pi/3$ (4) $\pi/6$
92. If $C_1: x^2 + y^2 - 20x + 64 = 0$ and $C_2: x^2 + y^2 + 30x + 144 = 0$, then the length of the shortest line segment PQ which touches C_1 at P and C_2 at Q is
 (1) 20 (2) 15 (3) 22 (4) 27
93. The circles having radii r_1 and r_2 intersect orthogonally. The length of their common chord is
 (1) $\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$ (2) $\frac{\sqrt{r_1^2 + r_2^2}}{2r_1r_2}$
 (3) $\frac{r_1r_2}{\sqrt{r_1^2 + r_2^2}}$ (4) $\frac{\sqrt{r_1^2 + r_2^2}}{r_1r_2}$
94. The two circles which pass through $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$ will intersect each other at right angle if
 (1) $a^2 = c^2(2m + 1)$ (2) $a^2 = c^2(2 + m^2)$
 (3) $c^2 = a^2(2 + m^2)$ (4) $c^2 = a^2(2m + 1)$
95. The locus of the center of the circle which touches the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ externally and also touches the y -axis is given by equation
 (1) $x^2 - 6x - 10y - 14 = 0$
 (2) $x^2 - 10x - 6y - 14 = 0$
 (3) $y^2 - 6x - 10y + 14 = 0$
 (4) $y^2 - 10x - 6y + 14 = 0$
96. If the chord of contact of tangents from a point P to a given circle passes through Q , then the circle on PQ as diameter
 (1) cuts the given circle orthogonally
 (2) touches the given circle externally
 (3) touches the given circle internally
 (4) none of these
97. If the angle of intersection of the circles $x^2 + y^2 + x + y = 0$ and $x^2 + y^2 + x - y = 0$ is θ , then the equation of the line passing through $(1, 2)$ and making an angle θ with the y -axis is
 (1) $x = 1$ (2) $y = 2$
 (3) $x + y = 3$ (4) $x - y = 3$
98. The coordinates of two points P and Q are (x_1, y_1) and (x_2, y_2) and O is the origin. If the circles are described on OP and OQ as diameters, then the length of their common chord is
 (1) $\frac{|x_1y_2 + x_2y_1|}{PQ}$ (2) $\frac{|x_1y_2 - x_2y_1|}{PQ}$
 (3) $\frac{|x_1x_2 + y_1y_2|}{PQ}$ (4) $\frac{|x_1x_2 - y_1y_2|}{PQ}$
99. If the circumference of the circle $x^2 + y^2 + 8x + 8y - b = 0$ is bisected by the circle $x^2 + y^2 - 2x + 4y + a = 0$, then $a + b$ equals
 (1) 50 (2) 56 (3) -56 (4) -34
100. Equation of circle which cuts the circle $x^2 + y^2 + 2x + 4y - 4 = 0$ and the lines $xy - 2x - y + 2 = 0$ orthogonally, is
 (1) $x^2 + y^2 - 2x - 4y - 6 = 0$
 (2) $x^2 + y^2 - 2x - 4y + 6 = 0$
 (3) $x^2 + y^2 - 2x - 4y - 12 = 0$
 (4) not possible to determine
101. The minimum radius of the circle which contains the three circles, $x^2 + y^2 - 4y - 5 = 0$, $x^2 + y^2 + 12x + 4y + 31 = 0$ and $x^2 + y^2 + 6x + 12y + 36 = 0$, is
 (1) $\frac{7}{28}\sqrt{900} + 3$ (2) $\frac{\sqrt{845}}{9} + 4$
 (3) $\frac{5}{36}\sqrt{949} + 3$ (4) none of these
102. A circle C_1 of radius b touches the circle $x^2 + y^2 = a^2$ externally and has its center on the positive x -axis; another circle C_2 of radius c touches the circle C_1 externally and has its center on the positive x -axis. Given $a < b < c$, the three circles have a common tangent if a, b , and c are in
 (1) AP (2) GP
 (3) HP (4) none of these
103. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = k^2$ orthogonally, then the equation of the locus of its center is
 (1) $2ax + 2by - (a^2 + b^2 + k^2) = 0$
 (2) $2ax + 2by - (a^2 - b^2 + k^2) = 0$
 (3) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - k^2) = 0$
 (4) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - k^2) = 0$
104. The centre of the smallest circle touching the circles $x^2 + y^2 - 2y - 3 = 0$ and $x^2 + y^2 - 8x - 18y + 93 = 0$ is
 (1) (3, 2) (2) (4, 4) (3) (2, 5) (4) (2, 7)
105. Two circles with radii a and b ($a > b$) touch each other externally. Let c be the radius of a circle that touches these two circles as well as a direct common tangent to these two circles. Then
 (1) $\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}$ (2) $c = \frac{2ab}{a + b}$
 (3) $\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}$ (4) $c = \frac{2ab}{\sqrt{a} + \sqrt{b}}$
106. Consider points $A(\sqrt{13}, 0)$ and $B(2\sqrt{13}, 0)$ lying on x -axis. These points are rotated in anticlockwise direction about the origin through an angle of $\tan^{-1}\left(\frac{2}{3}\right)$. Let the new position of A and B be A' and B' , respectively. With A' as centre and radius $\frac{2\sqrt{13}}{3}$, a circle C_1 is drawn and with B' as centre and radius $\frac{\sqrt{13}}{3}$, a circle C_2 is drawn. The radical axis of C_1 and C_2 is
 (1) $3x + 2y = 20$ (2) $3x + 3y = 10$
 (3) $9x + 6y = 65$ (4) none of these

107. The common chord of the circle $x^2 + y^2 + 6x + 8y - 7 = 0$ and a circle passing through the origin and touching the line $y = x$ always passes through the point

- (1) $(-1/2, 1/2)$ (2) $(1, 1)$
(3) $(1/2, 1/2)$ (4) none of these

108. The equation of the circle passing through the point of intersection of the circle $x^2 + y^2 = 4$ and the line $2x + y = 1$ and having minimum possible radius is

- (1) $5x^2 + 5y^2 + 18x + 6y - 5 = 0$
(2) $5x^2 + 5y^2 + 9x + 8y - 15 = 0$
(3) $5x^2 + 5y^2 + 4x + 9y - 5 = 0$
(4) $5x^2 + 5y^2 - 4x - 2y - 18 = 0$

109. The equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 4x - 2y = 8$ and $x^2 + y^2 - 2x - 4y = 8$ and the point $(-1, 4)$ is

- (1) $x^2 + y^2 + 4x + 4y - 8 = 0$
(2) $x^2 + y^2 - 3x + 4y + 8 = 0$
(3) $x^2 + y^2 + x + y - 8 = 0$
(4) $x^2 + y^2 - 3x - 3y - 8 = 0$

Multiple Correct Answers Type

1. The circle $x^2 + y^2 + 2a_1x + c = 0$ lies completely inside the circle $x^2 + y^2 + 2a_2x + c = 0$. Then,

- (1) $a_1a_2 > 0$ (2) $a_1a_2 < 0$ (3) $c > 0$ (4) $c < 0$

2. Consider the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. Let O be the center of the circle and tangents at $A(7, 3)$ and $B(5, 1)$ meet at C . Let $S = 0$ represents the family of circles passing through A and B . Then,

- (1) the area of quadrilateral $OACB$ is 4
(2) the radical axis for the family of circles $S = 0$ is $x + y = 10$
(3) the smallest possible circle of the family $S = 0$ is $x^2 + y^2 - 12x - 4y + 38 = 0$
(4) the coordinates of point C are $(7, 1)$

3. Tangents drawn from the point $(\alpha, 3)$ to the circle $2x^2 + 2y^2 = 25$ will be perpendicular to each other if α equals

- (1) 5 (2) -4 (3) 4 (4) -5

4. ABC is any triangle inscribed in the circle $x^2 + y^2 = r^2$ such that A is fixed point. If the external and internal bisectors of $\angle A$ intersect the circle at D and E , respectively, then which of the following statements is true about $\triangle ADE$?

- (1) Its centroid is a fixed point.
(2) Its circumcentre is a fixed point.
(3) Its orthocentre is a fixed point.
(4) none of these

5. The equations of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are

- (1) $x = 0$ (2) $y = 0$
(3) $(h^2 - r^2)x - 2rhy = 0$ (4) $(h^2 - r^2)x + 2rhy = 0$

6. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, and $S(x_4, y_4)$, then

- (1) $x_1 + x_2 + x_3 + x_4 = 0$ (2) $y_1 + y_2 + y_3 + y_4 = 0$
(3) $x_1x_2x_3x_4 = c^4$ (4) $y_1y_2y_3y_4 = c^4$

7. Let x and y be real variables satisfying $x^2 + y^2 + 8x - 10y - 40 = 0$. Let $a = \max \left\{ \sqrt{(x+2)^2 + (y-3)^2} \right\}$ and

$b = \min \left\{ \sqrt{(x+2)^2 + (y-3)^2} \right\}$. Then,

- (1) $a + b = 18$ (2) $a + b = \sqrt{2}$
(3) $a - b = 4\sqrt{2}$ (4) $a \cdot b = 73$

8. If the equation $x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle, then the condition for that circle to pass through three quadrants only but not passing through the origin is

- (1) $f^2 > c$ (2) $g^2 > c$ (3) $c > 0$ (4) $h = 0$

9. The points on the line $x = 2$ from which the tangents drawn to the circle $x^2 + y^2 = 16$ are at right angles is (are)

- (1) $(2, 2\sqrt{7})$ (2) $(2, 2\sqrt{5})$
(3) $(2, -2\sqrt{7})$ (4) $(2, -2\sqrt{5})$

10. The coordinates of the center of a circle, whose radius is 2 unit and which touches the line pair $x^2 - y^2 - 2x + 1 = 0$, are

- (1) $(4, 0)$ (2) $(1 + 2\sqrt{2}, 0)$
(3) $(4, 1)$ (4) $(1, 2\sqrt{2})$

11. If the circles $x^2 + y^2 - 9 = 0$ and $x^2 + y^2 + 2\alpha x + 2y + 1 = 0$ touch each other, then α is

- (1) $-4/3$ (2) 0 (3) 1 (4) $4/3$

12. Point M moved on the circle $(x - 4)^2 + (y - 8)^2 = 20$. Then it broke away from it and moving along a tangent to the circle cut the x -axis at point $(-2, 0)$. The coordinates of the point on the circle at which the moving point broke away is

- (1) $(42/5, 36/5)$ (2) $(-2/5, 44/5)$
(3) $(6, 4)$ (4) $(2, 4)$

13. The equation of the tangent to the circle $x^2 + y^2 = 25$ passing through $(-2, 11)$ is

- (1) $4x + 3y = 25$ (2) $3x + 4y = 38$
(3) $24x - 7y + 125 = 0$ (4) $7x + 24y = 250$

14. If the area of the quadrilateral formed by the tangents from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the radii corresponding to the points of contact is 15, then a value of c is

- (1) 9 (2) 4 (3) 5 (4) 25

15. The equation of the circle which touches the axes of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose center lies in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is

- (1) 1 (2) 2 (3) 3 (4) 6

16. Which of the following lines have the intercepts of equal lengths on the circle $x^2 + y^2 - 2x + 4y = 0$?

- (1) $3x - y = 0$ (2) $x + 3y = 0$
(3) $x + 3y + 10 = 0$ (4) $3x - y - 10 = 0$

17. The equation of the line(s) parallel to $x - 2y = 1$ which touch(es) the circle $x^2 + y^2 - 4x - 2y - 15 = 0$ is (are)
- $x - 2y + 2 = 0$
 - $x - 2y - 10 = 0$
 - $x - 2y - 5 = 0$
 - $x - 2y + 10 = 0$
18. The circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 + 4x + 4y - 1 = 0$
- touch internally
 - touch externally
 - have $3x + 4y - 1 = 0$ as the common tangent at the point of contact
 - have $3x + 4y + 1 = 0$ as the common tangent at the point of contact
19. The circles $x^2 + y^2 + 2x + 4y - 20 = 0$ and $x^2 + y^2 + 6x - 8y + 10 = 0$
- are such that the number of common tangents on them is 2
 - are orthogonal
 - are such that the length of their common tangent is $5(12/5)^{1/4}$
 - are such that the length of their common chord is $5\sqrt{3/2}$
20. A point $P(\sqrt{3}, 1)$ moves on the circle $x^2 + y^2 = 4$ and after covering a quarter of the circle leaves it tangentially. The equation of the line along which the point moves after leaving the circle is
- $y = \sqrt{3}x + 4$
 - $\sqrt{3}y = x + 4$
 - $y = \sqrt{3}x - 4$
 - $\sqrt{3}y = x - 4$
21. The equation of a circle of radius 1 touching the circles $x^2 + y^2 - 2|x| = 0$ is
- $x^2 + y^2 + 2\sqrt{2}x + 1 = 0$
 - $x^2 + y^2 - 2\sqrt{3}y + 2 = 0$
 - $x^2 + y^2 + 2\sqrt{3}y + 2 = 0$
 - $x^2 + y^2 - 2\sqrt{2} + 1 = 0$
22. The center(s) of the circle(s) passing through the points $(0, 0)$ and $(1, 0)$ and touching the circle $x^2 + y^2 = 9$ is (are)
- $(3/2, 1/2)$
 - $(1/2, 3/2)$
 - $(1/2, 2^{1/2})$
 - $(1/2, -2^{1/2})$
23. The equation(s) of straight lines which pass through the intersection of the lines $x - 2y - 5 = 0$ and $7x + y = 50$ and divide the circumference of the circle $x^2 + y^2 = 100$ into two arcs whose lengths are in the ratio 2 : 1 is/are
- $3x + 4y - 25 = 0$
 - $4x - 3y - 25 = 0$
 - $3x + 2y - 23 = 0$
 - $2x - 3y - 11 = 0$
24. Two lines through $(2, 3)$ from which the circle $x^2 + y^2 = 25$ intercepts chords of length 8 units have equations
- $y = 3$
 - $12x + 5y = 39$
 - $x = 2$
 - $9x - 11y = 51$
25. Normal to the circle $x^2 + y^2 = 4$ divides the circle having centre at $(2, 4)$ and radius 2 in the areas of ratio $(\pi - 2) : (3\pi + 2)$. Then the normal can be
- $y = x$
 - $y = 3x$
 - $y = 5x$
 - $y = 7x$

Linked Comprehension Type

For Problems 1–3

Each side of a square has length 4 units and its center is at $(3, 4)$. If one of the diagonals is parallel to the line $y = x$, then answer the following questions.

- Which of the following is not the vertex of the square?
 (1) $(1, 6)$ (2) $(5, 2)$ (3) $(1, 2)$ (4) $(4, 6)$
- The radius of the circle inscribed in the triangle formed by any three vertices is
 (1) $2\sqrt{2}(\sqrt{2} + 1)$ (2) $2\sqrt{2}(\sqrt{2} - 1)$
 (3) $2(\sqrt{2} + 1)$ (4) none of these
- The radius of the circle inscribed in the triangle formed by any two vertices of the square and the center is
 (1) $2(\sqrt{2} - 1)$ (2) $2(\sqrt{2} + 1)$
 (3) $\sqrt{2}(\sqrt{2} - 1)$ (4) none of these

For Problems 4–6

Tangents PA and PB are drawn to the circle $(x - 4)^2 + (y - 5)^2 = 4$ from the point P on the curve $y = \sin x$, where A and B lie on the circle. Consider the function $y = f(x)$ represented by the locus of the center of the circumcircle of triangle PAB . Then answer the following questions.

- The range of $y = f(x)$ is
 (1) $[-2, 1]$ (2) $[-1, 4]$ (3) $[0, 2]$ (4) $[2, 3]$
- The period of $y = f(x)$ is
 (1) 2π (2) 3π
 (3) π (4) not defined
- Which of the following is true?
 (1) $f(x) = 4$ has real roots.
 (2) $f(x) = 1$ has real roots.
 (3) The range of $y = f^{-1}(x)$ is $[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2]$.
 (4) None of these.

For Problems 7–9

Consider a family of circles passing through the points $(3, 7)$ and $(6, 5)$. Answer the following questions.

- The number of circles which belong to the family and also touch the x -axis are
 (1) 0 (2) 1
 (3) 2 (4) infinite
- If each circle in the family cuts the circle $x^2 + y^2 - 4x - 6y - 3 = 0$, then all the common chords pass through the fixed point which is
 (1) $(1, 23)$ (2) $(2, 23/2)$
 (3) $(-3, 3/2)$ (4) none of these
- If the circle which belongs to the given family cuts the circle $x^2 + y^2 = 29$ orthogonally, then the center of that circle is
 (1) $(1/2, 3/2)$ (2) $(9/2, 7/2)$
 (3) $(7/2, 9/2)$ (4) $(3, -7/9)$

For Problems 10–12

Consider the relation $4l^2 - 5m^2 + 6l + 1 = 0$, where $l, m \in R$.

10. Then the line $lx + my + 1 = 0$ touches a fixed circle whose center and radius are, respectively,
 (1) $(2, 0), 3$ (2) $(-3, 0), \sqrt{3}$
 (3) $(3, 0), \sqrt{5}$ (4) none of these
11. Tangents PA and PB are drawn to the above fixed circle from the points P on the line $x + y - 1 = 0$. Then the chord of contact AB passes through the fixed point
 (1) $(1/2, -5/2)$ (2) $(1/3, 4/3)$
 (3) $(-1/2, 3/2)$ (4) none of these
12. The number of tangents which can be drawn from the point $(2, -3)$ to the above fixed circle are
 (1) 0 (2) 1 (3) 2 (4) 1 or 2

For Problems 13–15

A circle C whose radius is 1 unit touches the x -axis at point A . The center Q of C lies in the first quadrant. The tangent from the origin O to the circle touches it at T and a point P lies on it such that $\triangle OAP$ is a right-angled triangle at A and its perimeter is 8 units.

13. The length of PQ is
 (1) $1/2$ (2) $4/3$
 (3) $5/3$ (4) none of these
14. The equation of circle C is
 (1) $(x - 2)^2 + (y - 1)^2 = 1$
 (2) $\{x - (\sqrt{3} - \sqrt{2})\}^2 + (y - 1)^2 = 1$
 (3) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$
 (4) none of these
15. The equation of tangent OT is
 (1) $3y = 4x$ (2) $x - \sqrt{2}y = 0$
 (3) $y - \sqrt{3}x = 0$ (4) none of these

For Problems 16–18

P is a variable point on the line $L = 0$. Tangents are drawn to the circles $x^2 + y^2 = 4$ from P to touch it at Q and R . The parallelogram $PQSR$ is completed.

16. If $L \equiv 2x + y - 6 = 0$, then the locus of the circumcenter of $\triangle PQR$ is
 (1) $2x - y = 4$ (2) $2x + y = 3$
 (3) $x - 2y = 4$ (4) $x + 2y = 3$
17. If $P \equiv (6, 8)$, then the area of $\triangle QRS$ is
 (1) $\frac{\sqrt{6}}{25}$ sq. units (2) $\frac{\sqrt{24}}{25}$ sq. units
 (3) $\frac{48\sqrt{6}}{25}$ sq. units (4) $\frac{192\sqrt{6}}{25}$ sq. units
18. If $P \equiv (3, 4)$, then the coordinates of S are
 (1) $(-46/25, 63/25)$ (2) $(-51/25, -68/25)$
 (3) $(-46/25, 68/25)$ (4) $(-68/25, 51/25)$

For Problems 19–21

To the circle $x^2 + y^2 = 4$, two tangents are drawn from $P(-4, 0)$, which touch the circle at T_1 and T_2 . A rhombus $PT_1P'T_2$ is completed.

19. The circumcenter of triangle PT_1T_2 is at
 (1) $(-2, 0)$ (2) $(2, 0)$
 (3) $(\sqrt{3}/2, 0)$ (4) none of these
20. The ratio of the area of triangle PT_1P' to that of triangle $P'T_1T_2$ is
 (1) $2 : 1$ (2) $1 : 2$
 (3) $\sqrt{3} : 2$ (4) none of these
21. If P is taken to be at $(h, 0)$ such that P' lies on the circle, the area of the rhombus is
 (1) $6\sqrt{3}$ (2) $2\sqrt{3}$
 (3) $3\sqrt{3}$ (4) none of these

For Problems 22–24

Let α chord of a circle be that chord of the circle which subtends an angle α at the center.

22. If $x + y = 1$ is a chord of $x^2 + y^2 = 1$, then α is equal to
 (1) $\pi/4$ (2) $\pi/2$
 (3) $\pi/6$ (4) $x + y = 1$ is not a chord
23. If the slope of a $\pi/3$ chord of $x^2 + y^2 = 4$ is 1, then its equation is
 (1) $x - y + \sqrt{6} = 0$ (2) $x - y = 2\sqrt{3}$
 (3) $x - y = \sqrt{3}$ (4) $x - y + \sqrt{3} = 0$
24. The distance of $2\pi/3$ chord of $x^2 + y^2 + 2x + 4y + 1 = 0$ from the center is
 (1) 1 (2) 2 (3) $\sqrt{2}$ (4) $1/\sqrt{2}$

For Problems 25–27

Two variable chords AB and BC of a circle $x^2 + y^2 = a^2$ are such that $AB = BC = a$. M and N are the midpoints of AB and BC , respectively, such that the line joining MN intersects the circles at P and Q , where P is closer to AB and O is the center of the circle.

25. $\angle OAB$ is
 (1) 30° (2) 60° (3) 45° (4) 15°
26. The angle between the tangents at A and C is
 (1) 90° (2) 120° (3) 60° (4) 150°
27. The locus of the point of intersection of tangents at A and C is
 (1) $x^2 + y^2 = a^2$ (2) $x^2 + y^2 = 2a^2$
 (3) $x^2 + y^2 = 4a^2$ (4) $x^2 + y^2 = 8a^2$

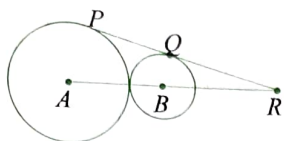
For Problems 28–30

Given two circles intersecting orthogonally having the length of common chord $24/5$ units. The radius of one of the circles is 3 units.

28. The radius of other circle is
 (1) 6 units (2) 4 units (3) 2 units (4) 4 units
29. The angle between direct common tangents is
 (1) $\sin^{-1} \frac{24}{25}$ (2) $\sin^{-1} \frac{4\sqrt{6}}{25}$
 (3) $\sin^{-1} \frac{4}{5}$ (4) $\sin^{-1} \frac{12}{25}$
30. The length of direct common tangent is
 (1) $\sqrt{12}$ (2) $4\sqrt{3}$ (3) $2\sqrt{6}$ (4) $3\sqrt{6}$

For Problems 31–33

In the given figure, there are two circles with centers A and B . The common tangent to the circles touches them, respectively, at P and Q . AR is 40 cm and AB is divided by the point of contact of the circles in the ratio 5 : 3.



31. What is ratio of the length of AB to that of BR ?
 (1) 1 : 4 (2) 2 : 3 (3) 2 : 5 (4) 7 : 4
32. The radius of the circle with center B is
 (1) 10 cm (2) 3 cm (3) 6 cm (4) 8 cm
33. The length of QR is
 (1) $10\sqrt{15}$ cm (2) $5\sqrt{15}$ cm
 (3) $4\sqrt{15}$ cm (4) $6\sqrt{15}$ cm

For Problems 34–36

Let each of the circles

$$S_1 \equiv x^2 + y^2 + 4y - 1 = 0$$

$$S_2 \equiv x^2 + y^2 + 6x + y + 8 = 0$$

$$S_3 \equiv x^2 + y^2 - 4x - 4y - 37 = 0$$

touch the other two. Also, let P_1 , P_2 and P_3 be the points of contact of S_1 and S_2 ; S_2 and S_3 ; and S_3 and S_1 , respectively. C_1 , C_2 and C_3 are the centres of S_1 , S_2 and S_3 , respectively.

34. The coordinates of P_1 are
 (1) (2, -1) (2) (-2, -1) (3) (-2, 1) (4) (2, 1)
35. The ratio $\frac{\text{area}(\Delta P_1 P_2 P_3)}{\text{area}(\Delta C_1 C_2 C_3)}$ is equal to
 (1) 3 : 2 (2) 2 : 3 (3) 5 : 3 (4) 2 : 5
36. P_2 and P_3 are images of each other with respect to the line
 (1) $y = x$ (2) $y = -x$
 (3) $y = x + 1$ (4) $y = -x + 2$

For Problems 37 and 38

The line $x + 2y + a = 0$ intersects the circle $x^2 + y^2 - 4 = 0$ at two distinct points A and B . Another line $12x - 6y - 41 = 0$ intersects the circle $x^2 + y^2 - 4x - 2y + 1 = 0$ at two distinct points C and D .

37. The value of a for which the four points A , B , C , and D are concyclic is
 (1) 1 (2) 3 (3) 4 (4) 2
38. The equation of the circle passing through the points A , B , C , and D is
 (1) $5x^2 + 5y^2 - 8x - 16y - 36 = 0$
 (2) $5x^2 + 5y^2 + 8x - 16y - 36 = 0$
 (3) $5x^2 + 5y^2 + 8x + 16y - 36 = 0$
 (4) $5x^2 + 5y^2 - 8x - 16y + 36 = 0$

For Problems 39 and 40

Let A , B and C be three sets such that

$$A = \left\{ (x, y) \mid \frac{x}{\cos \theta} = \frac{y}{\sin \theta} = 5, \text{ where } \theta \text{ is parameter} \right\}$$

$$B = \left\{ (x, y) \mid \frac{x-3}{\cos \phi} = \frac{y-4}{\sin \phi} = r \right\}$$

$$C = \left\{ (x, y) \mid (x-3)^2 + (y-4)^2 \leq R^2 \right\}$$

39. If $A \cap C = A$, then minimum value of R is
 (1) 5 (2) 6
 (3) 10 (4) 11
40. If ϕ is fixed and r varies and $n(A \cap B) = 1$, then $\sec \phi$ is equal to
 (1) $\frac{5}{4}$ (2) $\frac{-5}{4}$ (3) $\frac{5}{3}$ (4) $\frac{-5}{3}$

For Problems 41–43

Consider the family of circles $x^2 + y^2 - 2x - 2ay - 8 = 0$ passing through two fixed points A and B . Also, $S = 0$ is a circle of this family, the tangent to which at A and B intersect on the line $x + 2y + 5 = 0$.

41. The distance between the points A and B , is
 (1) 4 (2) $4\sqrt{2}$ (3) 6 (4) 8
42. The radius of circle S is
 (1) 3 (2) 6 (3) $2\sqrt{3}$ (4) $3\sqrt{2}$
43. If the circle $x^2 + y^2 - 10x + 2y + c = 0$ is orthogonal to $S = 0$, then the value of c is
 (1) 8 (2) 9 (3) 10 (4) 12

For Problems 44–46

A circle S of radius 1 is inscribed in an equilateral triangle PQR . The points of contact of S with the sides PQ , QR , and RP are D , E , and F , respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $(3\sqrt{3}/2, 3/2)$. Further, it is given that the origin and the center of S are on the same side of the line PQ .

44. The equation of circle S is
 (1) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$
 (2) $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$
 (3) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$
 (4) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$
45. Points E and F are given by
 (1) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$ (2) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$
 (3) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ (4) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
46. The equations of the sides QR and RP are, respectively,
 (1) $y = \frac{2}{\sqrt{3}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$
 (2) $y = \frac{1}{\sqrt{3}}x, y = 0$
 (3) $y = \frac{\sqrt{3}}{2}x + 1, y = -\frac{\sqrt{3}}{2}x - 1$
 (4) $y = \sqrt{3}x, y = 0$

Matrix Match Type

1. Match the following lists:

List I	List II
a. The number of circles touching the given three non-concurrent lines	p. 1
b. The number of circles touching $y = x$ at $(2, 2)$ and also touching the line $x + 2y - 4 = 0$	q. 2
c. The number of circles touching the lines $x \pm y = 2$ and passing through the point $(4, 3)$	r. 4
d. The number of circles intersecting the given three circles orthogonally	s. infinite

2. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be an equation of circle. Match the following lists:

List I	List II
a. If the circle lies in the first quadrant, then	p. $g < 0$
b. If the circle lies above the x -axis, then	q. $g > 0$
c. If the circle lies on the left of the y -axis, then	r. $g^2 - c < 0$
d. If the circle touches the positive x -axis and does not intersect the y -axis, then	s. $c > 0$

3. Match the following lists:

List I	List II
a. If $ax + by - 5 = 0$ is the equation of the chord of the circle $(x - 3)^2 + (y - 4)^2 = 4$, which passes through $(2, 3)$ and at the greatest distance from the center of the circle, then $ a + b $ is equal to	p. 6
b. Let O be the origin and P be a variable point on the circle $x^2 + y^2 + 2x + 2y = 0$. If the locus of midpoint of OP is $x^2 + y^2 + 2gx + 2fy + c = 0$, then $(g + f)$ is equal to	q. 3
c. The x -coordinates of the center of the smallest circle which cuts the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x + 12y + 52 = 0$ orthogonally is	r. 2
d. If θ be the angle between two tangents which are drawn to the circles $x^2 + y^2 - 6\sqrt{3}x - 6y + 27 = 0$ from the origin. Then $2\sqrt{3} \tan \theta$ equals	s. 1

4. Match the following lists:

List I	List II
a. If two circles $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2x + b = 0$ touch each other, then the triplet (a_1, a_2, b) can be	p. $(2, 2, 2)$
b. If two circles $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2y + b = 0$ touch each other, then the triplet (a_1, a_2, b) can be	q. $(1, 1, \frac{1}{2})$

c. If the straight line $a_1x - by + b^2 = 0$ touches the circle $x^2 + y^2 = a_2x + by$, then the triplet (a_1, a_2, b) can be	r. $(2, 1, 0)$
d. If the line $3x + 4y - 4 = 0$ touches the circle $(x - a_1)^2 + (y - a_2)^2 = b^2$, then the triplet (a_1, a_2, b) can be	s. $(1, 1, 3/5)$

5. Match the following lists and then choose the correct code.

List I	List II
a. The length of the common chord of two circles of radii 3 units and 4 units which intersect orthogonally is $k/5$. Then k is equal to	p. 1
b. The circumference of the circle $x^2 + y^2 + 4x + 12y + p = 0$ is bisected by the circle $x^2 + y^2 - 2x + 8y - q = 0$. Then $p + q$ is equal to	q. 24
c. The number of distinct chords of the circle $2x(x - \sqrt{2}) + y(2y - 1) = 0$, where the chords are passing through the point $(\sqrt{2}, 1/2)$ and are bisected on the x -axis, is	r. 32
d. One of the diameters of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points $(-3, 4)$ and $(5, 4)$, respectively, then the area of the rectangle is	s. 36

Codes:

	a	b	c	d
(1)	r	s	p	q
(2)	s	p	r	q
(3)	q	s	p	r
(4)	p	r	s	q

6. Let C_1 and C_2 be two circles whose equations are $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 2x = 0$. $P(\lambda, \lambda)$ is a variable point. Then match the following lists and choose the correct code.

List I	List II
a. P lies inside C_1 but outside C_2 .	p. $\lambda \in (-\infty, -1) \cup (0, \infty)$
b. P lies inside C_2 but outside C_1 .	q. $\lambda \in (-\infty, -1) \cup (1, \infty)$
c. P lies outside C_1 and outside C_2 .	r. $\lambda \in (-1, 0)$
d. P does not lie inside C_2 .	s. $\lambda \in (0, 1)$

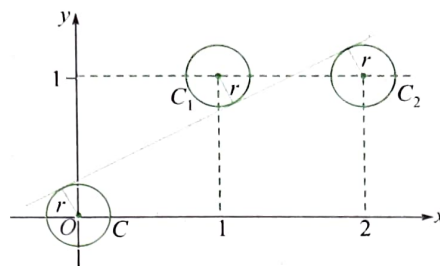
Codes:

	a	b	c	d
(1)	r	s	p	q
(2)	p	s	q	r
(3)	q	p	s	r
(4)	s	r	q	p

Numerical Value Type

- Let the lines $(y - 2) = m_1(x - 5)$ and $(y + 4) = m_2(x - 3)$ intersect at right angles at P (where m_1 and m_2 are parameters). If the locus of P is $x^2 + y^2 + gx + fy + 7 = 0$, then the value of $|f + g|$ is _____.
- Consider the family of circles $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$ passing through two fixed points A and B . Then the distance between the points A and B is _____.
- The number of points $P(x, y)$ lying inside or on the circle $x^2 + y^2 = 9$ and satisfying the equation $\tan^4 x + \cot^4 x + 2 = 4 \sin^2 y$ is _____.
- If real numbers x and y satisfy $(x + 5)^2 + (y - 12)^2 = (14)^2$, then the minimum value of $\sqrt{x^2 + y^2}$ is _____.
- The line $3x + 6y = k$ intersects the curve $2x^2 + 2xy + 3y^2 = 1$ at points A and B . The circle on AB as diameter passes through the origin. Then the value of k^2 is _____.
- The sum of the slopes of the lines tangent to both the circles $x^2 + y^2 = 1$ and $(x - 6)^2 + y^2 = 4$ is _____.
- A circle $x^2 + y^2 + 4x - 2\sqrt{2}y + c = 0$ is the director circle of the circle S_1 , and S_1 is the director circle of circle S_2 , and so on. If the sum of radii of all these circles is 2, then the value of c is $k\sqrt{2}$, where the value of k is _____.
- Two circles are externally tangent. Lines PAB and $PA'B'$ are common tangents with A and A' on the smaller circle and B and B' on the larger circle. If $PA = AB = 4$, then the square of the radius of the smaller circle is _____.
- The length of a common internal tangent to two circles is 7 and that of a common external tangent is 11. Then the product of the radii of the two circles is _____.
- Line segments AC and BD are diameters of the circle of radius one. If $\angle BDC = 60^\circ$, the length of line segment AB is _____.

- As shown in the figure, three circles which have the same radius r have centers at $(0, 0)$, $(1, 1)$, and $(2, 1)$. If they have a common tangent line, as shown, then the value of $10\sqrt{5}r$ is _____.



- The acute angle between the line $3x - 4y = 5$ and the circle $x^2 + y^2 - 4x + 2y - 4 = 0$ is θ . Then $9 \cos \theta =$ _____.
- If two perpendicular tangents can be drawn from the origin to the circle $x^2 - 6x + y^2 - 2py + 17 = 0$, then the value of $|p|$ is _____.
- Let $A \equiv (-4, 0)$ and $B \equiv (4, 0)$. The number of points on the circle $x^2 + y^2 = 16$ such that the area of the triangle whose vertices are A , B , and C is positive is _____.
- If the circles $x^2 + y^2 + (3 + \sin \beta)x + (2 \cos \alpha)y = 0$ and $x^2 + y^2 + (2 \cos \alpha)x + 2cy = 0$ touch each other, then the maximum value of c is _____.
- Two circles C_1 and C_2 both pass through the points $A(1, 2)$ and $E(2, 1)$ and touch the line $4x - 2y = 9$ at B and D , respectively. The possible coordinates of a point C , such that the quadrilateral $ABCD$ is a parallelogram, are (a, b) . Then the value of $|ab|$ is _____.
- Difference in values of the radius of a circle whose center is at the origin and which touches the circle $x^2 + y^2 - 6x - 8y + 21 = 0$ is _____.
- If the length of a common internal tangent to two circles is 7, and that of a common external tangent is 11, then the product of the radii of the two circles is _____.

Archives

JEE MAIN

Single Correct Answer Type

- If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P , Q , and $(1, 1)$ for
 - all values of p .
 - all except one value of p .
 - all except two values of p .
 - exactly one value of p .

(AIEEE 2009)

- Three distinct points A , B and C are given in the two dimensional coordinate plane such that the ratio of the distance of any one of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point

- $(0, 0)$
- $\left(\frac{5}{4}, 0\right)$
- $\left(\frac{5}{2}, 0\right)$
- $\left(\frac{5}{3}, 0\right)$ (AIEEE 2009)

3. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

(1) $35 < m < 85$ (2) $-85 < m < -35$
 (3) $-35 < m < 15$ (4) $15 < m < 65$

(AIEEE 2010)

4. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ ($c > 0$) touch each other, if

(1) $|a| = 2c$ (2) $2|a| = c$ (3) $|a| = c$ (4) $a = 2c$

(AIEEE 2011)

5. The length of the diameter of the circle which touches the x -axis at the point $(1, 0)$ and passes through the point $(2, 3)$ is

(1) $\frac{10}{3}$ (2) $\frac{3}{5}$ (3) $\frac{6}{5}$ (4) $\frac{5}{3}$

(AIEEE 2012)

6. The circle passing through $(1, -2)$ and touching the axis of x at $(3, 0)$ also passes through the point

(1) $(-5, 2)$ (2) $(2, -5)$
 (3) $(5, -2)$ (4) $(-2, 5)$ (JEE Main 2013)

7. Let C be the circle with centre at $(1, 1)$ and radius = 1. If T is the circle centred at $(0, y)$, passing through origin and touching the circle C externally, then the radius of T is equal to

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

(JEE Main 2014)

8. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is

(1) 1 (2) 2 (3) 3 (4) 4

(JEE Main 2015)

9. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x -axis, lie on

(1) an ellipse which is not a circle
 (2) a hyperbola
 (3) a parabola
 (4) a circle

(JEE Main 2016)

10. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S , whose centre is at $(-3, 2)$, then the radius of S is

(1) $5\sqrt{3}$ (2) 5 (3) 10 (4) $5\sqrt{2}$

(JEE Main 2016)

(3) $x^2 + y^2 - 2x + 6y - 20$

(4) $x^2 + y^2 - 6x - 4y + 19 = 0$

(IIT-JEE 2009)

2. The circle passing through the point $(-1, 0)$ and touching the y -axis at $(0, 2)$ also passes through the point

(1) $(-3/2, 0)$ (2) $(-5/2, 2)$
 (3) $(-3/2, 5/2)$ (4) $(-4, 0)$ (IIT-JEE 2011)

3. The locus of the midpoint of the chord of contact of tangents drawn from the points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is

(1) $20(x^2 + y^2) - 36x + 45y = 0$
 (2) $20(x^2 + y^2) + 36x - 45y = 0$
 (3) $36(x^2 + y^2) - 20x + 45y = 0$
 (4) $36(x^2 + y^2) + 20x - 45y = 0$

(IIT-JEE 2012)

Multiple Correct Answers Type

1. Circle(s) touching the x -axis at a distance of 3 units from the origin and having an intercept of length $2\sqrt{7}$ on the y -axis is (are)

(1) $x^2 + y^2 - 6x + 8y + 9 = 0$
 (2) $x^2 + y^2 - 6x + 7y + 9 = 0$
 (3) $x^2 + y^2 - 6x - 8y + 9 = 0$
 (4) $x^2 + y^2 - 6x - 7y + 9 = 0$

(JEE Advanced 2013)

2. A circle S passes through the point $(0, 1)$ and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then

(1) radius of S is 8
 (2) radius of S is 7
 (3) centre of S is $(-7, 1)$
 (4) centre of S is $(-8, 1)$

(JEE Advanced 2014)

3. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point $(1, 0)$. Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q . The normal to the circle at P intersects a line drawn through Q parallel to RS at point E . Then the locus of E passes through the point(s)

(1) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (2) $\left(\frac{1}{4}, \frac{1}{2}\right)$
 (3) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (4) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

(JEE Advanced 2016)

4. Let T be the line passing through the points $P(-2, 7)$ and $Q(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q , and also such that S_1 and S_2 touch each other at a point, say, M . Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1 . Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $R(1, 1)$ be F_2 . Let E_2 be the set of the mid-points of the line segments in the set F_2 . Then, which of the following statement(s) is (are) TRUE?

JEE ADVANCED

Single Correct Answer Type

1. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at points A and B . The equation of the circumcircle of triangle PAB is

(1) $x^2 + y^2 + 4x - 6y + 19 = 0$
 (2) $x^2 + y^2 - 4x - 10y + 19 = 0$

- (1) The point $(-2, 7)$ lies in E_1
- (2) The point $(4/5, 7/5)$ does NOT lie in E_2
- (3) The point $(1/2, 1)$ lies in E_2
- (4) The point $(0, 3/2)$ does NOT lie in E_1

(JEE Advanced 2018)

Linked Comprehension Type**For Problems 1 and 2**

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L , perpendicular to PT , is a tangent to the circle $(x-3)^2 + y^2 = 1$.

(IIT-JEE 2012)

1. A possible equation of
- L
- is

- (1) $x - \sqrt{3}y = 1$
- (2) $x + \sqrt{3}y = 1$
- (3) $x - \sqrt{3}y = -1$
- (4) $x + \sqrt{3}y = 5$

2. A common tangent of the two circles is

- (1) $x = 4$
- (2) $y = 2$
- (3) $x + \sqrt{3}y = 4$
- (4) $x + 2\sqrt{2}y = 6$

For Problems 3 and 4

Let S be the circle in the xy -plane defined by the equation $x^2 + y^2 = 4$.

(JEE Advanced 2018)

3. Let E_1E_2 and F_1F_2 be the chords of S passing through the point $P_0(1, 1)$ and parallel to the x -axis and the y -axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slope -1 . Let the tangents to S at E_1 and E_2 meet at E_3 , the tangents to S at F_1 and F_2 meet at F_3 , and the tangents to S at G_1 and G_2 meet at G_3 . Then, the points E_3, F_3 , and G_3 lie on the curve

- (1) $x + y = 4$
- (2) $(x-4)^2 + (y-4)^2 = 16$
- (3) $(x-4)(y-4) = 4$
- (4) $xy = 4$

4. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N . Then, the mid-point of the line segment MN must lie on the curve

- (1) $(x+y)^2 = 3xy$
- (2) $x^{2/3} + y^{2/3} = 2^{4/3}$
- (3) $x^2 + y^2 = 2xy$
- (4) $x^2 + y^2 = x^2y^2$

Matrix Match Type

1. Match the conics in List I with the statements/expressions in List II.

List I	List II
a. Circle	p. The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
b. Parabola	q. Points z in the complex plane satisfying $ z+2 - z-2 = \pm 3$
c. Ellipse	r. Points of the conic have parametric representation $x = \sqrt{3} \left(\frac{1-t^2}{1+t^2} \right)$, $y = \frac{2t}{1+t^2}$
d. Hyperbola	s. The eccentricity of the conic lies in the interval $1 \leq e < \infty$
	t. Points z in the complex plane satisfying $\operatorname{Re}(z+1)^2 = z ^2 + 1$

(IIT-JEE 2009)

Numerical Value Type

1. The centers of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the midpoint of the line segment joining the centers of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C , then find the radius of circle C .

(IIT-JEE 2009)

2. The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \leq 6$ into two parts.

$$\text{If } S = \left\{ \left(2, \frac{3}{4} \right), \left(\frac{5}{2}, \frac{3}{4} \right), \left(\frac{1}{4}, -\frac{1}{4} \right), \left(\frac{1}{8}, \frac{1}{4} \right) \right\}$$

then the number of point(s) in S lying inside the smaller part is _____.

(IIT-JEE 2011)

3. For how many values of p , the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points?

(JEE Advanced 2017)

Answers Key**EXERCISES****Single Correct Answer Type**

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (1) | 2. (1) | 3. (2) | 4. (1) | 5. (1) |
| 6. (1) | 7. (1) | 8. (1) | 9. (4) | 10. (2) |
| 11. (3) | 12. (3) | 13. (4) | 14. (1) | 15. (3) |
| 16. (1) | 17. (1) | 18. (2) | 19. (2) | 20. (1) |
| 21. (1) | 22. (1) | 23. (2) | 24. (4) | 25. (2) |
| 26. (4) | 27. (2) | 28. (3) | 29. (3) | 30. (2) |

- | | | | | |
|---------|---------|---------|---------|---------|
| 31. (1) | 32. (3) | 33. (3) | 34. (4) | 35. (2) |
| 36. (1) | 37. (1) | 38. (3) | 39. (3) | 40. (4) |
| 41. (2) | 42. (3) | 43. (1) | 44. (1) | 45. (3) |
| 46. (1) | 47. (3) | 48. (2) | 49. (2) | 50. (1) |
| 51. (1) | 52. (1) | 53. (4) | 54. (4) | 55. (3) |
| 56. (1) | 57. (2) | 58. (4) | 59. (4) | 60. (3) |
| 61. (3) | 62. (2) | 63. (2) | 64. (1) | 65. (4) |
| 66. (3) | 67. (2) | 68. (1) | 69. (1) | 70. (2) |

4.54 Coordinate Geometry

- | | | | | |
|----------|----------|----------|----------|----------|
| 71. (2) | 72. (3) | 73. (3) | 74. (1) | 75. (1) |
| 76. (3) | 77. (2) | 78. (1) | 79. (2) | 80. (1) |
| 81. (2) | 82. (1) | 83. (1) | 84. (2) | 85. (2) |
| 86. (4) | 87. (3) | 88. (3) | 89. (3) | 90. (1) |
| 91. (4) | 92. (1) | 93. (1) | 94. (3) | 95. (4) |
| 96. (1) | 97. (2) | 98. (2) | 99. (3) | 100. (1) |
| 101. (3) | 102. (2) | 103. (1) | 104. (3) | 105. (3) |
| 106. (3) | 107. (3) | 108. (4) | 109. (4) | |

Multiple Correct Answers Type

- | | |
|------------------------|------------------------|
| 1. (1), (3) | 2. (1), (3), (4) |
| 3. (2), (3) | 4. (1), (2), (3) |
| 5. (1), (3) | 6. (1), (2), (3), (4) |
| 7. (1), (3), (4) | 8. (1), (2), (3), (4) |
| 9. (1), (3) | 10. (2), (4) |
| 11. (1), (4) | 12. (2), (3) |
| 13. (1), (3) | 14. (1), (4) |
| 15. (1), (4) | 16. (1), (2), (3), (4) |
| 17. (2), (4) | 18. (2), (3) |
| 19. (1), (2), (3), (4) | 20. (2), (4) |
| 21. (2), (3) | 22. (3), (4) |
| 23. (1), (2) | 24. (1), (2) |
| 25. (1), (4) | |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (4) | 2. (2) | 3. (1) | 4. (4) | 5. (3) |
| 6. (3) | 7. (3) | 8. (4) | 9. (3) | 10. (3) |
| 11. (1) | 12. (3) | 13. (3) | 14. (1) | 15. (1) |
| 16. (2) | 17. (4) | 18. (2) | 19. (1) | 20. (4) |
| 21. (1) | 22. (2) | 23. (1) | 24. (1) | 25. (2) |
| 26. (3) | 27. (3) | 28. (2) | 29. (2) | 30. (3) |
| 31. (2) | 32. (3) | 33. (4) | 34. (2) | 35. (4) |
| 36. (1) | 37. (4) | 38. (1) | 39. (3) | 40. (2) |

- | | | | | |
|---------|---------|---------|---------|---------|
| 41. (3) | 42. (4) | 43. (4) | 44. (4) | 45. (1) |
| 46. (4) | | | | |

Matrix Match Type

- $a \rightarrow r; b \rightarrow q; c \rightarrow q; d \rightarrow p$
- $a \rightarrow p, r, s; b \rightarrow r, s; c \rightarrow q, s; d \rightarrow p, s$
- $a \rightarrow r; b \rightarrow s; c \rightarrow q; d \rightarrow q$
- $a \rightarrow r; b \rightarrow p, q; c \rightarrow q, r; d \rightarrow p, s$
- (3)
- (4)

Numerical Value Type

- | | | | | |
|---------|---------|----------|----------|---------|
| 1. (6) | 2. (6) | 3. (8) | 4. (1) | 5. (9) |
| 6. (0) | 7. (4) | 8. (2) | 9. (18) | 10. (1) |
| 11. (5) | 12. (3) | 13. (5) | 14. (62) | 15. (1) |
| 16. (4) | 17. (4) | 18. (18) | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

- | | | | | |
|--------|--------|--------|--------|---------|
| 1. (2) | 2. (2) | 3. (3) | 4. (3) | 5. (1) |
| 6. (3) | 7. (4) | 8. (3) | 9. (3) | 10. (1) |

JEE ADVANCED

Single Correct Answer Type

- | | | |
|--------|--------|--------|
| 1. (2) | 2. (4) | 3. (1) |
|--------|--------|--------|

Multiple Correct Answers Type

- | | |
|-------------|-------------|
| 1. (1), (3) | 2. (2), (3) |
| 3. (1), (3) | 4. (2), (4) |

Linked Comprehension Type

- | | | | |
|--------|--------|--------|--------|
| 1. (1) | 2. (4) | 3. (1) | 4. (4) |
|--------|--------|--------|--------|

Matrix Match Type

- $a \rightarrow p$

Numerical Value Type

- | | | |
|--------|--------|--------|
| 1. (8) | 2. (2) | 3. (2) |
|--------|--------|--------|

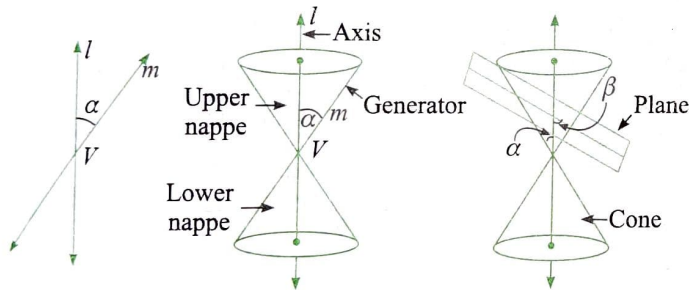
5

Parabola

CONIC SECTION: INTRODUCTION

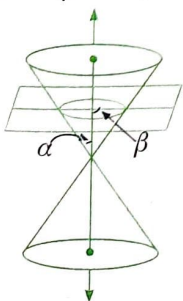
The four curves — circle, parabola, ellipse and hyperbola are called conic sections because they can be formed by intersecting a double right circular cone with a plane. Let us first see how double right circular cone is created.

Let l be a fixed vertical line and m be another line intersecting it at a fixed point V and inclined to it at an angle α . If we rotate the line m around the line l in such a way that the angle α remains constant, then the surface generated is a double-napped right circular hollow cone herein after referred as cone and extending indefinitely far in both directions. The point V is called the *vertex*; the line l is the *axis* of the cone. The rotating line m is called a *generator* of the cone. The *vertex* separates the cone into two parts called *nappes*.

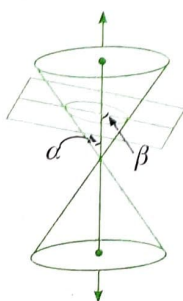


If we take the intersection of a plane with a cone, the section so obtained is called a *conic section*. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane. We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and by the angle β made by it with the vertical axis of the cone. When the plane cuts the nappe (other than the vertex) of the cone, we have the following conic sections:

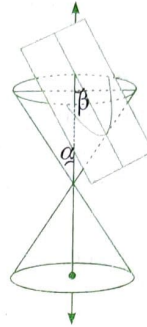
Circle ($\beta = 90^\circ$)



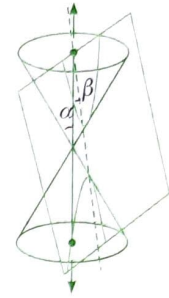
Ellipse ($\alpha < \beta < 90^\circ$)



Parabola ($\beta = \alpha$)



Hyperbola ($0 \leq \beta < \alpha$)



CONIC SECTION AS LOCUS OF A POINT

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is constant, is called conic section or conic.

This fixed point is called the *focus* of the conic and the fixed line is called the *directrix* of the conic.

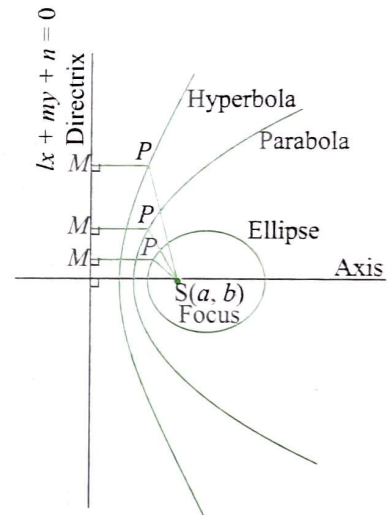
In the figure, focus is $S(a, b)$ and directrix is $lx + my + n = 0$.

Point $P(x, y)$ is moving in the plane such that ratio $\frac{SP}{PM}$ is constant.

This constant ratio is called eccentricity e of the conic.

$$\therefore \frac{SP}{PM} = e \text{ or } SP = e \cdot PM$$

Now, depending upon the values of e we have different characteristic curves.



Value of e	Curve or Locus of point P
$e = 1$, i.e., $SP = PM$	Parabola
$0 < e < 1$, i.e., $SP < PM$	Ellipse
$e > 1$, i.e., $SP > PM$	Hyperbola

Equation of Conic

For any point P on the conic, we have

$$SP = e \cdot PM$$

$$\therefore \sqrt{(x-a)^2 + (y-b)^2} = e \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

$$\Rightarrow (x-a)^2 + (y-b)^2 = e^2 \frac{(lx + my + n)^2}{l^2 + m^2}$$

This equation takes the form

$Ax^2 + By^2 + 2Hxy + 2Gx + 2Fy + C = 0$, which is general second-degree equation in variables x and y .

Elements of Conic

We have following elements of the conic

Axis of conic: The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

Each conic is symmetrical about its axis.

Vertex: The points of intersection of the conic section and the axis is (are) called vertex (vertices) of the conic section.

Focal Chord: Any chord passing through the focus is called focal chord of the conic section.

Double Ordinate: A straight line drawn perpendicular to the axis and terminated at both end of the curve is a double ordinate of the conic section.

Latus Rectum: The double ordinate passing through the focus is called the latus rectum of the conic section.

Length of the latus rectum: If double ordinate through focus S is terminated by conic at points P and P' on the curve, then PP' called length of latus rectum or simply latus rectum.

Now $PP' = 2 \times SP$

$$= 2 \times e \times PM \quad (\text{from the definition of the conic})$$

$$= 2 \times e \times (\text{distance of focus from directrix})$$

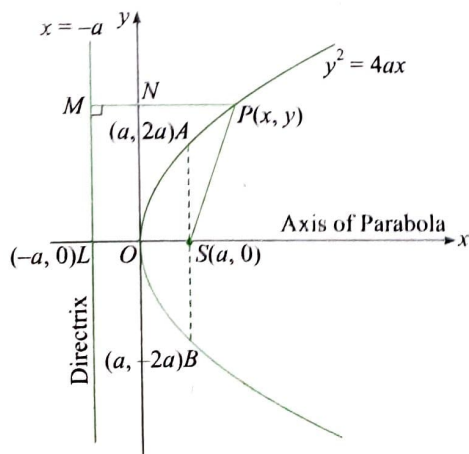
Centre: The point which bisects every chord of the conic passing through it, is called the centre of the conics section.

Parabola does not have centre, but circle, ellipse, hyperbola have centres.

STANDARD EQUATION OF PARABOLA

Standard equation of parabola is the simplest form of equation of parabola. The study of properties of parabola becomes easier with this form of equation.

For this parabola we consider focus 'S' to be at $(a, 0)$ and directrix to be $x = -a$.



So, from the definition, equation of parabola is locus of point $P(x, y)$ such that $SP = PM$.

$$\therefore SP = PN + NM$$

$$\Rightarrow \sqrt{(x-a)^2 + (y-0)^2} = x + a$$

$$\Rightarrow (x-a)^2 + y^2 = (a+x)^2$$

$$\Rightarrow y^2 = (a+x)^2 - (x-a)^2$$

$$\Rightarrow y^2 = 4ax$$

Since $y^2 \geq 0$, $4ax \geq 0$. So, $x \geq 0$. (as $a > 0$).

Therefore, curve lies in first and fourth quadrants.

Also, curve opens to the right.

The part of the curve above x -axis represents function $y = \sqrt{4ax}$ and the part of the curve below x -axis represents the function $y = -\sqrt{4ax}$.

From the figure,

Vertex: $O(0, 0)$

Axis of the parabola: x -axis or $y = 0$

Tangent at vertex: y -axis or $x = 0$

End points of Latus Rectum: $A(a, 2a)$ and $B(a, -2a)$

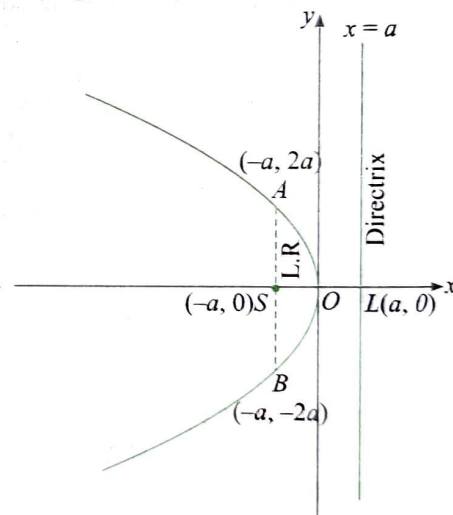
Length of Latus Rectum (L.R.): $4a$

Distance of point P from focus S (focal distance or focal radius): $SP = PM = PN + NM = x + a$

Other Standard Forms of Parabola

I. Parabola with focus S at $(-a, 0)$ and directrix $x = a$.

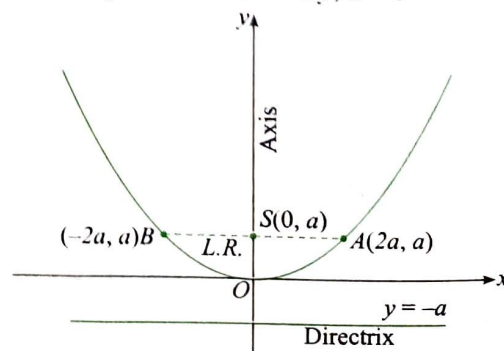
Equation of parabola is $y^2 = -4ax$, $a > 0$



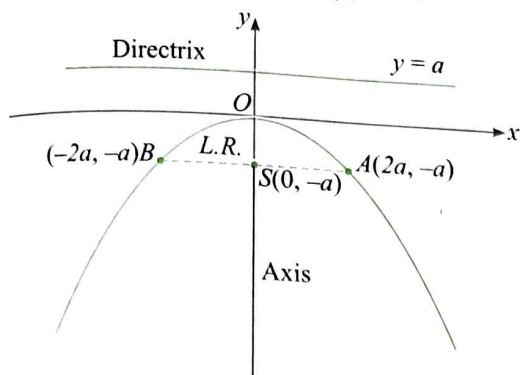
The part of the curve above x -axis represents function $y = \sqrt{-4ax}$ and the part of the curve below x -axis represents the function $y = -\sqrt{-4ax}$.

II. Parabola with focus S at $(0, a)$ and directrix $y = -a$.

Equation of parabola is $x^2 = 4ay$, $a > 0$



- III. Parabola with focus S at $(0, -a)$ and directrix $y = a$.
Equation of parabola is $x^2 = -4ay$, $a > 0$



In all above parabolas vertex is $O(0, 0)$ and length of latus rectum is ' $4a$ '.

Note: Two parabolas are called equal parabolas if they have same length of latus rectum.

ILLUSTRATION 5.1

Find the equation of parabola

- having focus at $(0, -3)$ and its directrix is $y = 3$.
- having end points of latus rectum $(5, 10)$ and $(5, -10)$ and which opens towards right.
- having vertex at origin and focus at $(0, 2)$

Sol.

- (i) Here, focus $S(0, -3)$ and directrix $y = 3$ are at same distance from origin.

So, origin is vertex of the parabola and parabola opens downwards.

So, we consider equation $x^2 = -4ay$.

Here focus is $(0, -3) \equiv (0, -a)$

$$\therefore a = 3$$

So, equation of parabola is $y^2 = -12x$.

- (ii) End points of latus rectum are $A(5, 10)$ and $B(5, -10)$.

Now, focus S is midpoint of AB .

$$\therefore S \equiv (5, 0)$$

Also, parabola opens towards right.

So, we consider equation $y^2 = 4ax$.

$$\therefore \text{Vertex is } (0, 0) \text{ and } a = 5.$$

So, equation of parabola is $y^2 = 20x$.

- (iii) Vertex is $(0, 0)$ focus $S(0, 2)$.

So, directrix is $y = -2$.

So, we consider equation $x^2 = 4ay$, where $a = 2$.

$$\therefore \text{Equation of parabola is } x^2 = 8y.$$

ILLUSTRATION 5.2

An arch is in the form of a parabola with its axis vertical. The arch is 12 m high and 6 m wide at the base. How wide is it 6 m from the vertex of the parabola?

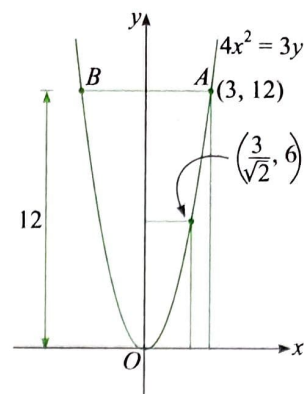
Sol. From the figure, AB is base and height is 12 m.

Consider parabola $x^2 = 4ay$.

From the figure, point $(3, 12)$ lies on the parabola.

$$\therefore 9 = 4 \times a \times 12$$

$$\therefore 4a = \frac{3}{4}$$



Therefore, equation of parabola is $x^2 = \frac{3}{4}y$.

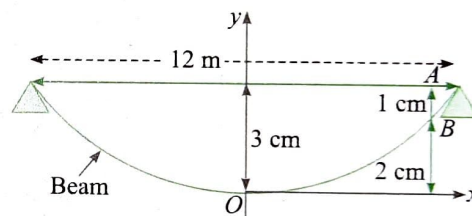
$$\text{When } y = 6, x^2 = \frac{9}{2}.$$

$$\therefore \text{Width} = 2 \times \frac{3}{\sqrt{2}} = 3\sqrt{2} \text{ m}$$

ILLUSTRATION 5.3

A beam is supported at its ends by two supports which are 12 m apart. Since the load is concentrated at its center, there is a deflection of 3 cm at the center and the deflected beam is in the shape of a parabola. How far from the center is the deflection 1 cm?

Sol.



Let the vertex be at the lowest point and the axis vertical. Let the coordinate axes be chosen as shown in figure.

The equation of the parabola takes the form $x^2 = 4ay$. Since it passes through $(6, 3/100)$, we have

$$6^2 = 4a \times \frac{3}{100}$$

$$\text{i.e., } a = 300$$

Let AB be the deflection of the beam which is $\frac{1}{100}$ m. The coordinates of B are $(x, 2/100)$. Therefore,

$$x^2 = 4 \times 300 \times \frac{2}{100} = 24$$

$$\text{or } x = 2\sqrt{6} \text{ m}$$

ILLUSTRATION 5.4

Find the coordinates of a point on the parabola $y^2 = 8x$ whose distance from the focus is 10.

Sol. Distance of point $P(x, y)$ from the focus $S(a, 0)$ of the parabola $y^2 = 4ax$ is $SP = a + x$.

For parabola $y^2 = 8x$, $a = 2$.

Also, given that $SP = 10$.

$$\therefore 10 = 2 + x$$

$$\therefore x = 8$$

So, from $y^2 = 8x$,

$$y^2 = 64$$

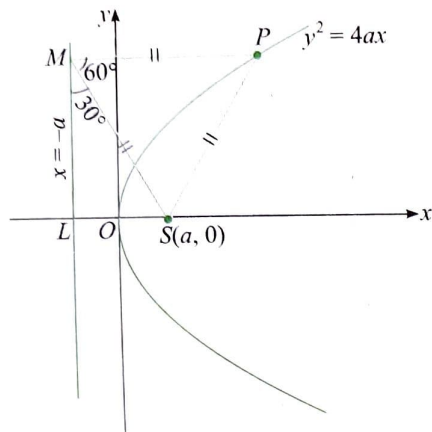
$$\therefore y = \pm 8$$

So, coordinates of point P are $(8, 8)$ or $(8, -8)$.

ILLUSTRATION 5.5

If M is the foot of the perpendicular from a point P on a parabola $y^2 = 4ax$ to its directrix and triangle SPM is equilateral, where S is the focus, then find SP .

Sol.



From the definition of the parabola, we have

$$SP = PM$$

Now, triangle SPM is equilateral.

Therefore, $SP = PM = SM$.

From the figure, in triangle SLM ,

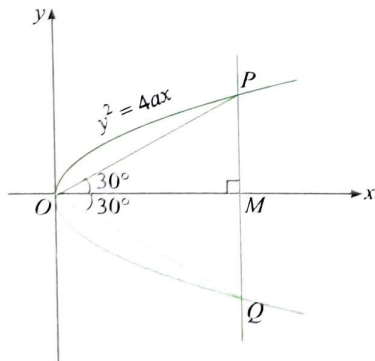
$$SM = SL \operatorname{cosec} 30^\circ = 2a \times 2 = 4a$$

$$\therefore SP = 4a$$

ILLUSTRATION 5.6

An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, such that one vertex of this triangle coincides with the vertex of the parabola. Then find the side length of this triangle.

Sol. As shown in the figure, equilateral triangle OPQ is symmetrical about x -axis.



Let $OP = l$.

In triangle OMP ,

$$OM = OP \cos 30^\circ = \frac{\sqrt{3}l}{2}$$

$$\text{and } MP = OP \sin 30^\circ = \frac{l}{2}$$

$$\therefore P \equiv \left(\frac{\sqrt{3}l}{2}, \frac{l}{2} \right)$$

P lies on parabola.

$$\therefore \left(\frac{l}{2} \right)^2 = 4a \left(\frac{\sqrt{3}l}{2} \right)$$

$$\therefore l = 8\sqrt{3}a \Rightarrow OP = 8\sqrt{3}a$$

EQUATION OF CHORD OF PARABOLA BISECTED AT GIVEN POINT

Consider parabola $y^2 = 4ax$.

Let QR be the chord joining two points $Q(x_2, y_2)$ and $R(x_3, y_3)$.

Let midpoint of QR be $P(x_1, y_1)$.

Since Q and R lie on parabola,

$$y_2^2 = 4ax_2 \text{ and } y_3^2 = 4ax_3$$

$$\Rightarrow y_3^2 - y_2^2 = 4a(x_3 - x_2)$$

$$\Rightarrow \text{Slope of } QR = \frac{y_3 - y_2}{x_3 - x_2} = \frac{4a}{y_3 + y_2} = \frac{4a}{2y_1}$$

($\because P(x_1, y_1)$ is midpoint of QR)

Therefore, equation of QR is

$$y - y_1 = \frac{2a}{y_1} (x - x_1)$$

$$\Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$$

Subtracting $2ax_1$ from both sides, we have

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

$$\text{or } T = S_1, \text{ where } T = yy_1 - 2a(x + x_1) \text{ and } S_1 = y_1^2 - 4ax_1$$

This is the equation of chord of parabola $y^2 = 4ax$ which is bisected at point $P(x_1, y_1)$.

Similarly, for parabola $x^2 = 4ay$, equation of chord which is bisected at point $P(x_1, y_1)$ is

$$xx_1 - 2a(y + y_1) = x_1^2 - 4ay_1$$

ILLUSTRATION 5.7

Find the equation of the chord of the parabola $y^2 = 8x$ having slope 2 if midpoint of the chord lies on the line $x = 4$.

Sol. Let $M(4, y_1)$ be midpoint of the chord.

So, equation of chord using equation $T = S_1$ is

$$yy_1 - 4(x + 4) = y_1^2 - 8 \times 4$$

$$\text{or } yy_1 - 4x = y_1^2 - 16$$

$$\text{Slope} = \frac{4}{y_1} = 2$$

$$\therefore y_1 = 2$$

So, equation of the chord is

$$2y - 4x = -12$$

$$\text{or } y = 2x - 6$$

(Given)

ILLUSTRATION 5.8

Find the locus of midpoint of family of chords $\lambda x + y = 5$ (λ is parameter) of the parabola $x^2 = 20y$.

Sol. Equation of family of chord is

$$(y - 5) + \lambda(x - 0) = 0, \text{ which are concurrent at } (0, 5).$$

The given parabola is $x^2 - 20y = 0$.

Let $M(h, k)$ be midpoint of chord.

Therefore, equation of such chord is

$$hx - 10(y + k) = h^2 - 20k$$

(Using $T = S_1$)

This chord is passing through the point $(0, 5)$.

$$h^2 = 10k - 50$$

∴ So, locus of $M(h, k)$ is $x^2 = 10(y - 5)$.

POSITION OF A POINT WITH RESPECT TO A PARABOLA

Parabola drawn on the plane divides the plane into two regions, one is concave (interior) region where focus lies and other is convex (exterior) region.

Consider parabola having equation $y^2 = 4ax$.

Let $P(x_1, y_1)$ be a point in the plane (in first or fourth quadrant).

From point P , draw perpendicular to x -axis meeting x -axis at M and parabola at Q .

Let the coordinates of Q be (x_1, y_2) .

$$\therefore y_2^2 = 4ax_1$$

Now, point $P(x_1, y_1)$ will be outside, on or inside the parabola $y^2 = 4ax$ according as

$$PM >, = \text{ or } < QM$$

$$\Rightarrow PM^2 >, = \text{ or } < QM^2$$

$$\Rightarrow y_1^2 >, = \text{ or } < y_2^2$$

$$\Rightarrow y_1^2 >, = \text{ or } < 4ax_1$$

If point P lies in second or third quadrant, $y_1^2 - 4ax_1 > 0$ (as $x_1 < 0$).

Thus, $y^2 - 4ax < 0$ represents the concave region and $y^2 - 4ax > 0$ represents the convex region of the parabola.

Similarly, $y^2 + 4ax < 0$, $x^2 - 4ay < 0$ and $x^2 + 4ay < 0$ represent the concave (interior) region of the parabolas. Whereas, $y^2 + 4ax > 0$, $x^2 - 4ay > 0$ and $x^2 + 4ay > 0$ represent the convex (exterior) region of the parabolas ($a > 0$).

ILLUSTRATION 5.9

Find the position of points $P(1, 3)$ w.r.t. parabolas $y^2 = 4x$ and $x^2 = 8y$.

Sol. We have parabolas $y^2 - 4x = 0$ and $x^2 - 3y = 0$.

Let $S_1 \equiv y^2 - 4x$ and $S_2 \equiv x^2 - 3y$

Now, $S_1(1, 3) = 3^2 - 4(1) > 0$

And $S_2(1, 3) = 1^2 - 3(3) < 0$

Thus, $(1, 3)$ lies in the exterior region of the parabola $y^2 = 4x$ and interior region of the parabola $x^2 = 3y$.

ILLUSTRATION 5.10

The point $(a, 2a)$ is interior region bounded by the parabola $y^2 = 16x$ and the double ordinate through the focus. Then find the values of a .

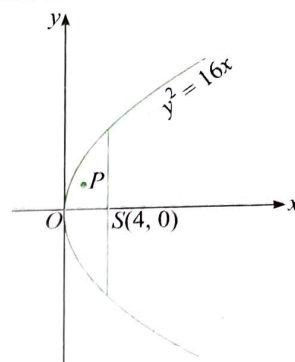
Sol. $P(a, 2a)$ is in interior region of the parabola $y^2 - 16x = 0$ if

$$(2a)^2 - 16a < 0$$

$$\text{or } a^2 - 4a < 0$$

$$\Rightarrow 0 < a < 4$$

(1)



Now, double ordinate line through focus is $x - 4 = 0$.

If $(a, 2a)$ lies to the left of $x - 4 = 0$ then

$$a - 4 < 0$$

$$\text{or } a < 4$$

(2)

From (1) and (2), $0 < a < 4$.

PARAMETRIC FORM OF PARABOLA

Parametric coordinates (or parametric form) of different standard forms of parabolas are listed below.

Equation of Parabola	Parametric form
$y^2 = 4ax$	$x = at^2, y = 2at$
$y^2 = -4ax$	$x = -at^2, y = 2at$
$x^2 = 4ay$	$x = 2at, y = at^2$
$x^2 = -4ay$	$x = 2at, y = -at^2$

Note:

- For parabola $y^2 = 4ax$, any point P on it has coordinates $(at^2, 2at)$, where t is parameter.

Sometimes, we use notation $P(t)$ for point P , which means that point P has parameter t .

For example, for parabola $y^2 = 8x$, point $P(3)$ means $P \equiv (at^2, 2at) \equiv (2(3)^2, 2(2)(3)) \equiv (18, 12)$.

Similarly, for other parabolas also, we can represent point P .

- Slope of line joining points $P(t_1)$ and $Q(t_2)$ on the parabola

$$y^2 = 4ax \text{ is } \frac{2at_1 - 2at_2}{at_1^2 - at_2^2} = \frac{2}{t_1 + t_2}$$

- For parabola $y^2 = 4ax$, if $t_1 t_2 = \text{Constant}$, then chord joining points $P(t_1)$ and $Q(t_2)$ passes through fixed point on x -axis.

Proof:

Equation of chord joining point $P(t_1)$ and $Q(t_2)$ is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$\therefore (t_1 + t_2)y = 2x + 2at_1 t_2$$

For $t_1 t_2 = c$ (constant), we have

$(t_1 + t_2)y = 2x + 2ac$, which passes through fixed point $(-ac, 0)$ on x -axis.

- If chord of parabola $y^2 = 4ax$ joining points $P(t_1)$ and $Q(t_2)$ subtends right angle at vertex, then $t_1 t_2 = -4$.

ILLUSTRATION 5.11

Find the locus of the middle points of chords of a parabola $y^2 = 4ax$ which subtend right angle at the vertex.

Sol. Let the chord joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ subtends right angle at vertex of the parabola $y^2 = 4ax$.

$$\therefore t_1 t_2 = -4$$

If midpoint of chord PQ is $R(h, k)$, then

$$h = \frac{at_1^2 + at_2^2}{2}, k = \frac{2at_1 + 2at_2}{2}$$

$$\therefore \frac{2h}{a} = (t_1 + t_2)^2 - 2t_1 t_2 = (t_1 + t_2)^2 + 8 \text{ and } \frac{k}{a} = t_1 + t_2$$

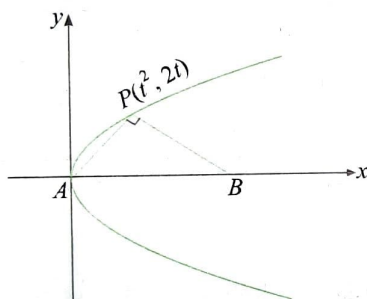
Eliminating $t_1 + t_2$, we get

$$\frac{2h}{a} = \frac{k^2}{a^2} + 8$$

$$\therefore 2ax = y^2 + 8a^2, \text{ which is the required locus.}$$

ILLUSTRATION 5.12

In the following figure, find the locus of centroid of triangle PAB , where AP is perpendicular to PB .



Sol.

$$\therefore \text{Slope of } AP = \frac{2}{t}$$

$$\Rightarrow \text{Slope of } BP = -\frac{t}{2}$$

$$\text{So, equation of line } BP \text{ is } y - 2t = -\frac{t}{2}(x - t^2).$$

Putting $y = 0$, we get point B as $(t^2 + 4, 0)$.

Now, let centroid of $\triangle PAB$ be (h, k) .

$$\therefore h = \frac{t^2 + t^2 + 4}{3} \text{ and } k = \frac{2t}{3}$$

Eliminating 't', we get

$$3h - 4 = 2\left(\frac{3k}{2}\right)^2$$

$$\therefore 3x - 4 = \frac{9y^2}{2}, \text{ which is the required locus.}$$

ILLUSTRATION 5.13

A quadrilateral is inscribed in a parabola $y^2 = 4ax$ and three of its sides pass through fixed points on the axis. Show that the fourth side also passes through a fixed point on the axis of the parabola.

Sol. Consider quadrilateral joining points $A(t_1), B(t_2), C(t_3)$ and $D(t_4)$ is inscribed in the parabola $y^2 = 4ax$.

Equation of line AB is

$$y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$\text{or } 2x - (t_1 + t_2)y + 2at_1 t_2 = 0$$

Let AB pass through fixed point $(c, 0)$ on x -axis.

$$\therefore t_1 t_2 = -\frac{c}{a} \quad (1)$$

Similarly, for chord BC ,

$$t_2 t_3 = -\frac{d}{a} \quad (2)$$

and for chord CD ,

$$t_3 t_4 = -\frac{e}{a} \quad (3)$$

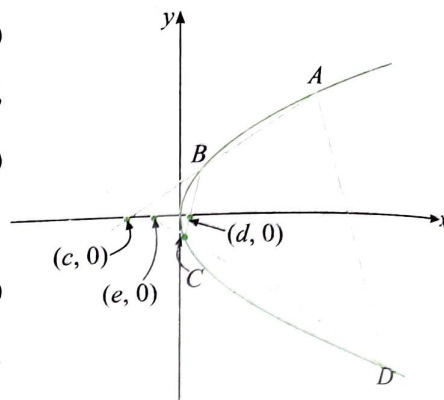
Multiplying (1) and (3),

$$t_1 t_2 t_3 t_4 = \frac{ec}{a^2}$$

$$\Rightarrow t_1 t_4 \left(-\frac{d}{a}\right) = \frac{ec}{a^2}$$

$$\Rightarrow t_1 t_4 = -\frac{ec}{ad}$$

Thus, chord AD also passes through a fixed point on x -axis.

**CONCEPT APPLICATION EXERCISE 5.1**

- Find the angle made by a double ordinate of length $8a$ at the vertex of the parabola $y^2 = 4ax$.
- If focal distance of a point P on the parabola $y^2 = 4ax$ whose abscissa is 5 is 10, then find the value of a .
- Analyse the following equations if they represent parabola(s) or part of parabola(s)?
 - $y = x|x|$
 - $x = \sqrt{-y}$
 - $x^2 = y^4$
 - $x = e^t, 2t = \log_e y$
- Find the range of values of λ for which the point $(\lambda, -1)$ is exterior to both the parabolas $y^2 = |x|$.
- Find the locus of a point on the variable parabola $y^2 = 4ax$ whose distance from focus is always equal to k .
- Find the locus of midpoint of chord of the parabola $y^2 = 4ax$ that passes through the point $(3a, a)$.
- If chord BC subtends right angle at the vertex A of the parabola $y^2 = 4x$ with $AB = \sqrt{5}$, then find the area of triangle ABC .
- PQ is a chord of the parabola $y^2 = 4x$ whose perpendicular bisector meets the axis at M and the ordinate of the midpoint PQ meets the axis at N . Find length MN .

9. LOL' and MOM' are two chords of parabola $y^2 = 4ax$ with vertex A passing through a point O on its axis. Prove that the radical axis of the circles described on LL' and MM' as diameters passes through the vertex of the parabola.

ANSWERS

- | | |
|--------------------------------|---------------------------|
| 1. 90° | 2. 5 |
| 4. $-1 < \lambda < 1$ | 5. $4x^2 + y^2 - 4kx = 0$ |
| 6. $y^2 - 2ax - ay + 6a^2 = 0$ | |
| 7. 20 sq. units | 8. 2 |

OTHER FORMS OF PARABOLA

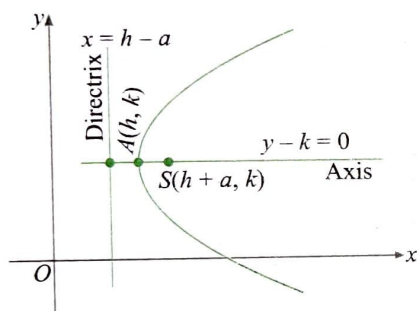
EQUATION OF PARABOLA HAVING AXIS PARALLEL TO X-AXIS

The standard equation of parabola $y^2 = 4ax$ ($a > 0$) can be written as $(y - 0)^2 = 4a(x - 0)$.

Here, vertex of the parabola is $(0, 0)$.

Now, if the parabola is shifted (translated) with vertex at (h, k) , then its equation becomes

$$(y - k)^2 = 4a(x - h)$$



The axis of this parabola is $y - k = 0$, which is parallel to x -axis.

The focus is $(h + a, k)$ and directrix is $x = h - a$.

Parametric form of this parabola is $x = h + at^2$ and $y = k + 2at$.

The part of the curve above axis $y = k$ represents the function $y - k = \sqrt{4a(x - h)}$ and part of the curve below $y = k$ represents the function $y - k = -\sqrt{4a(x - h)}$.

The general form of the parabola having axis parallel to x -axis is $x = Ay^2 + By + C$ ($A \neq 0$).

EQUATION OF PARABOLA HAVING AXIS PARALLEL TO Y-AXIS

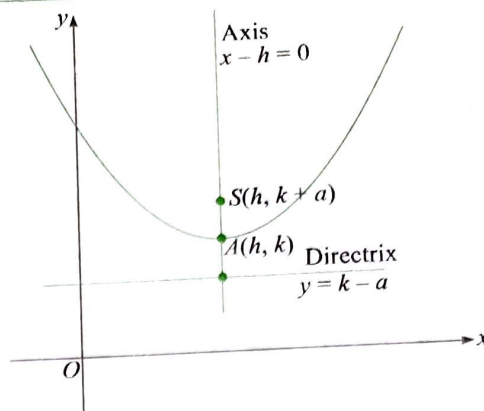
If parabola having equation $x^2 = 4ay$ ($a > 0$) or $(x - 0)^2 = 4a(y - 0)$ is shifted (translated) with vertex at (h, k) , then its equation becomes

$$(x - h)^2 = 4a(y - k)$$

The axis of this parabola is $x - h = 0$, which is parallel to y -axis.

The focus is $(h, k + a)$ and directrix is $y = k - a$.

Parametric form of this parabola is $x = h + 2at$ and $y = k + at^2$.



The general form of the parabola having axis parallel to y -axis is $y = Ax^2 + Bx + C$ ($A \neq 0$), which represents the quadratic function.

Now, $y = Ax^2 + Bx + C$

$$= A \left(x^2 + \frac{B}{A}x + \frac{C}{A} \right)$$

$$= A \left(\left(x + \frac{B}{2A} \right)^2 - \frac{B^2}{4A^2} + \frac{C}{A} \right)$$

$$= A \left(\left(x + \frac{B}{2A} \right)^2 - \frac{B^2 - 4AC}{4A^2} \right)$$

$$\text{or } \left(x + \frac{B}{2A} \right)^2 = \frac{1}{A} \left(y + \frac{B^2 - 4AC}{4A} \right)$$

Comparing it with $(x - h)^2 = 4a(y - k)$, we have vertex at $\left(-\frac{B}{2A}, -\frac{B^2 - 4AC}{4A} \right)$ and length of latus rectum $\frac{1}{|A|}$.

Curve opens upwards if $A > 0$ and opens downwards if $A < 0$. Similarly, $x = Ay^2 + By + C$ simplifies to

$$\left(y + \frac{B}{2A} \right)^2 = \frac{1}{A} \left(x + \frac{B^2 - 4AC}{4A} \right)$$

Comparing it with $(y - k)^2 = 4a(x - h)$, we have vertex at $\left(-\frac{B^2 - 4AC}{4A}, -\frac{B}{2A} \right)$ and length of latus rectum $\frac{1}{|A|}$.

Curve opens to the right if $A > 0$ and opens to the left if $A < 0$.

ILLUSTRATION 5.14

Find the equation of parabola

- having its vertex at $A(1, 0)$ and focus at $S(3, 0)$
- having its focus at $S(2, 5)$ and one of the extremities of latus rectum is $A(4, 5)$

Sol.

- Vertex is at $A(1, 0)$ and focus is at $S(3, 0)$.

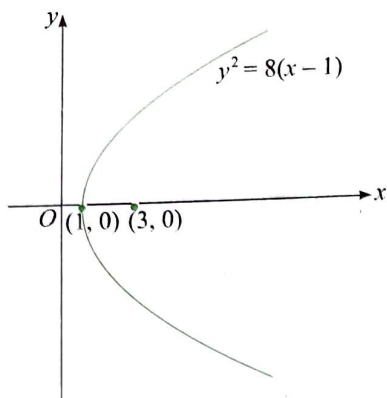
Clearly, the axis of parabola is the x -axis.

So, we consider equation of the parabola as $(y - k)^2 = 4a(x - h)$.

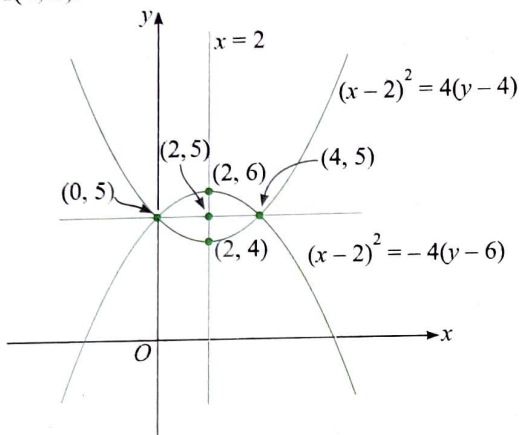
Here, vertex is $(1, 0) \equiv (h, k)$

and a = distance between vertex and focus = 2

Therefore, equation of parabola is $y^2 = 8(x - 1)$.



- (ii) Focus is at $S(2, 5)$ and one of the extremities of latus rectum is $A(4, 5)$.



Clearly, the other extremity of latus rectum is $B(0, 5)$ and axis of the parabola is $x = 2$.

So, we consider equation of parabolas as

$$(x-h)^2 = 4a(y-k)$$

or $(x-h)^2 = -4a(y-k)$, where $h = 2$.

Length of latus rectum is $AB = 4a = 4$.

$$\therefore a = 1$$

Now, we have following two cases:

Parabola opens upwards:

In this case, vertex lies on the line $x = 2$ at distance 1 unit below the focus.

So, vertex is $(2, 5 - 1)$ or $(2, 4)$.

Thus, equation of parabola is $(x-2)^2 = 4(y-4)$.

Parabola opens downwards:

In this case, vertex lies on the line $x = 2$ at distance 1 unit above the focus.

So, vertex is $(2, 5 + 1)$ or $(2, 6)$.

Thus, equation of parabola is $(x-2)^2 = -4(y-6)$.

ILLUSTRATION 5.15

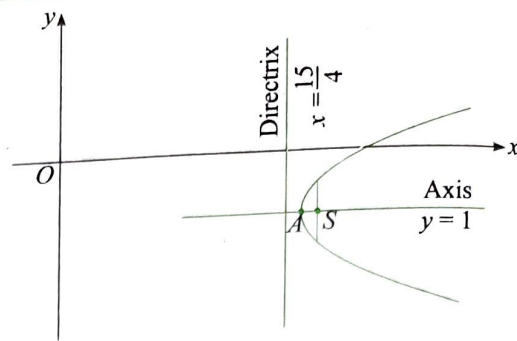
Equation $y^2 + 2y - x + 5 = 0$ represents a parabola. Find its vertex, equation of axis, equation of latus rectum, coordinates of the focus, equation of the directrix, extremities of the latus rectum, and the length of the latus rectum.

Sol. We have $y^2 + 2y - x + 5 = 0$

$$\text{or } x = y^2 + 2y + 5 \quad (1)$$

$$\therefore (y+1)^2 = (x-4) \quad (2)$$

This is parabola having vertex at $A(4, -1)$ and axis parallel to x -axis having equation $y + 1 = 0$.



Equation of tangent at vertex is $x - 4 = 0$.

Also, coefficient of y^2 in (1) is positive.

So, parabola opens to the right.

Comparing (2) with $(y-k)^2 = 4a(x-h)$, we have $4a = 1$ or $a = \frac{1}{4}$.

Thus, length of latus rectum is 1 unit.

Focus is on the axis at distance a units from the vertex.

$$\therefore \text{Focus, } S \equiv \left(4 + \frac{1}{4}, -1\right) \equiv \left(\frac{17}{4}, -1\right)$$

Also, equation of directrix is $x = 4 - \frac{1}{4}$ or $x = \frac{15}{4}$.

End points of latus rectum lies on either side of focus at distance $2a$ units on the latus rectum line.

Thus, end points of latus rectum are $\left(\frac{17}{4}, -1 \pm \frac{1}{2}\right) \equiv \left(\frac{17}{4}, -\frac{1}{2}\right)$ and $\left(\frac{17}{4}, -\frac{3}{2}\right)$.

ILLUSTRATIONS 5.16

The parametric equation of a parabola is $x = t^2 + 1$, $y = 2t + 1$. Find the equation of directrix

Sol. We have $x = t^2 + 1$, $y = 2t + 1$

Eliminating t , we get

$$x - 1 = \left(\frac{y-1}{2}\right)^2$$

$$\text{or } (y-1)^2 = 4(x-1)$$

Comparing with $(y-k)^2 = 4a(x-h)$, we have vertex $(1, 1)$ and axis parallel to x -axis having equation $y - 1 = 0$.

Also, $4a = 4$. So, $a = 1$

Directrix is parallel to y -axis and lies to the left of the vertex at distance a units from it.

So, equation of directrix is $x = 1 - 1$ or $x = 0$.

ILLUSTRATION 5.17

Find points on the parabola $y^2 - 2y - 4x = 0$ whose focal length is 6.

Sol. We have $y^2 - 2y - 4x = 0$

$$\text{or } (y-1)^2 = 4(x + 1/4) \quad (1)$$

Vertex of the parabola is $(-1/4, 1)$.

So, focus is $S(-1/4 + 1, 1) \equiv S(3/4, 1)$.

Let point $P(x, y)$ be on parabola.

$$SP = \sqrt{(x-3/4)^2 + (y-1)^2} = 6 \quad (\text{Given})$$

$$x^2 - \frac{3}{2}x + \frac{9}{16} + 4x + 1 = 36 \quad (\text{Using (1)})$$

$$x^2 + \frac{5}{2}x + \frac{25}{16} = 36$$

$$\left(x + \frac{5}{4}\right)^2 = 36$$

$$x + \frac{5}{4} = 6 \quad \left(\because x + \frac{5}{4} = -6 \text{ is not possible}\right)$$

$$x = \frac{19}{4}$$

$$(y-1)^2 = 20 \quad (\text{Using (1)})$$

$$y = 1 \pm 2\sqrt{5}$$

Hence, points on the parabola whose focal distance is 6 are

$$\left(\frac{19}{4}, 1 \pm 2\sqrt{5}\right).$$

ILLUSTRATION 5.18

Find the value of p such that the vertex of $y = x^2 + 2px + 13$ is 4 units above the x -axis.

Sol. The given parabola is

$$y = x^2 + 2px + 13$$

$$\text{or } y = (x+p)^2 + 13 - p^2$$

$$\text{or } y - (13 - p^2) = (x+p)^2$$

So, the vertex is $(-p, 13 - p^2)$.

Since the parabola is at a distance of 4 units above the x -axis,

$$13 - p^2 = 4$$

$$\text{or } p^2 = 9$$

$$\text{or } p = \pm 3$$

ILLUSTRATION 5.19

Find equation of parabola which has axis parallel to y -axis and which passes through the points $(0, 2)$, $(-1, 0)$ and $(1, 6)$.

Sol. General equation of parabola having axis parallel to y -axis is

$$y = Ax^2 + Bx + C$$

It passes through the points $(0, 2)$, $(-1, 0)$ and $(1, 6)$.

Then we have,

$$C = 2 \quad (1)$$

$$A - B + C = 0 \quad (2)$$

$$A + B + C = 6 \quad (3)$$

Solving (1), (2) and (3), we get $C = 2$, $A = 1$ and $B = 3$.

Therefore, equation of parabola is $y = x^2 + 3x + 2$.

ILLUSTRATION 5.20

Prove that the focal distance of the point (x, y) on the parabola $x^2 - 8x + 16y = 0$ is $|y + 5|$.

Sol. $x^2 - 8x + 16y = 0$

$$\text{i.e., } (x-4)^2 = -16y + 16$$

$$\text{i.e., } (x-4)^2 = -16(y-1) \quad (1)$$

Therefore, the focus is $(4, -3)$.

The focal distance of any point on a parabola is its distance from the focus.

Therefore, the focal distance of point $P(x, y)$ on the parabola is

$$\begin{aligned} \sqrt{(x-4)^2 + (y+3)^2} &= \sqrt{-16(y-1) + (y+3)^2} \\ &= \sqrt{(y+5)^2} = |y+5| \end{aligned}$$

ILLUSTRATION 5.21

If the focus of a parabola is $(2, 3)$ and its latus rectum is 8, then find the locus of the vertex of the parabola.

Sol. Latus rectum $= 4a = 8$.

$$\therefore a = 2.$$

Now, the vertex lies at distance a units from the focus.

Let the vertex be (x, y) .

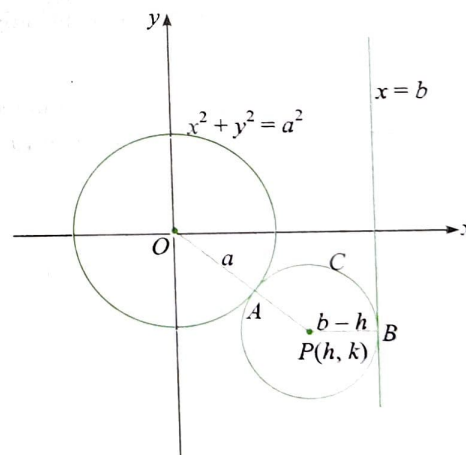
$$\therefore (x-2)^2 + (y-3)^2 = 4,$$

which is the equation of locus of the vertex.

ILLUSTRATION 5.22

Prove that the locus of centre of circle which touches the given circle externally and given line is parabola.

Sol.



Let the given circle be $x^2 + y^2 = a^2$ and given line be $x = b$.

One of the variable circles C having centre $P(h, k)$ touches the given circle at A and line at B .

From the figure, radius of circle C is $(b-h)$.

Also, $OP = OA + AP$

$$\therefore OP = OA + PB$$

$$\Rightarrow \sqrt{h^2 + k^2} = a + (b-h)$$

$$\Rightarrow (a+b)^2 - 2(a+b)h + h^2 = h^2 + k^2$$

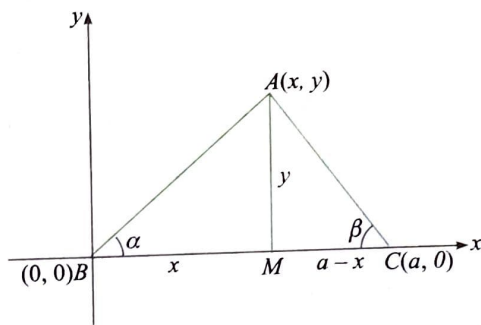
$$\Rightarrow y^2 = (a+b)^2 - 2(a+b)x,$$

which is required equation of locus.

Clearly, this is equation of parabola.

ILLUSTRATION 5.23

In triangle ABC , base BC is fixed. Then prove that the locus of vertex A such that $\tan B + \tan C = \text{Constant}$ is parabola.

Sol.

Let base BC lie on x -axis with B at $(0, 0)$ and C at $(a, 0)$. Here, a is constant.

Now, according to the question,

$$\tan \alpha + \tan \beta = c \text{ (constant)}$$

$$\Rightarrow \frac{y}{x} + \frac{y}{a-x} = c$$

$$\Rightarrow \frac{ay}{x(a-x)} = c$$

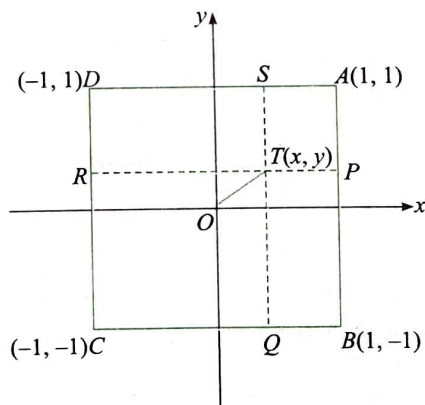
$$\Rightarrow ay = c(ax - x^2), \text{ which is equation of locus of vertex } A.$$

Clearly, this is equation of parabola.

ILLUSTRATION 5.24

Consider a square with vertices at $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$. Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S .

Sol. Let us consider any point $T(x, y)$ inside the square such that its distance from origin is less than or equal to its distance from any of the edges.



$$\therefore OT < TP, TQ, TR, TS$$

Consider $OT < TP$

$$\Rightarrow \sqrt{x^2 + y^2} < 1 - x$$

$$\Rightarrow y^2 \leq -2 \left(x - \frac{1}{2} \right) \quad (1)$$

So, point T lies on or inside the parabola $y^2 = -2(x - 1/2)$, having vertex at $(1/2, 0)$ and which is concave to the left.

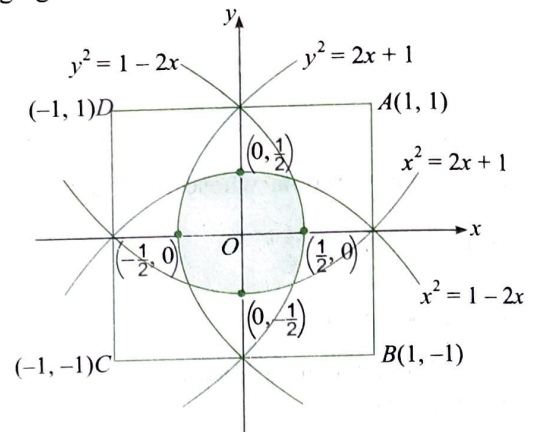
Also, parabola passes through the points $(0, \pm 1)$ on y -axis. Similarly, we have other inequalities as follows:

$$\text{For } OT < TQ, \text{ we get } x^2 \leq 2 \left(y + \frac{1}{2} \right).$$

$$\text{For } OT < TR, \text{ we get } y^2 \leq 2 \left(x + \frac{1}{2} \right).$$

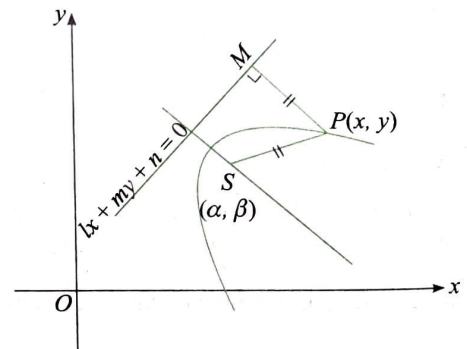
$$\text{For } OT < TS, \text{ we get } x^2 \leq -2 \left(y - \frac{1}{2} \right).$$

Common region of all the inequalities is plotted as shown in the following figure.

**EQUATION OF PARABOLA WITH GIVEN FOCUS AND DIRECTRIX**

Let $S(\alpha, \beta)$ be the focus, and $lx + my + n = 0$ is the equation of the directrix of the parabola.

Let $P(x, y)$ be any point on the parabola.



Then using definition of parabola

$$SP = PM$$

$$\therefore \sqrt{(x - \alpha)^2 + (y - \beta)^2} = \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

$$\Rightarrow (x - \alpha)^2 + (y - \beta)^2 = \frac{(lx + my + n)^2}{l^2 + m^2}$$

$$\Rightarrow m^2x^2 + l^2y^2 - 2lmxy + px + qy + r = 0$$

This is second-degree equation in variables x and y .

We observe that second-degree terms form a perfect square.

$$\text{i.e., } m^2x^2 + l^2y^2 - 2lmxy = (mx - ly)^2$$

Now, consider general second-degree equation,

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

If this equation represents parabola, then we must have
 $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$,
 for which curve does not represent pair of straight lines or point
 circle;
 and $h^2 - ab = 0$,
 for second-degree terms to form perfect square.

ILLUSTRATION 5.25

Find the value of λ if the equation $9x^2 + 4y^2 + 2\lambda xy + 4x - 2y + 3 = 0$ represents a parabola.

Sol. Comparing the given equation with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, we have $a = 9$, $b = 4$, $c = 3$, $h = \lambda$, $g = 2$, and $f = -1$. If the equation $9x^2 + 4y^2 + 2\lambda xy + 4x - 2y + 3 = 0$ represents a parabola, then its second-degree terms must form the perfect square. Therefore,

$$\lambda^2 = 36$$

$$\text{(Using } h^2 - ab = 0 \text{)}$$

$$\text{or } \lambda = \pm 6$$

Also, for these values of λ , $\Delta \neq 0$.

ILLUSTRATION 5.26

Does equation $(5x-5)^2 + (5y+10)^2 = (3x+4y+5)^2$ represent a parabola?

Sol. We have $(5x-5)^2 + (5y+10)^2 = (3x+4y+5)^2$

$$\therefore \sqrt{(x-1)^2 + (y+2)^2} = \frac{|3x+4y+5|}{5}$$

L.H.S. = distance of variable point $P(x, y)$ from fixed point $(1, -2)$

R.H.S. = distance of point $P(x, y)$ from the fixed line $3x + 4y + 5 = 0$

But point $(1, -2)$ satisfies the line $3x + 4y + 5 = 0$.

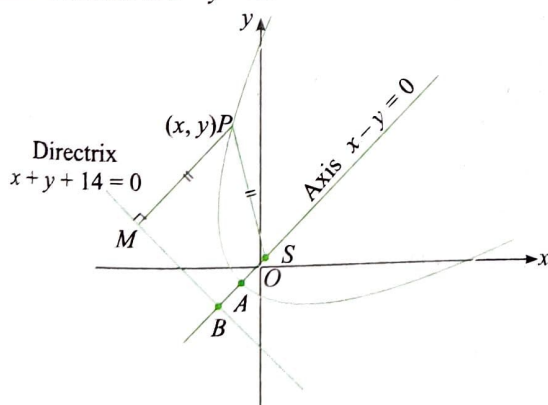
So, equation does not represent a parabola.

ILLUSTRATION 5.27

Find the equation of parabola having focus at $(1, 1)$ and vertex at $(-3, -3)$.

Sol. Focus is at $S(1, 1)$ and vertex is at $A(-3, -3)$.

So, equation of axis is $x - y = 0$.



Let the directrix meet the axis at B .

Since A is midpoint of BS , coordinates of point B are $(-7, -7)$.

Directrix is perpendicular to the axis and passes through the point $B(-7, -7)$.

So, equation of directrix is $x + y + 14 = 0$.

Now, equation of parabola is locus of point $P(x, y)$ which is equidistant from focus and directrix.

$$\therefore \sqrt{(x-1)^2 + (y-1)^2} = \frac{|x+y+14|}{\sqrt{2}}$$

or $x^2 + y^2 - 2xy - 32x - 32y - 192 = 0$, which is required equation of parabola.

ILLUSTRATION 5.28

Find the value of λ if the equation $(x-1)^2 + (y-2)^2 = \lambda(x+y+3)^2$ represents a parabola. Also, find its focus, vertex, the equation of its directrix, the equation of axis, the equation of latus rectum, the length of the latus rectum, and the extremities of the latus rectum.

Sol. We have equation $(x-1)^2 + (y-2)^2 = \lambda(x+y+3)^2$

$$\text{or } \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{2\lambda} \frac{|x+y+3|}{\sqrt{1^2+1^2}}$$

This represents parabola if $\sqrt{2\lambda} = 1$

$$\therefore \lambda = \frac{1}{2}$$

Focus of the parabola is $S(1, 2)$ and directrix is $x + y + 3 = 0$.

Axis passes through focus and it is perpendicular to directrix.

So, equation of axis is $x - y + 1 = 0$.

Axis and directrix meet at $B(-2, -1)$.

Vertex A is midpoint of BS .

$$\therefore \text{Vertex, } A \equiv \left(\frac{1-2}{2}, \frac{2-1}{2} \right) \equiv \left(-\frac{1}{2}, \frac{1}{2} \right)$$

Latus rectum line is parallel to directrix and passes through focus.

So, equation of latus rectum is $x + y - 3 = 0$.

Length of latus rectum = $2 \times$ Distance of focus from directrix

$$= 2 \times \frac{|1+2+3|}{\sqrt{2}} = 6\sqrt{2}$$

Extremities of latus rectum lie at distance $\frac{6\sqrt{2}}{2} = 3\sqrt{2}$ from the

focus on the latus rectum line whose equation is $x + y - 3 = 0$.

Therefore, extremities of latus rectum are

$$(1 \pm 3\sqrt{2} \cos 135^\circ, 2 \pm 3\sqrt{2} \sin 135^\circ)$$

$$\equiv (1 \mp 3, 2 \pm 3)$$

$$\equiv (4, -1) \text{ and } (-2, 5)$$

ILLUSTRATION 5.29

Show that the curve whose parametric coordinates are $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents a parabola.

Sol. From the given relations, we have

$$\frac{x+y}{2} = t^2 + 1, \quad \frac{x-y}{2} = t$$

Eliminating t , we get

$$2(x+y) = (x-y)^2 + 4$$

This is second-degree equation in which second-degree terms form perfect square.

Rewriting the equation, we have

$$x^2 + y^2 - 2xy - 2x - 2y + 4 = 0$$

Comparing with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, we find that

$$abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0.$$

So, given equation represents a parabola.

CONCEPT APPLICATION EXERCISE 5.2

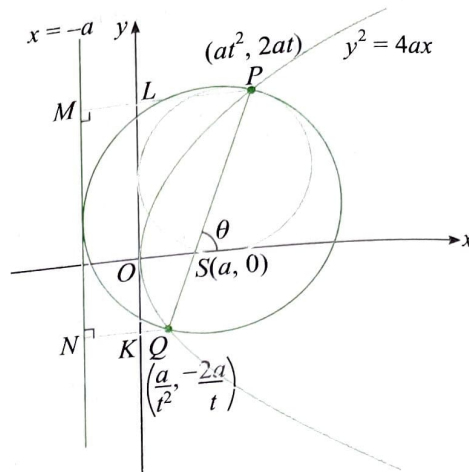
1. If the focus and vertex of a parabola are the points (0, 2) and (0, 4) respectively, then find its equation.
2. Find the equation of parabola whose focus is (0, 1) and the directrix is $x + 2 = 0$. Also, find the vertex of the parabola.
3. Find the vertex, focus and directrix of the parabola $x^2 = 2(2x + y)$.
4. The vertex of a parabola is (2, 2) and the coordinates of extremities of its latus rectum are (-2, 0) and (6, 0). Then find the equation of the parabola.
5. A parabola passes through the points (1, 2), (2, 1), (3, 4) and (4, 3). Find the equation of the axis of parabola.
6. Find the length of the common chord of the parabola $y^2 = 4(x + 3)$ and the circle $x^2 + y^2 + 4x = 0$.
7. The equation of the latus rectum of a parabola is $x + y = 8$ and the equation of the tangent at the vertex is $x + y = 12$. Then find the length of the latus rectum.
8. Find the length of the latus rectum of the parabola whose focus is at (2, 3) and directrix is the line $x - 4y + 3 = 0$.
9. If (a, b) is the midpoint of a chord passing through the vertex of the parabola $y^2 = 4(x + 1)$, then prove that $2(a + 1) = b^2$.
10. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4x$ internally in the ratio 1 : 2 is a parabola. Also, find the vertex of this parabola.
11. Plot the region in first quadrant in which points are nearer to the origin than to the line $x = 3$.
12. Prove that the locus of a point, which moves so that its distance from a fixed line is equal to the length of the tangent drawn from it to a given circle, is a parabola.
13. Prove that the locus of the centre of a circle, which intercepts a chord of given length $2a$ on the axis of x and passes through a given point on the axis of y distant b from the origin, is a parabola.
14. Find the equation of the parabola whose focus is $S(-1, 1)$ and directrix is $4x + 3y - 24 = 0$. Also find its axis, the vertex, the length, and the equation of the latus rectum.
15. The axis of a parabola is along the line $y = x$ and the distance of its vertex and focus from origin are $\sqrt{2}$ and $2\sqrt{2}$, respectively. If vertex and focus both lie in the first quadrant, then find the equation of the parabola.

ANSWERS

1. $x^2 + 8y = 32$
2. $(y - 1)^2 = 4(x + 1)$, Vertex $\equiv (-1, 1)$
3. Vertex $\equiv (2, -2)$, Focus $\equiv (2, -3/2)$, Directrix: $y = -\frac{5}{2}$
4. $(x - 2)^2 = -8(y - 2)$
5. $x - y = 0$
6. 4
7. $8\sqrt{2}$
8. $\frac{14}{\sqrt{17}}$
10. Vertex $\equiv \left(\frac{2}{9}, \frac{8}{9}\right)$
14. $9x^2 + 16y^2 - 24xy + 242x + 94y - 526 = 0$
15. $(x - y)^2 = 8(x + y - 2)$

FOCAL CHORD AND ITS PROPERTIES

The chord of the parabola which passes through its focus is called focal chord. Focus divides focal chord in two segments which are called segments of focal chord. We have following properties of focal chord:



- I. If on the parabola $y^2 = 4ax$, chord joining $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is focal chord then $t_1 t_2 = -1$.

Proof: Chord PQ passes through the focus $S(a, 0)$.

$\therefore$ Slope of PS = Slope of QS

$$\Rightarrow \frac{2at_1 - 0}{at_1^2 - a} = \frac{0 - 2at_2}{a - at_1^2}$$

$$\Rightarrow \frac{2t_1}{t_1^2 - 1} = \frac{2t_2}{t_2^2 - 1}$$

$$\Rightarrow t_1(t_2^2 - 1) = t_2(t_1^2 - 1)$$

$$\Rightarrow t_1 t_2 (t_2 - t_1) + (t_2 - t_1) = 0$$

$$\Rightarrow t_1 t_2 + 1 = 0$$

$$\Rightarrow t_1 t_2 = -1$$

$$\Rightarrow t_2 = -\frac{1}{t_1}$$

Thus, if parameter of point P is t then parameter of point Q is $-\frac{1}{t}$.

Hence, if point $P \equiv (at^2, 2at)$ then point $Q \equiv \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$. Here, P and Q are extremities of focal chord.

- II. In any parabola, semi latus rectum ($2a$) is harmonic mean of segments of focal chord (SP and SQ).

Proof: This is general result. Let us prove this for standard parabola $y^2 = 4ax$.

From the figure, $SP = PM = PL + LM = at^2 + a$

$$SQ = QN = QK + KN = \frac{a}{t^2} + a$$

$$\text{Now, } \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a + at^2} + \frac{1}{a + a/t^2}$$

$$= \frac{1}{a + at^2} + \frac{t^2}{at^2 + a}$$

$$= \frac{1}{a}$$

$$\Rightarrow 2a = \frac{2SP \cdot SQ}{SP + SQ}$$

III. The length of the focal chord through point $P(at^2, 2at)$ on parabola $y^2 = 4ax$ is $a\left(t + \frac{1}{t}\right)^2$.

Proof:

$$PQ = SP + SQ = a + at^2 + a + \frac{a}{t^2} = a\left(t + \frac{1}{t}\right)^2$$

IV. The length of the focal chord of the parabola $y^2 = 4ax$ which makes an angle θ with x positive direction of x -axis is $4a \operatorname{cosec}^2 \theta$.

Proof: Length of focal chord $PQ = a\left(t + \frac{1}{t}\right)^2$

$$\text{Now, slope of } PQ = \frac{2at - 0}{at^2 - a} = \frac{2}{t - \frac{1}{t}} = \tan \theta$$

$$\Rightarrow 2 \cot \theta = t - \frac{1}{t}$$

$$\begin{aligned} \Rightarrow PQ &= a\left(t + \frac{1}{t}\right)^2 = a\left[\left(t - \frac{1}{t}\right)^2 + 4\right] \\ &= a[4 \cot^2 \theta + 4] \\ &= 4a \operatorname{cosec}^2 \theta \end{aligned}$$

From this we can conclude that minimum length of focal chord is $4a$ which is length of latus rectum.

V. In any parabola, circle described on focal length as diameter touches the tangent at vertex.

Proof: This is general result. Let us prove this for standard parabola $y^2 = 4ax$.

From the figure, equation of circle described on SP as diameter is

$$(x - at^2)(x - a) + (y - 2at)(y - 0) = 0$$

To solve it with y -axis, we put $x = 0$

$$\therefore (0 - at^2)(0 - a) + (y - 2at)(y - 0) = 0$$

$$\text{or } y^2 - 2aty + a^2t^2 = 0, \text{ which has equal roots.}$$

Hence, circle touches y -axis.

Also, point of contact is $(0, at)$.

VI. In any parabola, circle described on focal chord as diameter touches the directrix.

Proof: This is general result. Let us prove this for standard parabola $y^2 = 4ax$.

From the figure, equation of circle described on PQ as diameter is

$$(x - at^2)(x - a/t^2) + (y - 2at)(y + 2a/t) = 0$$

To solve this with directrix, we put $x = -a$.

$$\therefore (-a - at^2)(-a - a/t^2) + (y - 2at)(y + 2a/t) = 0$$

$$\text{or } y^2 - 2a\left(t - \frac{1}{t}\right)y + a^2\left(t - \frac{1}{t}\right)^2 = 0,$$

which has equal roots.

Hence, circle touches the directrix.

ILLUSTRATION 5.30

If $(2, -8)$ is an end of a focal chord of the parabola $y^2 = 32x$, then find the other end of the chord.

Sol. We have parabola $y^2 = 32x$.

One end of focal chord is $P(2, -8) \equiv P(8t^2, 16t)$.

$$\therefore t = -1/2$$

Thus, parameter of the other end Q of the focal chord PQ is 2.

Therefore, coordinates of Q are $(8(2)^2, 16(2)) \equiv (32, 32)$.

ILLUSTRATION 5.31

If X is the foot of the directrix on the axis of a parabola. PP' is a double ordinate of the curve and PX meets the curve again in Q . Then prove that $P'Q$ passes through the focus of the parabola.

Sol. Consider parabola $y^2 = 4ax$

$$X \equiv (-a, 0)$$

Let point P be $(at^2, 2at)$.

$$\therefore P' \equiv (at^2, -2at)$$

Equation of line PX is

$$y = \frac{2at - 0}{at^2 - a}(x + a)$$

$$\text{or } (1 + t^2)y = 2t(x + a) \quad (1)$$

Solving this line with given parabola, we get

$$\frac{4t^2(x + a)^2}{(1 + t^2)^2} = 4ax$$

$$\Rightarrow t^2(x + a)^2 = ax(1 + t^2)^2$$

$$\Rightarrow t^2(x^2 + a^2 + 2ax) = a(t^4 + 1 + 2t^2)x$$

$$\Rightarrow t^2x^2 + t^2a^2 = xat^4 + ax$$

$$\Rightarrow t^2x^2 - (a + at^4)x + at^2 = 0$$

$$\Rightarrow xt^2(x - at^2) - a(x - at^2) = 0$$

$$\Rightarrow (x - at^2)(xt^2 - a) = 0$$

$$\Rightarrow x = at^2, a/t^2$$

Putting $x = a/t^2$ into (1), we get

$$(1 + t^2)y = 2at\left(\frac{a}{t^2} + a\right)$$

$$\therefore y = 2a/t$$

Thus, point Q lies at $(a/t^2, 2a/t)$.

Clearly, points Q and P' are extremities of the focal chord.

So, $P'Q$ passes through the focus.

ILLUSTRATION 5.32

Find the length of a focal chord of the parabola $y^2 = 4ax$ which is at a distance b from the vertex.

Sol. In the figure, $OM = b$

$=$ distance of focal chord from vertex.

Let focal chord PQ make an angle θ with positive x -axis.

$$\therefore PQ = 4a \operatorname{cosec}^2 \theta$$

Now, in right angled triangle OMS ,

$$\sin \theta = OM/OS = b/a$$

$$\therefore PQ = 4a(a/b)^2 = 4a^3/b^2$$

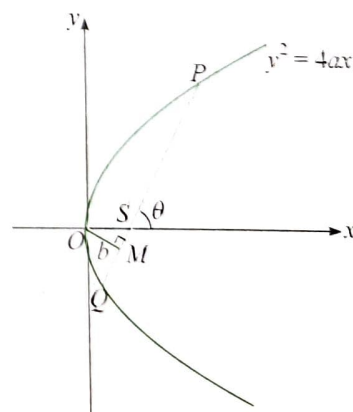


ILLUSTRATION 5.33

If AB is a focal chord of $x^2 - 2x + y - 2 = 0$ whose focus is S and $AS = l_1$, then find BS .

Sol. $x^2 - 2x + y - 2 = 0$

$$\text{or } x^2 - 2x + 1 = 3 - y$$

or $(x-1)^2 = -(y-3)$

The length of its latus rectum is 1 unit.

Since AS , $1/2$, and BS are in HP, we have

$$\frac{1}{2} = \frac{2AS \times BS}{AS + BS}$$

or $BS = \frac{l_1}{(4/l_1 - 1)}$

ILLUSTRATION 5.34

Circles drawn with diameter as extremities of any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line. Find the equation of line.

Sol. $y^2 - 4x - y - 4 = 0$

or $y^2 - y + \frac{1}{4} = 4x + \frac{17}{4}$

or $\left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$

Circle drawn with diameter as extremities of any focal chord of the parabola always touches the directrix of the parabola.

Thus, the circle will touch the line

$$x + \frac{17}{16} = -1$$

or $16x + 33 = 0$

CONCEPT APPLICATION EXERCISE 5.3

1. If t_1 and t_2 are the parameters of the ends of the focal chord $y^2 = 4ax$, then prove that roots of the equation $t_1x^2 + ax + t_2 = 0$ are real.
2. If the line passing through the focus S of the parabola $y = ax^2 + bx + c$ meets the parabola at P and Q and if $SP = 4$ and $SQ = 6$, then find the values of a .
3. If a focal chord of $y^2 = 4ax$ makes an angle $\alpha \in [\pi/4, \pi/2]$ with the positive direction of the x -axis, then find the maximum length of this focal chord.
4. If the length of focal chord of $y^2 = 4ax$ is ℓ , then find the angle between the axis of the parabola and the focal chord.
5. If the length of focal chord PQ is ' ℓ ', and ' p ' is the perpendicular distance of PQ from the vertex of the parabola, then prove that $\ell \propto 1/p^2$.
6. Circle drawn having its diameter equal to the focal distance of any point lying on the parabola $x^2 - 4x + 6y + 10 = 0$ will touch a fixed line. Find the equation of line.
7. A circle is drawn to pass through the extremities of the latus rectum of the parabola $y^2 = 8x$. It is given that this circle also touches the directrix of the parabola. Find the radius of this circle.

ANSWERS

2. $\pm \frac{5}{48}$

3. $8a$ units

4. $\pm \sin^{-1} \sqrt{\frac{4a}{l}}$

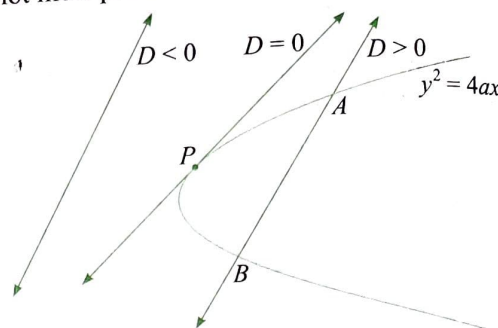
6. $y + 1 = 0$

7. 4

TANGENT TO PARABOLA

INTERSECTION OF A LINE AND A PARABOLA

A line may meet the parabola in one point or two distinct points or it may not meet parabola at all.



Consider the standard equation of parabola

$$y^2 = 4ax \quad (1)$$

and the line having equation

$$y = mx + c \quad (2)$$

Solving line and parabola, we get

$$(mx + c)^2 = 4ax \quad (\text{eliminating } y)$$

or $m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad (3)$

Discriminant of this equation is

$$D = 4(mc - 2a)^2 - 4m^2c^2$$

or $D = 4(4a^2 - 4amc)$

Now, this equation has either two distinct real roots or two equal roots or has non-real roots.

If line meets the parabola in two distinct points (A and B) then equation (3) has two distinct real roots.

$$\therefore D > 0 \text{ or } c < a/m$$

If line meets the parabola in one point (P), i.e., touches the parabola then equation (3) has two equal roots.

$$\therefore D = 0 \text{ or } c = a/m$$

If line doesn't meet the parabola then equation (3) has imaginary roots.

$$\therefore D < 0 \text{ or } c > a/m$$

EQUATION OF TANGENT TO STANDARD PARABOLA

We have already learned that if line $y = mx + c$ meet the parabola

$y^2 = 4ax$ in one point, i.e., touches the parabola then $c = \frac{a}{m}$.

In that case, equation of line is

$$y = mx + \frac{a}{m} \quad (1)$$

which is tangent to parabola $y^2 = 4ax$.

Equation (1) is equation of tangent to parabola $y^2 = 4ax$ having slope m .

Here, m can take any real value except zero.

Solving (1) with the parabola, we get

$$\left(mx + \frac{a}{m}\right)^2 = 4ax$$

or $\left(mx - \frac{a}{m}\right)^2 = 0$

or $x = \frac{a}{m^2}$

So, from (1), $y = m\left(\frac{a}{m^2}\right) + \frac{a}{m} = \frac{2a}{m}$

Thus, point of contact on the parabola is $P\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.
Also, $P \equiv (at^2, 2at)$ for some t .

$$\therefore m = \frac{1}{t}$$

So, equation of (1) becomes

$$ty = x + at^2 \quad (2)$$

This is the equation of tangent to parabola $y^2 = 4ax$ at $(at^2, 2at)$ or at point whose parameter is t .

From (2), we have

$$2aty = 2a(x + at^2)$$

$$\text{or } yy_1 = 2a(x + x_1)$$

$$\text{or } yy_1 - 2a(x + x_1) = 0$$

$$\text{or } T = 0, \text{ where } T = yy_1 - 2a(x + x_1)$$

$$\text{Here, } (at^2, 2at) \equiv (x_1, y_1)$$

Thus, for parabola $y^2 = 4ax$, equation of tangent at point (x_1, y_1) on it is obtained by replacing y^2 by yy_1 and $2x$ by $x + x_1$. With the similar replacement mechanism, we obtained the equation of tangent to circle at a point (x_1, y_1) on it.

If the equation of parabola is $x^2 = 4ay$ then equation of tangent at point (x_1, y_1) on it is $xx_1 - 2a(y + y_1) = 0$ or $T = 0$.

Here, x^2 is replaced by xx_1 and $2y$ is replaced by $y + y_1$.

All above forms of equation of tangent can be obtained by calculus method also in which we differentiate the equation of parabola to get the slope of tangent to parabola at any point on it.

Differentiating equation $y^2 = 4ax$, w.r.t. x , we get

$$2y \frac{dy}{dx} = 4a \text{ or } \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \text{Slope of tangent to parabola at } (x_1, y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2a}{y_1}$$

So, equation of tangent to parabola at (x_1, y_1) is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\text{or } yy_1 - y_1^2 = 2ax - 2ax_1$$

$$\text{or } yy_1 - 4ax_1 = 2ax - 2ax_1$$

(as point (x_1, y_1) on the curve, so $y_1^2 = 4ax_1$)

$$\text{or } yy_1 = 2a(x + x_1)$$

Putting $x_1 = at^2$ and $y_1 = 2at$, we get

$$2aty = 2a(x + at^2)$$

$$\text{or } ty = x + at^2$$

$$\text{or } y = \frac{1}{t}x + at$$

$$\text{or } y = mx + \frac{a}{m} \quad \left(\text{Putting } \frac{1}{t} = m = \text{slope}\right)$$

The equations of tangents to all standard parabolas in terms of parameter of the point of contact and the slope of the tangents are tabulated as shown below.

Equation of Parabola	Point on the parabola	Equation of tangent
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$

$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
	$(2am, am^2)$	$y = mx - am^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$
	$(-2am, -am^2)$	$y = mx + am^2$

Equation of tangent to parabola $(y - k)^2 = 4a(x - h)$ having slope m is $y - k = m(x - h) + \frac{a}{m}$.

ILLUSTRATION 5.35

Find the equation of the tangent to the parabola $y^2 = 8x$ having slope 2 and also find the point of contact.

Sol. The equation of the tangent to $y^2 = 4ax$ having slope m is

$$y = mx + \frac{a}{m}$$

Hence, for the given parabola, the equation of the tangent is

$$y = 2x + \frac{2}{2} \text{ or } y = 2x + 1$$

and the point of contact is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{2}{2^2}, \frac{2(2)}{2}\right) = \left(\frac{1}{2}, 2\right)$$

ILLUSTRATION 5.36

A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Then find one of the points of contact.

Sol. The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is

$$ty = x + at^2$$

Here, $a = 2$. So the equation of the tangent at $(2t^2, 4t)$ to the parabola $y^2 = 8x$ is

$$ty = x + 2t^2 \quad (1)$$

The slope of (1) is $1/t$ and that of the given line is 3. Therefore,

$$\frac{1}{t} - 3 = \pm \tan 45^\circ = \pm 1$$

$$1 + \frac{1}{t} \times 3$$

$$\text{i.e., } t = -\frac{1}{2} \text{ or } 2$$

For $t = -1/2$, the tangent is

$$\left(-\frac{1}{2}\right)y = x + 2\left(\frac{1}{4}\right)$$

$$\text{i.e., } 2x + y + 1 = 0$$

and the point of contact is $(1/2, -2)$.

For $t = 2$, the tangent is $2y = x + 8$ and the point of contact is $(8, 8)$.

ILLUSTRATION 5.37

Show that $x \cos \alpha + y \sin \alpha = p$ touches the parabola $y^2 = 4ax$ if $p \cos \alpha + a \sin^2 \alpha = 0$ and that the point of contact is $(a \tan^2 \alpha, -2a \tan \alpha)$.

Sol. The given line is

$$x \cos \alpha + y \sin \alpha = p$$

or $y = -x \cot \alpha + p \operatorname{cosec} \alpha$

$\therefore m = -\cot \alpha$ and $c = p \operatorname{cosec} \alpha$

Since the given line touches the parabola, we have

$$c = \frac{a}{m}$$

or $cm = a$

or $(p \operatorname{cosec} \alpha)(-\cot \alpha) = a$

or $p \cos \alpha + a \sin^2 \alpha = 0$

The point of contact is

$$\left(\frac{a}{\cot^2 \alpha}, -\frac{2a}{\cot \alpha} \right) = (a \tan^2 \alpha, -2a \tan \alpha)$$

ILLUSTRATION 5.38

Parabola $y^2 = 4x$ and the circle having its centre at $(6, 5)$ intersect at right angle. Then find the possible points of intersection of these curves.

Sol. Since circle intersects parabola orthogonally, the tangent to parabola at its point of intersection with circle must be normal to the circle.

Normal to the circle passes through its centre.

Let the point of intersection of circle and parabola be $P(t^2, 2t)$.

Equation of tangent at P to the parabola is $yt = x + t^2$.

This is normal to circle.

So, it must pass through centre of the circle $(6, 5)$.

$$\therefore t^2 - 5t + 6 = 0$$

$$\therefore t = 2, 3$$

So, possible points on the parabola are $(4, 4)$ and $(9, 6)$.

ILLUSTRATION 5.39

Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point $(2, 5)$.

Sol. The equation of the given parabola is $y^2 = 12x$ (1)

Comparing with $y^2 = 4ax$, we get $a = 3$.

Therefore, the equation of the tangent from $(2, 5)$ is

$$y = mx + \frac{3}{m} \quad (2)$$

Tangent passes through the point $(2, 5)$. Then,

$$5 = 2m + \frac{3}{m}$$

or $2m^2 - 5m + 3 = 0$

or $m = 1, \frac{3}{2}$

Therefore, from (2), the equations of the required tangents are

$$y = x + 3 \text{ and } 2y = 3x + 4$$

ILLUSTRATION 5.40

The tangents to the parabola $y^2 = 4ax$ at the vertex V and any point P meet at Q . If S is the focus, then prove that SP , SQ , and SV are in GP.

Sol. Let the parabola be $y^2 = 4ax$.

Q is the intersection of the line $x = 0$ and the tangent at point $P(at^2, 2at)$, $ty = x + at^2$.

Solving these, we get $Q = (0, at)$. Also, $S = (a, 0)$.

Now, focal length

$$SP = a + at^2$$

$$SQ^2 = a^2 + a^2 t^2 = a^2(t^2 + 1)$$

and $SV = a$

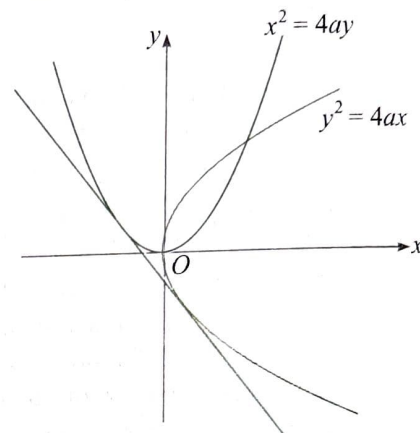
$$\therefore SQ^2 = SP \times SV$$

Therefore, SP , SQ , and SV are in GP.

ILLUSTRATION 5.41

Find the equation of common tangent of $y^2 = 4ax$ and $x^2 = 4by$.

Sol. To find the equation of common tangent to two curves, we first consider the standard equation of tangent to one of the curves and then use the condition of tangency for the other curve.



Equation of tangent to $y^2 = 4ax$ having slope m is

$$y = mx + \frac{a}{m} \quad (1)$$

Solving this line with the parabola $x^2 = 4by$, we have

$$x^2 = 4b \left(mx + \frac{a}{m} \right)$$

or $x^2 - 4bmx - \frac{4ab}{m} = 0$

If line (1) touches the parabola $x^2 = 4ay$, then above equation has equal roots.

Thus, discriminant is zero.

or $16b^2 m^2 + 16 \frac{ab}{m} = 0$

$$\Rightarrow m^3 = -\frac{a}{b}$$

$$\Rightarrow m = -\left(\frac{a}{b} \right)^{\frac{1}{3}}$$

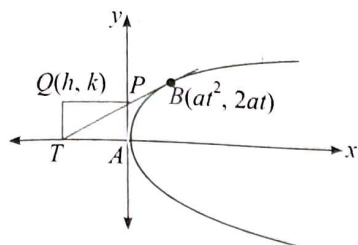
Thus, equation of common tangent is $y = -\left(\frac{a}{b} \right)^{\frac{1}{3}} x - \left(\frac{b}{a} \right)^{\frac{1}{3}} a$.

ILLUSTRATION 5.42

If a tangent to the parabola $y^2 = 4ax$ meets the x -axis at T and intersects the tangent at vertex A at P , and rectangle $TAPQ$ is completed, then find the locus of point Q .

Sol. The tangent at any point $B(at^2, 2at)$ to the parabola is $ty = x + at^2$ (1)

Since the tangent at vertex $A(0, 0)$ is the y -axis, T and P are $(-at^2, 0)$ and $(0, at)$, respectively.



If Q is (h, k) , then $h = -at^2$ and $k = at$.

Eliminating t , we get $k^2 + ah = 0$.

Hence, the locus of Q is $y^2 + ax = 0$, which is a parabola.

ILLUSTRATION 5.43

Two tangents are drawn from the point $(-2, -1)$ to the parabola $y^2 = 4x$. If α is the angle between these tangents, then find the value of $\tan \alpha$.

Sol. Here $a = 1$. Any tangent having slope m is $y = mx + \frac{1}{m}$

It passes through $(-2, -1)$. Therefore,

$$2m^2 - m - 1 = 0$$

$$\text{or } m = 1, -\frac{1}{2}$$

$$\text{So, } \tan \alpha = \frac{1 + (1/2)}{1 - (1/2)} = 3$$

ILLUSTRATION 5.44

If two tangents drawn from the point (α, β) to the parabola $y^2 = 4x$ are such that the slope of one tangent is double of the other, then prove that $\alpha = \frac{2}{9}\beta^2$.

Sol. Tangent to the parabola $y^2 = 4x$ having slope m is $y = mx + \frac{1}{m}$

It passes through (α, β) . Therefore,

$$\beta = m\alpha + \frac{1}{m}$$

$$\text{or } \alpha m^2 - \beta m + 1 = 0$$

According to the question, it has roots m_1 and $2m_1$. Now,

$$m_1 + 2m_1 = \frac{\beta}{\alpha} \text{ and } m_1 \cdot 2m_1 = \frac{1}{\alpha}$$

$$\text{or } 2\left(\frac{\beta}{3\alpha}\right)^2 = \frac{1}{\alpha}$$

$$\text{or } \alpha = \frac{2}{9}\beta^2$$

ILLUSTRATION 5.45

If the tangent at the point $P(2, 4)$ to the parabola $y^2 = 8x$ meets the parabola $y^2 = 8x + 5$ at Q and R , then find the midpoint of QR .

Sol. The equation of the tangent to parabola $y^2 = 8x$ at $P(2, 4)$ is $4y = 4(x + 2)$
 or $x - y + 2 = 0$ (1)
 This chord meets the parabola $y^2 = 8x + 5$ at Q and R .

Let (x_1, y_1) be the midpoint of chord QR .

Then equation of QR is

$$yy_1 - 4(x + x_1) - 5 = y_1^2 - 8x_1 - 5 \quad (\text{Using } T = S_1)$$

$$\text{or } 4x - yy_1 - 4x_1 + y_1^2 = 0 \quad (2)$$

Clearly, (1) and (2) represent the same line.

$$\text{So, } \frac{4}{1} = \frac{-y_1}{-1} = \frac{-4x_1 + y_1^2}{2}$$

$$\Rightarrow y_1 = 4 \text{ and } 8 = -4x_1 + y_1^2$$

$$\Rightarrow y_1 = 4 \text{ and } x_1 = 2$$

ILLUSTRATION 5.46

Find the locus of the foot of the perpendiculars drawn from the vertex on a variable tangent to the parabola $y^2 = 4ax$.

Sol. Let foot of the perpendicular from vertex on tangent be $M(h, k)$.

$$\text{Slope of } OM = \frac{k}{h}$$

$$\therefore \text{Slope of tangent} = -\frac{h}{k}$$

Thus, equation of tangent is

$$y - k = -\frac{h}{k}(x - h)$$

$$\text{or } y = -\frac{h}{k}x + \frac{h^2 + k^2}{k}$$

Comparing with $y = mx + \frac{a}{m}$, we have

$$m = -\frac{h}{k} \text{ and } \frac{a}{m} = \frac{h^2 + k^2}{k}$$

$$\therefore -\frac{ak}{h} = \frac{h^2 + k^2}{k} \quad (\text{eliminating } m)$$

So, $x(x^2 + y^2) + ay^2 = 0$ is the required locus.

EQUATION OF TANGENT TO ANY PARABOLA

To find the equation of tangent to standard parabola, we use standard equation of tangent. But if the equation of parabola is not in standard form, we use calculus method to find the equation of tangent. In this method, we differentiate the equation of parabola to find the slope of tangent or point of contact on the parabola.

ILLUSTRATION 5.47

Find the equation of the tangent to the parabola $y = x^2 - 2x + 3$ at point $(2, 3)$.

Sol. Since the equation of the parabola is not in the standard form, we use calculus method to find the equation of tangent.

$$y = x^2 - 2x + 3$$

Differentiating w.r.t. x , we have

$$\frac{dy}{dx} = 2x - 2$$

We want to find the tangent at point $(2, 3)$. Then,

$$\left(\frac{dy}{dx}\right)_{(2,3)} = 2(2) - 2 = 2$$

Hence, using point-slope form, the equation of tangent is
 $y - 3 = 2(x - 2)$ or $y = 2x - 1$

ILLUSTRATION 5.48

Find the equation of the tangent to the parabola $x = y^2 + 3y + 2$ having slope 1.

Sol. $x = y^2 + 3y + 2$

Differentiating both sides w.r.t. x , we have

$$1 = 2y \frac{dy}{dx} + 3 \frac{dy}{dx}$$

or $\frac{dy}{dx} = \frac{1}{2y + 3}$

Now, the slope of the tangent is 1. Therefore,

or $\frac{dy}{dx} = \frac{1}{2y + 3} = 1$

or $y = -1$

which is the ordinate of the point on the curve where the slope of the tangent is 1.

Putting $y = -1$ in the equation of parabola, we get $x = 0$.

Hence, using point-slope form, we have

$$y - (-1) = 1(x - 0)$$

or $x - y - 1 = 0$

ILLUSTRATION 5.49

Find the equations of tangents drawn to the parabola $y = x^2 - 3x + 2$ from the point $(1, -1)$.

Sol. Tangents are drawn to the parabola from the point $(1, -1)$.

Now, the equation of line from $(1, -1)$ having slope m is

$$y - (-1) = m(x - 1)$$

or $y = mx - m - 1$

Since this line touches the parabola, when we solve line and parabola, the resulting quadratic will have equal roots.

Solving, we have

$$mx - m - 1 = x^2 - 3x + 2$$

or $x^2 - (3 + m)x + 3 + m = 0$

This equation has equal roots, i.e.,

$$(m + 3)^2 - 4(m + 3) = 0$$

i.e., $m = -3$ or $m = 1$

Hence, the equations of tangents are

$$y + 1 = -3(x - 1) \text{ and } y + 1 = x - 1$$

or $3x + y - 2 = 0$ and $x - y - 2 = 0$

ILLUSTRATION 5.50

Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

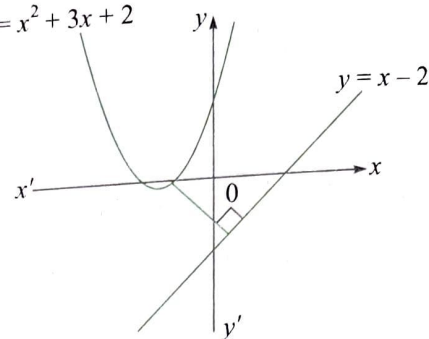
Sol. Let $P(x_1, y_1)$ be the point closest to the line $y = x - 2$. Then

$$\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \text{Slope of given line}$$

$$2x_1 + 3 = 1$$

or $x_1 = -1$ and $y_1 = 0$

$$y = x^2 + 3x + 2$$



Hence, point $(-1, 0)$ is the closest and its perpendicular distance from the line $y = x - 2$ will be the shortest distance. Therefore,

$$\text{Shortest distance} = \frac{3}{\sqrt{2}}$$

ILLUSTRATION 5.51

If the lines L_1 and L_2 are tangents to $4x^2 - 4x - 24y + 49 = 0$ and are normals for $x^2 + y^2 = 72$, then find the slopes of L_1 and L_2 .

Sol. As L_1 and L_2 are normals to $x^2 + y^2 = 72$, they must be of the form $y = mx$.

Solving it with parabola $4x^2 - 4x - 24y + 49 = 0$, we have

$$4x^2 - 4x - 24mx + 49 = 0$$

or $4x^2 - 4x(1 + 6m) + 49 = 0$

Since the lines touch the parabola,

$$D = 0$$

or $16(6m + 1)^2 - 16 \times 49 = 0$

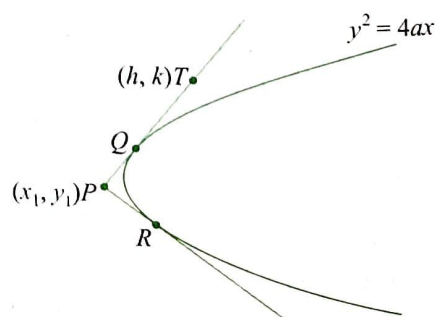
or $6m + 1 = \pm 7$

i.e., $m = 1$ or $-\frac{4}{3}$

EQUATION OF PAIR OF TANGENTS

Consider parabola $y^2 = 4ax$.

To get the equation of pair of tangents, we find the equation of locus of point $T(h, k)$ on the pair of tangents PQ and PR drawn from point $P(x_1, y_1)$ to the parabola.



Equation of tangent PT is

$$y - y_1 = \frac{k - y_1}{h - x_1} (x - x_1)$$

$$\text{or } y = \left(\frac{k - y_1}{h - x_1} \right) x + \left(\frac{hy_1 - kx_1}{h - x_1} \right) \quad (1)$$

Recall that if the line $y = mx + c$ touches the parabola $y^2 = 4ax$,

$$\text{then } c = \frac{a}{m} \text{ or } a = cm$$

So, from (1), we have

$$\left(\frac{hy_1 - kx_1}{h - x_1} \right) \left(\frac{k - y_1}{h - x_1} \right) = a$$

$$\Rightarrow (k - y_1)(hy_1 - kx_1) = a(h - x_1)^2$$

Thus, locus of (h, k) is

$$(y - y_1)(xy_1 - x_1y) = a(x - x_1)^2$$

$$\text{or } (y^2 - 4ax)(y_1^2 - 4ax_1) = (yy_1 - 2a(x + x_1))^2 \quad (2)$$

This is the equation of pair of tangents to parabola $y^2 = 4ax$ from point $P(x_1, y_1)$.

This equation can be expressed as

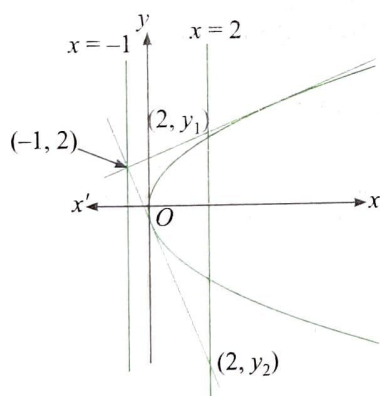
$$SS_1 = T^2$$

where, $S = y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$

ILLUSTRATION 5.52

Tangents are drawn from the point $(-1, 2)$ on the parabola $y^2 = 4x$. Find the length that these tangents will intercept on the line $x = 2$.

Sol.



The equation of the pair of tangents from point (x_1, y_1) is

$$SS_1 = T^2$$

$$\text{or } (y^2 - 4x)(y_1^2 - 4x_1) = (yy_1 - 2(x + x_1))^2$$

Then equation of the pair of tangents from $(-1, 2)$ is

$$(y^2 - 4x)(4 + 4) = [2y - 2(x - 1)]^2 = 4(y - x + 1)^2$$

$$\text{or } 2(y^2 - 4x) = (y - x + 1)^2$$

Solving with the line $x = 2$, we get

$$2(y^2 - 8) = (y - 1)^2$$

$$\text{or } y^2 + 2y - 17 = 0$$

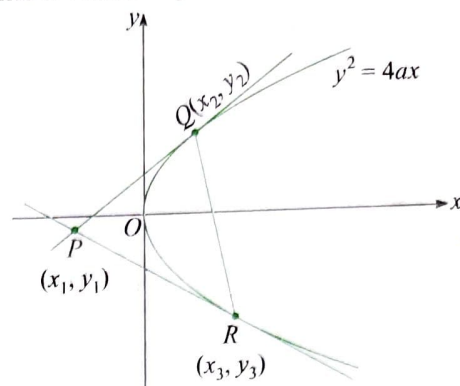
where $y_1 + y_2 = -2$ and $y_1 y_2 = -17$

$$\text{Now, } |y_1 - y_2|^2 = (y_1 + y_2)^2 - 4y_1 y_2 = 4 - 4(-17) = 72$$

$$\therefore |y_1 - y_2| = \sqrt{72} = 6\sqrt{2}$$

EQUATION OF CHORD OF CONTACT

Let tangents drawn from point $P(x_1, y_1)$ to the parabola touch the parabola at points $Q(x_2, y_2)$ and $R(x_3, y_3)$. Then line QR is called the chord of contact. Consider parabola $y^2 = 4ax$.



Equation of the tangent PQ is

$$yy_2 = 2a(x + x_2) \quad (1)$$

and equation of the tangent

$$PR \text{ is } yy_3 = 2a(x + x_3) \quad (2)$$

Since (1) and (2) pass through $P(x_1, y_1)$,

$$y_1 y_2 = 2a(x_1 + x_2) \quad (3)$$

$$\text{and } y_1 y_3 = 2a(x_1 + x_3) \quad (4)$$

Thus, points Q and R lie on the line

$$yy_1 = 2a(x + x_1) \quad (5)$$

$$\text{or } yy_1 - 2a(x + x_1) = 0$$

$$\text{or } T = 0, \text{ where } T = yy_1 - 2a(x + x_1).$$

Here, if point P lies on the curve then equation (5) represents the tangent at point P . In that case, points P , Q and R merge.

Similarly, for parabola $x^2 = 4ay$, chord of contact w.r.t. point $P(x_1, y_1)$ is $xx_1 = 2a(y + y_1)$.

ILLUSTRATION 5.53

Tangents are drawn to $y^2 = 4ax$ at point where the line $lx + my + n = 0$ meets this parabola. Find the point of intersection of these tangents.

Sol. Given line is

$$lx + my + n = 0 \quad (1)$$

Let this line intersect parabola $y^2 = 4ax$ at points P and Q .

Now, tangents drawn to parabola at points P and Q , intersect at $R(h, k)$.

So, PQ is chord of contact of parabola w.r.t. point $R(h, k)$.

So, equation of line PQ is

$$ky = 2a(x + h)$$

$$\text{or } 2ax - ky + 2ah = 0 \quad (2)$$

Equations (1) and (2) represent the same straight line

$$\therefore \frac{2a}{l} = \frac{-k}{m} = \frac{2ah}{n}$$

$$\Rightarrow h = \frac{n}{l}, k = -\frac{2am}{l}$$

So, required point of intersection is $\left(\frac{n}{l}, -\frac{2am}{l} \right)$.

ILLUSTRATION 5.54

If the chord of contact of tangents from a point P to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$, then find the locus of P .

Sol. Chord of contact of parabola $y^2 = 4ax$ w.r.t. point $P(x_1, y_1)$ is

$$y_1 = 2a(x + x_1).$$

Solving this line with parabola $x^2 = 4by$, we get

$$x^2 = 4b \times \frac{2a}{y_1} (x + x_1)$$

$$\text{or } y_1 x^2 - 8abx - 8abx_1 = 0$$

Since the line touches the parabola, this equation will have equal roots.

$$\therefore \text{Discriminant, } D = 0$$

$$\Rightarrow 64a^2b^2 + 32abx_1y_1 = 0$$

$$\Rightarrow x_1y_1 = -2ab$$

$$\Rightarrow xy = -2ab, \text{ which is the required locus.}$$

ILLUSTRATION 5.55

From a variable point on the tangent at the vertex of a parabola $y^2 = 4ax$, a perpendicular is drawn to its chord of contact. Show that these variable perpendicular lines pass through a fixed point on the axis of the parabola.

Sol. Equation of chord of contact to parabola $y^2 = 4ax$ w.r.t. point $P(0, \lambda)$ is

$$\lambda y = 2ax$$

$$\text{or } 2ax - \lambda y = 0 \quad (1)$$

Equation of line perpendicular to this line is $\lambda x + 2ay = c$.

If it passes through $P(0, \lambda)$ then $c = 2a\lambda$.

Therefore, equation of line perpendicular to (1) through point P is

$$\lambda x + 2ay = 2a\lambda$$

$$\text{or } 2ay + \lambda(x - 2a) = 0$$

This equation represents the equation of family of straight lines concurrent at point of intersection of lines $y = 0$ and $x - 2a = 0$.

So, lines are concurrent at $(2a, 0)$.

CONCEPT APPLICATION EXERCISE 5.4

- Find the point on the curve $y^2 = ax$ the tangent at which makes an angle of 45° with the x -axis.
- Find the equation of the straight lines touching both $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$.
- Find the angle at which the parabolas $y^2 = 4x$ and $x^2 = 32y$ intersect.
- If line $y = 3x + c$ touches the parabola $y^2 = 12x$ at point P , then find the equation of tangent at point Q , where PQ is focal chord.
- If the line $x + y = a$ touches the parabola $y = x - x^2$, then find the value of a .
- Find the slopes of the tangents to the parabola $y^2 = 8x$ which are normal to the circle $x^2 + y^2 + 6x + 8y - 24 = 0$.
- Find the equation of the tangent to the parabola $9x^2 + 12x + 18y - 14 = 0$ which passes through the point $(0, 1)$.
- Find the locus of the point from which the two tangents drawn to the parabola $y^2 = 4ax$ are such that the slope of one is thrice that of the other.
- From an external point P , a pair of tangents is drawn to the parabola $y^2 = 4x$. If θ_1 and θ_2 are the inclinations of these tangents with the x -axis such that $\theta_1 + \theta_2 = \pi/4$, then find the locus of P .

- Show that the common tangents to the parabola $y^2 = 4x$ and the circle $x^2 + y^2 + 2x = 0$ form an equilateral triangle.
- TP and TQ are tangents to the parabola, $y^2 = 4ax$ at P and Q . If the chord PQ passes through the fixed point $(-a, b)$, then find the locus of T .
- Tangent is drawn to parabola $y^2 - 2y - 4x + 5 = 0$ at a point P which cuts the directrix at the point Q . A point R is such that it divides QP externally in the ratio $1/2 : 1$. Find the locus of point R .
- If the distance of the point $(h, 2)$ from its chord of contact w.r.t. parabola $y^2 = 4x$ is 4, then find the value of h .

ANSWERS

- $(a/4, a/2)$
- $x \pm y + 2a = 0$
- $\tan^{-1}\left(\frac{3}{5}\right)$
- $x + 3y + 27 = 0$
- $a = 1$
- $\frac{2 \pm \sqrt{10}}{3}$
- $4x + 3y - 3 = 0$ and $y - 1 = 0$
- $3y^2 = 16ax$
- $y = x - 1$
- $by = 2a(x - a)$
- $(x + 1)(y - 1)^2 + 4 = 0$
- $1 - 2\sqrt{2}$

PROPERTIES OF TANGENT

We have some interesting properties of tangents of parabola. These properties are discussed as follows:

- Let us find the point of intersection of tangents at any two points on the parabola $y^2 = 4ax$.

Let two points on the parabola be $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.

Equation of tangent at P is

$$t_1y = x + at_1^2 \quad (1)$$

Equation of tangent at Q is

$$t_2y = x + at_2^2 \quad (2)$$

Solving (1) and (2), we get

$$x = at_1t_2, y = a(t_1 + t_2)$$

Thus, abscissa and ordinate of point of intersection of tangents at P and Q are, respectively, G.M. and A.M. of abscissas and ordinates of P and Q .

These coordinates are useful in problems about locus of point of intersection of tangents under different conditions.

- In any parabola, foot of perpendicular from focus upon any tangent lies on the tangent at vertex.

Proof: This is general result. Let us prove this for standard parabola $y^2 = 4ax$.

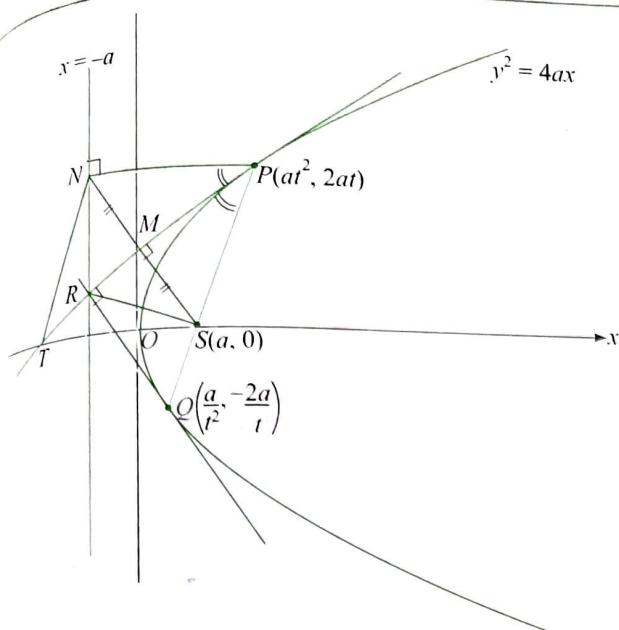
Equation of tangent at point P is $ty = x + at^2$.

It meets y -axis at $M(0, at)$.

$$\text{Now, slope of } SM = \frac{at - 0}{0 - a} = -t$$

$$\text{Also, slope of tangent } PM \text{ is } \frac{1}{t}.$$

Hence, SM is perpendicular to PM .



In other words, locus of feet of perpendicular from focus upon any tangent to parabola is tangent at vertex.

- III. In any parabola, image of focus in any tangent lies on the directrix.

Proof: This is general result. Let us prove this for standard parabola $y^2 = 4ax$.

In the figure, coordinates of point N are $(-a, 2at)$.

$$\text{Slope of } MN = \frac{2at - at}{-a - 0} = -t = \text{slope of } SM$$

Therefore, points S , M and N are collinear.

Also, from the definition of the parabola, $SP = PN$.

Thus, triangle PSN is isosceles in which $SM \perp PM$ and so $SN \perp PM$.

$$\therefore SM = MN$$

So, image of focus S in the tangent at point P is N .

- IV. In any parabola, tangent at any point P on it bisects the angle between the focal chord through P and the perpendicular from P to the directrix.

Proof: This is general result. Let us prove this for standard parabola $y^2 = 4ax$.

In the figure, clearly, tangent PM is the angle bisector of $\angle SPM$.

- V. In any parabola, length of tangent between the point of contact on the curve and point where it meets the directrix subtends right angle at focus

Proof: This is general result. Let us prove this for standard parabola $y^2 = 4ax$.

Equation of tangent at point P is $ty = x + at^2$.

It meets directrix $x = -a$ at $R\left(-a, \frac{at^2 - a}{t}\right)$.

$$\text{Now, slope of } SP, m_{SP} = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1}$$

$$\text{Slope of } SR, m_{SR} = \frac{\frac{at^2 - a}{t} - 0}{-a - a} = \frac{1 - t^2}{2t}$$

$$\text{Clearly, } m_{SP} \times m_{SR} = -1.$$

Hence, SP is perpendicular to SR .

- VI. Tangents at extremities of focal chord are perpendicular and intersect on directrix.

Proof: Parameter of point P is t and slope of tangent at point P is $\frac{1}{t}$.

PQ is focal chord.

So, parameter of point Q is $-\frac{1}{t}$ (by property of focal chord).

$\therefore$ Slope of tangent at point $Q = -t$

Thus, tangents at point P and Q are perpendicular.

Also, coordinates of point of intersection of tangents to parabola $y^2 = 4ax$ whose parameters are t_1 and t_2 are $(at_1t_2, a(t_1 + t_2))$.

Therefore, coordinates of point R are $\left(a\frac{1}{t}(-t), a\left(t - \frac{1}{t}\right)\right) \equiv \left(-a, a\left(t - \frac{1}{t}\right)\right)$.

Thus, tangents intersect on the directrix.

In other words, locus of point of intersection of tangents at the extremities of the focal chord of the parabola is directrix.

Or

Chord of contact w.r.t. any point on the directrix of the parabola is focal chord.

Note:

It should be noted that tangent at P meets x -axis at $T(-at^2, 0)$. Since $SP = PN = a + at^2$ and $ST = a + at^2$, quadrilateral $TSPN$ is rhombus. So, M is midpoint of TP .

ILLUSTRATION 5.56

Find the points of contact Q and R of tangents from the point $P(2, 3)$ on the parabola $y^2 = 4x$.

Sol. Let points Q and R be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$.

Point of intersection of tangent at Q and R is $(t_1t_2, (t_1 + t_2)) \equiv (2, 3)$.

$$\therefore t_1t_2 = 2 \text{ and } t_1 + t_2 = 3.$$

Solving, we get, $t_1 = 1$ and $t_2 = 2$.

So, Q is $(1, 2)$ and R is $(4, 4)$.

ILLUSTRATION 5.57

Tangents are drawn from any point on the line $x + 4a = 0$ to the parabola $y^2 = 4ax$. Then find the angle subtended by their chord of contact at the vertex

Sol. Let $P(t_1)$ and $Q(t_2)$ be the points of contact of tangents drawn from any point on the line $x + 4a = 0$.

The point of intersection of these tangents lies on the line $x + 4a = 0$.

$$\therefore at_1t_2 + 4a = 0$$

$$\text{or } t_1t_2 = -4$$

This is the condition of chord PQ to subtend a right angle at the vertex.

So, required angle is 90° .

ILLUSTRATION 5.58

Two straight lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents to $y^2 = 4ax$. Prove that $m_1 m_2 = -1$.

Sol. Clearly, both the lines pass through $(-a, b)$, which is a point lying on the directrix of the parabola.

Thus, $m_1 m_2 = -1$, because tangents drawn from any point on the directrix are always mutually perpendicular.

ILLUSTRATION 5.59

Mutually perpendicular tangents TA and TB are drawn to $y^2 = 4ax$. Then find the minimum length of AB .

Sol. Chord of contact of mutually perpendicular tangents is always a focal chord.

Thus, minimum length of AB is latus rectum $= 4a$.

ILLUSTRATION 5.60

Tangents PA and PB are drawn from point P on the directrix to the parabola $(x - 2)^2 + (y - 3)^2 = \frac{(5x - 12y + 3)^2}{169}$. Find the least radius of the circumcircle of the triangle PAB .

Sol. We have equation of parabola,

$$\sqrt{(x - 2)^2 + (y - 3)^2} = \frac{|5x - 12y + 3|}{\sqrt{5^2 + (-12)^2}}$$

Focus of the parabola is $(2, 3)$ and directrix is $5x - 12y + 3 = 0$.

Now, tangents drawn to parabola from point P on the directrix are perpendicular and the corresponding chord of contact AB is focal chord which is diameter of the circumcircle of the triangle PAB .

So, least value of diameter is latus rectum.

Here, L.R. $= 2 \times$ Distance of focus from directrix

$$= 2 \times \frac{|10 - 36 + 3|}{13} = \frac{23}{13}$$

So, required radius $= \frac{46}{13}$.

ILLUSTRATION 5.61

Tangents are drawn to the parabola $(x - 3)^2 + (y + 4)^2 = \frac{(3x - 4y - 6)^2}{25}$ at the extremities of the chord $2x - 3y - 18 = 0$.

Find the angle between the tangents.

Sol. We have equation of parabola,

$$\sqrt{(x - 3)^2 + (y + 4)^2} = \frac{|3x - 4y - 6|}{\sqrt{3^2 + (-4)^2}}$$

Focus of the parabola is $(3, -4)$ and directrix is $3x - 4y - 6 = 0$.

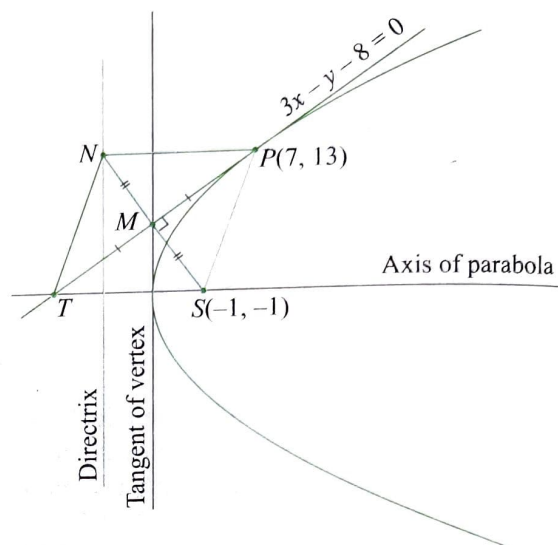
The given chord $2x - 3y - 18 = 0$ passes through the focus $(3, -4)$ of the parabola. So, it is focal chord.

Since tangents drawn at the extremities of focal chord are perpendicular, angle between tangents is 90° .

ILLUSTRATION 5.62

Let $y = 3x - 8$ be the equation of the tangent at the point $(7, 13)$ lying on a parabola whose focus is at $(-1, -1)$. Find the equation of directrix and the length of the latus rectum of the parabola.

Sol.



We know that

Length of latus rectum of parabola
 $= 2 \times$ Distance of focus from directrix

So, we need to find the equation of directrix.

Now, the image of focus in any tangent lies on the directrix

Image of focus $S(-1, -1)$ in the tangent $y = 3x - 8$ is the point $N(5, -3)$, which lies on the directrix.

Also, line joining point of contact $P(7, 13)$ and $N(5, -3)$ is perpendicular to the directrix.

$$\text{Slope of } NP = \frac{13 - (-3)}{7 - 5} = 8$$

$$\therefore \text{Slope of directrix} = -\frac{1}{8}$$

Since directrix passes through point N , its equation is $x + 8y + 19 = 0$.

$$\text{So, length of latus rectum} = 2 \times \frac{|-1 + 8(-1) + 19|}{\sqrt{1 + 64}} = \frac{20}{\sqrt{65}}$$

LOCUS OF POINT OF INTERSECTION OF TANGENT UNDER DIFFERENT CONDITIONS

Equation of tangents to parabola $y^2 = 4ax$ at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ are $t_1 y = x + at_1^2$ and $t_2 y = x + at_2^2$, respectively. These tangents intersect at point $R(at_1 t_2, a(t_1 + t_2))$. If we want to find the locus of point $R(h, k)$ under some conditions, then we let

$$h = at_1 t_2$$

$$\text{and } k = a(t_1 + t_2)$$

We can find third equation between t_1 and t_2 using given conditions.

By eliminating t_1 and t_2 from these equations, we get relation between h and k and hence, the required locus.

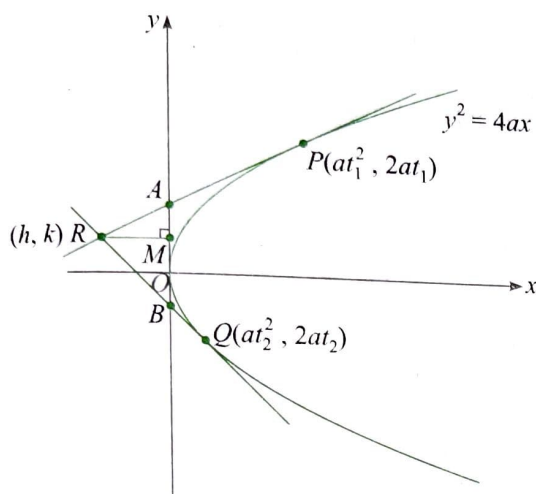
ILLUSTRATION 5.63

Find the locus of point of intersection of tangents to the parabola $y^2 = 4ax$

- which are inclined at an angle θ to each other
- which intercept constant length c on the tangent at vertex
- such that area of $\triangle ABR$ is constant c , where A and B are points of intersection of tangents with y -axis; R is point of intersection of tangents.

Sol.

(i)



Equations of tangents at points P and Q are, respectively, $t_1y = x + at_1^2$ and $t_2y = x + at_2^2$.

These tangents intersect at R .

$$\therefore t_1t_2 = \frac{h}{a} \text{ and } t_1 + t_2 = \frac{k}{a}$$

The angle between tangents is θ .

$$\therefore \tan \theta = \left| \frac{\frac{1}{t_1} - \frac{1}{t_2}}{1 + \frac{1}{t_1} \cdot \frac{1}{t_2}} \right| = \left| \frac{t_1 - t_2}{1 + t_1t_2} \right|$$

$$\Rightarrow \tan^2 \theta (1 + t_1t_2)^2 = (t_1 - t_2)^2$$

$$\Rightarrow \tan^2 \theta (1 + t_1t_2)^2 = (t_1 + t_2)^2 + 4t_1t_2$$

$$\Rightarrow \tan^2 \theta \left(1 + \frac{h}{a} \right)^2 = \left(\frac{k}{a} \right)^2 + 4 \frac{h}{a}$$

$$\Rightarrow \tan^2 \theta (x + a)^2 = y^2 + 4ax$$

(Replacing h by x and k by y)

This is the required equation of locus.

- Tangents at P and Q meet y -axis at $A(0, at_1)$ and $B(0, at_2)$, respectively.

Given that $AB = c$

$$\therefore |at_1 - at_2| = c$$

$$\Rightarrow a^2[(t_1 + t_2)^2 - 4t_1t_2] = c^2$$

$$\Rightarrow a^2 \left[\left(\frac{k}{a} \right)^2 - 4 \frac{h}{a} \right] = c^2$$

$$\Rightarrow y^2 - 4ax = c^2, \text{ which is the required equation of locus.}$$

- Area of triangle $ABR = c$

$$\Rightarrow \frac{1}{2} AB \cdot RM = c$$

$$\Rightarrow \frac{1}{2} |at_1 - at_2|(at_1t_2) = c^2$$

$$\Rightarrow a^4 [(t_1 + t_2)^2 - 4t_1t_2](t_1t_2)^2 = 4c^2$$

$$\Rightarrow a^4 \left[\left(\frac{k}{a} \right)^2 - 4 \frac{h}{a} \right] \left(\frac{h}{a} \right)^2 = 4c^2$$

$$\Rightarrow (y^2 - 4ax)x^2 = 4c^2, \text{ which is the required equation of locus.}$$

CONCEPT APPLICATION EXERCISE 5.5

- The tangents at the points P and Q on the parabola $y^2 = 4ax$ meet at T . If S is its focus, then prove that SP , ST and SQ are in G.P.
- If PQ is the focal chord of parabola $y = x^2 - 2x + 3$ such that $P \equiv (2, 3)$, then find the slope of tangent at Q .
- If there exists at least one point on the circle $x^2 + y^2 = a^2$ from which two perpendicular tangents can be drawn to parabola $y^2 = 2x$, then find the values of a .
- Find the angle between the tangents drawn to $y^2 = 4x$, where it is intersected by the line $y = x - 1$.
- Find the angle between the tangents drawn from the origin to the parabolas $y^2 = 4a(x - a)$.
- Find the locus of the point of intersection of the perpendicular tangents of the curve $y^2 + 4y - 6x - 2 = 0$.
- A tangent is drawn to the parabola $y^2 = 4ax$ at P such that it cuts the y -axis at Q . A line perpendicular to this tangent is drawn through Q which cuts the axis of the parabola at R . If the rectangle $PQRS$ is completed, then find the locus of S .
- Let $y = x + 1$ is axis of parabola, $y + x - 4 = 0$ is tangent of same parabola at its vertex and $y = 2x + 3$ is one of its tangents. Then find the focus of the parabola.

ANSWERS

- $-1/2$
- 90°
- $a \geq 1/2$
- $2x + 5 = 0$
- 90°
- $2a(x - a) - y^2 = (a - x) \left(a \left(\frac{x - a}{y} \right)^2 - x \right)$
- $\left(\frac{17}{9}, \frac{26}{9} \right)$

NORMAL TO PARABOLA

We know that line perpendicular to the tangent to the curve at the point of contact is normal.

Let us get the equation of normal to the standard parabola $y^2 = 4ax$ in different forms.

At any point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$ slope of tangent is $\frac{1}{t}$.

$$\therefore \text{Slope of normal at point } P = -t$$

So, equation of normal at point P is

$$y - 2at = -t(x - at^2) \quad (1)$$

$$\text{or } y = -tx + 2at + at^3 \quad (2)$$

Slope of the normal, $m = -t$

Therefore, equation of normal in terms of its slope is

$$y = mx - 2am - am^3 \quad (3)$$

This is the equation of normal at point $(am^2, -2am)$ on the curve.

If $(at^2, 2at) \equiv (x_1, y_1)$, then equation (1) becomes

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad (4)$$

Here, slope of normal is $-\frac{y_1}{2a}$.

We can get this slope using differentiation also.

Differentiating $y^2 = 4ax$, w.r.t. x , we get

$$\frac{dy}{dx} = \frac{2a}{y_1} = \text{Slope of tangent at point } (x_1, y_1)$$

$$\therefore \text{Slope of normal} = -\frac{y_1}{2a}$$

The equations of normals to all standard parabolas in terms of parameter of the point on the curve and the slope of the normal are tabulated as follows.

Equation of Parabola	Point on the parabola	Equation of Normal
$y^2 = 4ax$	$(at^2, 2at)$	$y = -tx + 2at + at^3$
	$(am^2, -2am)$	$y = mx - 2am - am^3$
$y^2 = -4ax$	$(-at^2, 2at)$	$y = tx + 2at + at^3$
	$(-am^2, 2am)$	$y = mx + 2am + am^3$
$x^2 = 4ay$	$(2at, at^2)$	$x = -ty + 2at + at^3$
	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx + 2a + \frac{a}{m^2}$
$x^2 = -4ay$	$(2at, -at^2)$	$x = ty + 2at + at^3$
	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx - 2a - \frac{a}{m^2}$

ILLUSTRATION 5.64

Find the equations of normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum.

Sol. End points of latus rectum of parabola $y^2 = 4ax$ are $P(a, 2a)$ and $Q(a, -2a)$.

Comparing with $(at^2, 2at)$, we get $t = 1$ for point P and $t = -1$ for point Q .

Now, equation of normal at point $(at^2, 2at)$ is $y = -tx + 2at + at^3$.

So, equation of normal at point P is $y = -x + 2a + a$ or $x + y = 3a$.

Also, equation of normal at point Q is $y = x - 2a - a$ or $x - y = 3a$.

ILLUSTRATION 5.65

If $y = x + 2$ is normal to the parabola $y^2 = 4ax$, then find the value of a .

Sol. Normal to the parabola $y^2 = 4ax$ having slope m is

$$y = mx - 2am - am^3$$

The given normal is $y = x + 2$. Therefore,

$$m = 1 \text{ and } -2am - am^3 = 2$$

$$\text{or } -2a(1) - a(1)^3 = 2$$

$$\text{or } a = -\frac{2}{3}$$

ILLUSTRATION 5.66

Find the equation of line which is normal to the parabola $x^2 = 4y$ and touches the parabola $y^2 = 12x$.

Sol. Normal to parabola $x^2 = 4y$ having slope m is

$$y = mx + 2 + \frac{1}{m^2} \quad (1)$$

It is tangent to $y^2 = 12x$.

Now, tangent to above parabola having slope m is

$$y = mx + \frac{3}{m} \quad (2)$$

Comparing (1) and (2), we get

$$\frac{1}{m^2} + 2 = \frac{3}{m}$$

$$\Rightarrow 2m^2 - 3m + 1 = 0$$

$$\Rightarrow (2m - 1)(m - 1) = 0$$

$$\Rightarrow m = \frac{1}{2} \text{ or } m = 1$$

Therefore, equations of lines are $2y = x + 12$ or $y = x + 3$.

ILLUSTRATION 5.67

Find the equation of normal to the parabola $y = x^2 - x - 1$ which has equal intercepts on axes. Also, find the point where this normal meets the curve again.

Sol. Here, given equation of parabola is not standard.

So, we use differentiation method to find the equation of normal.

Since normal has equal intercepts on axes, its slope is -1 .

Now, differentiating equation $y = x^2 - x - 1$ on both sides w.r.t. x , we get

$$\frac{dy}{dx} = 2x - 1$$

This is the slope of tangent to the curve at any point on it.

Thus, slope of normal at any point on it is,

$$-\frac{dx}{dy} = \frac{1}{1 - 2x} = -1 \text{ (given)}$$

$$\therefore x = 1$$

Putting $x = 1$ in the equation of curve, we get $y = -1$.

So, equation of normal at point $(1, -1)$ on the curve is

$$y - (-1) = -1(x - 1)$$

$$\text{or } x + y = 0$$

Solving this equation of normal with the equation of parabola we have

$$-x = x^2 - x - 1$$

$$\text{or } x^2 = 1$$

$$\text{or } x = \pm 1$$

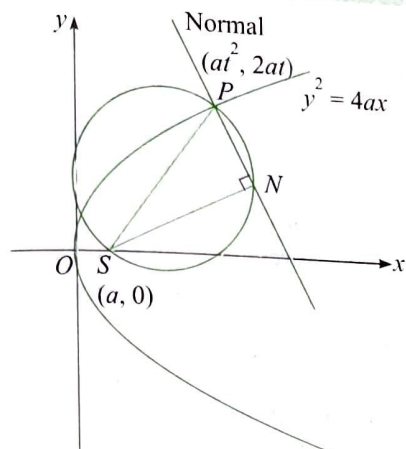
Hence, normal meets parabola again at point whose abscissa is -1 .

Putting $x = -1$ in the equation of curve, we get $y = 1$.

So, normal meets the parabola again at $(-1, 1)$.

ILLUSTRATION 5.68

Prove that the length of the intercept on the normal at the point $P(at^2, 2at)$ of a parabola $y^2 = 4ax$ made by the circle described on the line joining the focus and P as diameter is $a\sqrt{1+t^2}$.

Sol.

Equation of normal at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$y = -tx + 2at + at^3 \quad (1)$$

Let this normal meet the circle described on SP as diameter in N , then $\angle SNP = 90^\circ$ (angle in a semicircle).

$$\therefore PN^2 = SP^2 - SN^2$$

$$\text{Now } SP = a + at^2$$

and $SN = \text{Distance of focus from the normal}$

$$= \frac{|at - 2at - at^3|}{\sqrt{1+t^2}} = at\sqrt{1+t^2}$$

$$\begin{aligned} \therefore PN^2 &= a^2(1+t^2)^2 - a^2t^2(1+t^2) \\ &= a^2(1+t^2)[1+t^2-t^2] = a^2(1+t^2) \end{aligned}$$

$$\therefore PN = a\sqrt{1+t^2}$$

ILLUSTRATION 5.69

If a normal chord subtends a right angle at the vertex of the parabola $y^2 = 4ax$, then find its inclination to the axis.

Sol. Equation of normal to the parabola $y^2 = 4ax$ having slope 'm' is

$$y = mx - 2am - am^3 \quad (1)$$

This normal meets the parabola at P and Q .

Given that PQ subtends right angle at vertex.

Now, we can get the equation of pair of straight lines OP and OQ using homogenization.

$$\text{From (1), } 1 = \frac{mx - y}{2am + am^3}$$

Homogenizing the equation $y^2 = 4ax$, we get

$$y^2 = 4ax \left(\frac{mx - y}{2am + am^3} \right)^2$$

$$\text{or } 4amx^2 - (2am + am^3)y^2 - 4axy = 0 \quad (2)$$

This is the equation of pair of straight lines OP and OQ .

Since lines OP and OQ are perpendicular, sum of coefficients of x^2 and y^2 is zero.

$$\text{i.e., } 4am - 2am - am^3 = 0$$

$$\text{or } m^2 = 2$$

$$\therefore m = \pm\sqrt{2}$$

NORMAL TO PARABOLA FROM A POINT NOT LYING ON IT

There are different methods to find the equation of normal depending upon the equation of parabola.

If equation of parabola is in standard form say, $y^2 = 4ax$, then we consider standard equation of normal.

$$\text{i.e., } y = mx - 2am - am^3 \quad (1)$$

Let this normal pass through the point (x_1, y_1) which is not lying on the parabola.

$$\therefore y_1 = mx_1 - 2am - am^3$$

Solving this equation, we get values of m .

Since this equation is cubic in m , we get at least one real value of m .

Hence, from any point on the plane, we can draw at least one normal to the parabola.

ILLUSTRATION 5.70

How many normals can be drawn to parabola $y^2 = 4x$ from point $(15, 12)$? Find their equations. Also, find corresponding feet of normals on the parabola.

Sol. Equation of the normal to parabola $y^2 = 4x$ having slope m is

$$y = mx - 2m - m^3$$

If it passes through the point $(15, 12)$, then

$$12 = 15m - 2m - m^3$$

$$\text{or } m^3 - 13m + 12 = 0$$

Clearly, one root of the equation is 1.

$$\therefore (m-1)(m^2+m-12) = 0$$

$$\Rightarrow (m-1)(m-3)(m+4) = 0$$

$$\Rightarrow m = 1, 3, -4$$

Thus, three normals can be drawn from $(15, 12)$.

The equations of normal and corresponding points on the curve are given by

Value of 'm'	Equation of normal $y = mx - 2m - m^3$	Point on the parabola $(m^2, -2m)$
1	$y = x - 3$	$(1, -2)$
3	$y = 3x - 33$	$(9, -6)$
-4	$y = -4x + 72$	$(16, 8)$

ILLUSTRATION 5.71

Three normals are drawn from the point $(7, 14)$ to the parabola $x^2 - 8x - 16y = 0$. Find the coordinates of the feet of the normals.

Sol. The equation of the given parabola is

$$x^2 - 8x - 16y = 0$$

Differentiating w.r.t. x , we get

$$2x - 8 - 16 \frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = \frac{x-4}{8}$$

Let normal be drawn at point (h, k) on the curve.

$$\therefore k = \frac{h^2 - 8h}{16}$$

Also, slope of the normal at $(h, k) = \frac{8}{4-h}$

Therefore, equation of the normal at (h, k) is

$$y - k = \left(\frac{8}{4-h} \right) (x - h)$$

If the normal passes through $(7, 14)$, then we have

$$14 - \frac{h^2 - 8h}{16} = \left(\frac{8}{4-h} \right) (7-h)$$

$$\Rightarrow (4-h)(224 - h^2 + 8h) = 128(7-h)$$

$$\Rightarrow h^3 - 12h^2 - 64h = 0$$

$$\Rightarrow h(h-16)(h+4) = 0$$

$$\Rightarrow h = 0, -4, 16$$

For $h = 0, k = 0$

for $h = -4, k = 3$ and

for $h = 16, k = 8$.

Therefore, the coordinates of the feet of the normals are $(0, 0)$, $(-4, 3)$ and $(16, 8)$.

ILLUSTRATION 5.72

Find the minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.

Sol. We have parabola $y^2 = 4x$ and circle $x^2 + y^2 - 12x + 31 = 0$. Shortest distance between two curves occur along common normal line.

Normal to circle always passes through its centre.

So, we have to find the normal to parabola which goes through centre of the circle.

Given circle is $(x-6)^2 + y^2 = 5$.

So, centre is $T(6, 0)$ and radius is $\sqrt{5}$.

Equation of normal to parabola having slope m is

$$y = mx - 2m - m^3$$

If it passes through $(6, 0)$, then

$$0 = 6m - 2m - m^3$$

$$\therefore m^3 - 4m = 0$$

$$\therefore m = 0, \pm 2.$$

So, equations of normal and corresponding points on the parabola are tabulated below.

Value of m	Equation of normal	Point on the curve
0	$y = 0$	$P(0, 0)$
2	$y = 2x - 12$	$Q(4, -4)$
-2	$y = -2x + 12$	$R(4, 4)$

$$\therefore TQ = TR = \sqrt{20}$$

$$TP = 6$$

So, required shortest distance $= \sqrt{20} - \sqrt{5} = \sqrt{5}$

ILLUSTRATION 5.73

If normals drawn at three different points on the parabola $y^2 = 4ax$ pass through the point (h, k) , then show that $h > 2a$.

Sol. The equation of a normal to the parabola $y^2 = 4ax$ having slope m is

$$y = mx - 2am - am^3 \quad (1)$$

Since the normal drawn at three different points on the parabola pass through (h, k) , it must satisfy the equation (1).

$$\therefore k = mh - 2am - am^3$$

$$\Rightarrow am^3 + (2a - h)m + k = 0 \quad (2)$$

Now, it is given that three different normals pass through (h, k) .

Thus, equation (2) has three distinct real roots m_1, m_2 and m_3 .

From equation (2),

$$m_1 + m_2 + m_3 = 0 \quad (3)$$

$$\text{and } m_1m_2 + m_2m_3 + m_3m_1 = 2a - h \quad (4)$$

Now, $(m_1 + m_2 + m_3)^2 = 0$

$$\Rightarrow m_1^2 + m_2^2 + m_3^2 = -2(m_1m_2 + m_2m_3 + m_3m_1)$$

$$\Rightarrow m_1^2 + m_2^2 + m_3^2 = 2(h - 2a)$$

Since m_1, m_2 and m_3 are distinct and real, L.H.S. > 0

(As in case of imaginary roots L.H.S. may be negative)

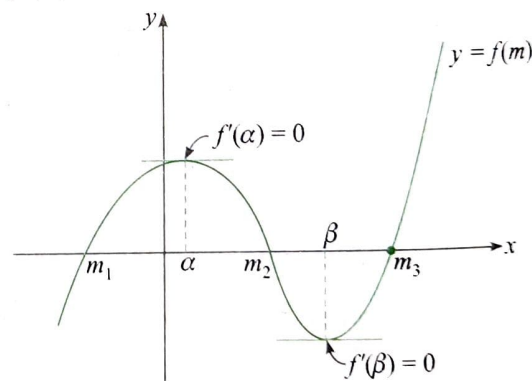
$$\therefore h - 2a > 0$$

$$\Rightarrow h > 2a$$

Alternative method:

Let $am^3 + (2a - h)m + k = f(m)$

Since equation (2), i.e., $f(m) = 0$ has three distinct real roots, then equation $f'(m) = 0$ must have two distinct real roots.



$$\therefore 3am^2 + (2a - h) = 0 \text{ has two distinct real roots.}$$

$$\text{So, } 2a - h < 0 \text{ or } h > 2a.$$

Note:

The converse of this is not true, i.e., if $h > 2a$ then it is not necessary that equation (2) has three distinct real roots. In other words, if $h > 2a$ then it is not necessary that three distinct normals can be drawn from (h, k) .

ILLUSTRATION 5.74

If three distinct normals to the parabola $y^2 - 2y = 4x - 9$ meet at point (h, k) , then prove that $h > 4$.

Sol. Given parabola is $(y-1)^2 = 4(x-2)$.

Equation of normal to parabola having slope m is

$$y - 1 = m(x - 2) - 2m - m^3$$

It passes through (h, k)

$$k - 1 = m(h - 2) - 2m - m^3$$

$$m^3 + (4 - h)m + k - 1 = 0$$

or
This equation has three distinct real roots.

So, its derivative equation must have two distinct real roots.

$$3m^2 + (4 - h) = 0 \text{ has two distinct real roots.}$$

$$4 - h < 0 \text{ or } h > 4$$

CONCEPT APPLICATION EXERCISE 5.6

1. Prove that the chord $y - \sqrt{2}x + 4a\sqrt{2} = 0$ is a normal chord of the parabola $y^2 = 4ax$. Also, find the point on the parabola where the given chord is normal to the parabola.
2. Find the equation of normal to parabola $y = x^2 - 3x - 4$
 - (a) at point $(3, -4)$
 - (b) having slope 5.
3. If $y = 2x + 3$ is a tangent to the parabola $y^2 = 24x$, then find its distance from the parallel normal.
4. Prove that for $\theta \in R$, the line $y = (x - 11) \cos \theta - \cos 3\theta$ is always normal to the parabola $y^2 = 16x$.
5. Find the locus of the midpoints of the portion of the normal to the parabola $y^2 = 4ax$ intercepted between the curve and the axis.
6. If the parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have a common normal other than x -axis (a, b, c being distinct positive real numbers), then prove that $\frac{b}{a - c} > 2$.
7. Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. Show that c must be greater than $1/2$. One normal is always the x -axis. Find c for which the other two normals are perpendicular to each other.

ANSWERS

1. $(2a, -2\sqrt{2}a)$
2. (a) $3x - y - 13 = 0$, (b) $5x - y - 20 = 0$
3. $\frac{75}{\sqrt{5}}$ 5. $y^2 = a(x - a)$ 7. $c > \frac{1}{2}, c = \frac{3}{4}$

PROPERTIES OF NORMAL

We have following properties of normal to the parabola.

- I. Normal other than axis of parabola never passes through the focus.

Proof: Consider normal to the parabola $y^2 = 4ax$ at point $P(at^2, 2at)$, which has equation $y = -tx + 2at + at^3$.

If this normal passes through the focus $S(a, 0)$, then

$$0 = -at + 2at + at^3$$

$$\therefore t^2 + 1 = 0, \text{ which is not possible.}$$

Thus, normal other than axis never passes through focus.

- II. Tangents and normal drawn to parabola $y^2 = 4ax$ at point $P(t)$, $t \neq 0$, meet the x -axis at points T and N , respectively, such that $SP = ST = SN$.

Proof: Equation of tangent at point $P(t)$ is

$$ty = x + at^2.$$

It meets the x -axis at $T(-at^2, 0)$.

Equation of normal at point $P(t)$ is

$$y = -tx + 2at + at^3$$

It meets x -axis at $N(2a + at^2, 0)$.

Focus is $S(a, 0)$.

$$\therefore SP = a + at^2 \text{ (focal length)}$$

$$ST = a + at^2 \text{ and } SN = a + at^2.$$

Thus, $SP = ST = SN$.

- III. Let us find the point of intersection of normal at two points $P(t_1)$ and $Q(t_2)$ on the parabola $y^2 = 4ax$. Consider two points on the parabola $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.

Equation of normal at points P and Q are given by

$$y = -t_1x + 2at_1 + at_1^3 \quad (1)$$

$$y = -t_2x + 2at_2 + at_2^3 \quad (2)$$

Solving these two equations, we get point of intersection as

$$(2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2))$$

These coordinates are useful in problems of locus of point of intersection of normals under different conditions.

- IV. Normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the curve again at point Q whose parameter is given

$$\text{by } t' = -t - \frac{2}{t}.$$

Proof: Equation of normal at point $P(at^2, 2at)$ is

$$y = -tx + 2at + at^3 \quad (1)$$

It meets the parabola again at point $Q(at'^2, 2at')$.

$$\therefore 2at' = -at'^2 + 2at + at^3$$

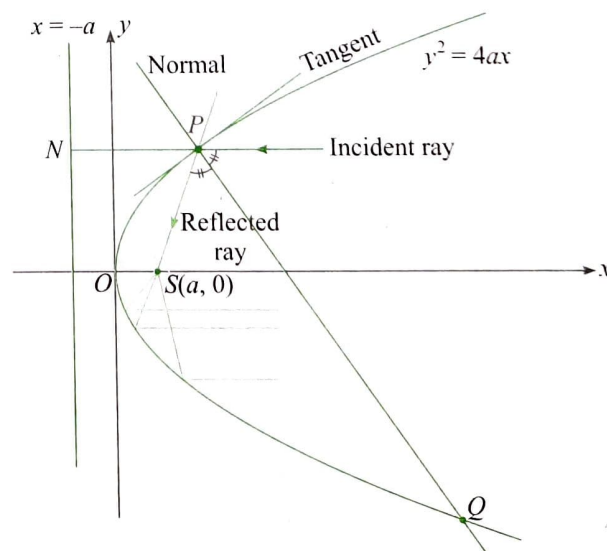
$$\Rightarrow 2a(t' - t) + at(t'^2 - t^2) = 0$$

$$\Rightarrow 2 + t(t' + t) = 0 \text{ (at } t' \neq t)$$

$$\therefore t' = -t - \frac{2}{t}$$

- V. In any parabola, normal at any point P on it bisects the external angle between the focal chord through P and the perpendicular from P to the directrix.

Proof:



In properties of tangent, we learned that tangent at any point P on the parabola bisects the angle between the focal chord through P and the perpendicular from P to the directrix ($\angle SPN$).

Since two angle bisectors of given two lines are always perpendicular, another bisector (of external angle) is normal at point P .

This also spells out the reflection property of parabola that "If incident ray is parallel to the axis of parabola, then after striking the parabola, the reflected ray passes through the focus."

This reflection property of parabola is used in reflectors which we use in day to day life, i.e., in torch, headlight, dish antenna etc.

If the source of the light is placed at the focus of parabolic surface, infinite light rays emanating from the light source get reflected by the parabolic surface and become parallel to the axis of the parabola creating a light beam.

ILLUSTRATION 5.75

In the parabola $y^2 = 4ax$, the tangent at P whose abscissa is equal to the latus rectum meets its axis at T , and normal at P cuts the curve again at Q . Show that $PT : PQ = 4 : 5$.

Sol. Let P be $(at^2, 2at)$. Since $at^2 = 4a$, $t = \pm 2$. Consider $t = 2$ and $P(4a, 4a)$. The tangent at P is $2y = x + 4a$ which meets the x -axis at $T(-4a, 0)$.

If the coordinates of Q are $(at_1^2, 2at_1)$, then

$$t_1 = -t - \frac{2}{t} = -3$$

So, Q is $(9a, -6a)$. Therefore,

$$(PQ)^2 = 125a^2 \text{ and } (PT)^2 = 80a^2$$

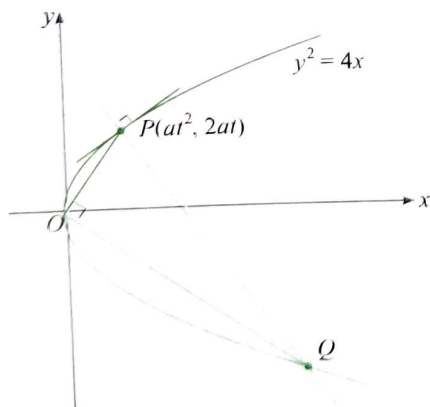
or $PT : PQ = 4 : 5$

ILLUSTRATION 5.76

Find the length of normal chord which subtends an angle of 90° at the vertex of the parabola $y^2 = 4x$.

Sol. We know that normal at point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the parabola again at point $Q(at'^2, 2at')$ such that

$$t' = -t - \frac{2}{t} \quad (1)$$



Also, PQ subtends right angle vertex $O(0, 0)$.

$$\therefore tt' = -4$$

$$\text{From (1), } tt' = -t^2 - 2$$

$$\Rightarrow -4 = -t^2 - 2$$

$$\Rightarrow t^2 = 2$$

$$\Rightarrow t = \sqrt{2}$$

$$\Rightarrow t' = -2\sqrt{2}$$

$$\therefore \text{Point } P \equiv (2, 2\sqrt{2}) \text{ and } Q \equiv (8, -4\sqrt{2})$$

$$\begin{aligned} \therefore PQ &= \sqrt{(8-2)^2 + (-4\sqrt{2} - 2\sqrt{2})^2} \\ &= \sqrt{36 + 72} \\ &= 6\sqrt{3} \end{aligned}$$

ILLUSTRATION 5.77

Prove that the locus of the point of intersection of the normals at the ends of a system of parallel chords of a parabola is a straight line which is a normal to the curve.

Sol. Consider chord PQ joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on parabola $y^2 = 4ax$.

$$\text{Slope of } PQ \text{ is } m = \frac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \frac{2}{t_2 + t_1}$$

Point of intersection of normals at points P and Q is

$$R(2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)) \equiv (h, k)$$

$$\therefore h - 2a = a((t_1 + t_2)^2 - t_1t_2) \text{ and } k = -at_1t_2(t_1 + t_2)$$

Putting $t_1 + t_2 = \frac{2}{m}$, we get

$$h - 2a = \frac{4a}{m^2} - at_1t_2 \text{ and } k = -\frac{2a}{m}t_1t_2$$

Eliminating t_1t_2 , we get

$$h - 2a = \frac{4a}{m^2} + \frac{mk}{2}$$

So, locus of R is

$$x - 2a = \frac{4a}{m^2} + \frac{m}{2}y$$

$$\text{or } y = \frac{2}{m}x - \frac{4a}{m} - \frac{8a}{m^3}$$

$$\text{or } y = m'x - 2am' - am'^3, \text{ where } m' = \frac{2}{m}$$

Thus, locus of point R is normal to the parabola.

ILLUSTRATION 5.78

Find the locus of midpoint of normal chord of parabola $y^2 = 4ax$.

Sol. Normal to parabola $y^2 = 4ax$ at point $P(t)$ meets the parabola again at point $Q(t')$ such that

$$t' = -t - \frac{2}{t} \quad (1)$$

Let midpoint of PQ be $R(h, k)$.

$$\therefore h = \frac{at^2 + at'^2}{2}$$

$$\text{and } k = \frac{2at + 2at'}{2} = a(t + t') = -\frac{2a}{t}$$

$$\therefore t = -\frac{2a}{k}$$

(Using (1))

(2)

$$\text{Now, } h = \frac{at^2 + a\left(-t - \frac{2}{t}\right)^2}{2}$$

$$\Rightarrow \frac{2h}{a} = \left(-\frac{2a}{k}\right)^2 + \left(\frac{2a}{k} + \frac{k}{a}\right)^2$$

$$\Rightarrow \frac{2x}{a} = \frac{4a^2}{y^2} + \left(\frac{2a}{y} + \frac{y}{a}\right)^2$$

$$\Rightarrow 2axy^2 = 4a^4 + (2a^2 + y^2)^2, \text{ which is required locus.}$$

Alternative method:

Equation of normal at point $P(t)$ is

$$y = -tx + 2at + at^3 \quad (1)$$

It meets the parabola at point Q .

Midpoint of PQ is $R(h, k)$.

So, equation of chord PQ , whose midpoint is $R(h, k)$, is

$$ky - 2a(x + h) = k^2 - 4ah \quad (\text{Using } T = S) \quad (2)$$

Equations (1) and (2) represent the same straight line.

$$\therefore \frac{t}{-2a} = \frac{1}{k} = \frac{2at + at^3}{k^2 - 2ah}$$

$$\Rightarrow t = -\frac{2a}{k} \text{ and } \frac{1}{k} = \frac{2a\left(-\frac{2a}{k}\right) + a\left(-\frac{2a}{k}\right)^3}{k^2 - 2ah}$$

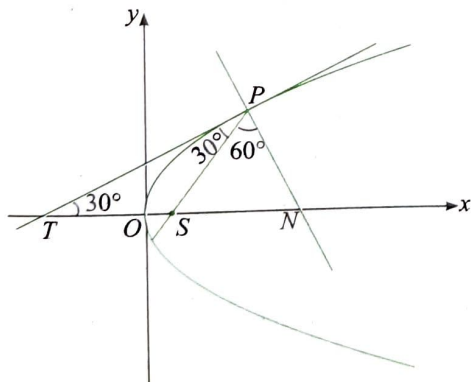
$$\Rightarrow \frac{y^2 - 2ax}{y} = -\frac{4a^2}{y} - \frac{8a^4}{y^3}$$

$$\Rightarrow 2axy^2 = 4a^4 + (2a^2 + y^2)^2, \text{ which is required locus.}$$

ILLUSTRATION 5.79

If the angle between the normal to the parabola $y^2 = 4ax$ at point P and the focal chord passing through P is 60° , then find the slope of the tangent at point P .

Sol.



By property,

$$SP = ST = SN.$$

Given that $\angle SPN = 60^\circ$

$$\therefore \angle SPT = 30^\circ$$

Since $ST = SP$, $\angle PTS = 30^\circ$

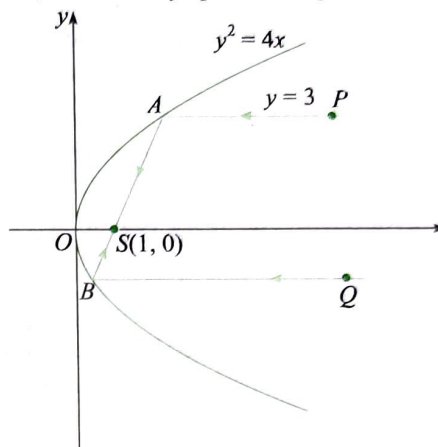
$$\text{Therefore, slope of tangent} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

ILLUSTRATION 5.80

A parabolic mirror is kept along $y^2 = 4x$ and two light rays, parallel to its axis, are reflected along one straight line. If one of the incident light rays is at 3 units distance from the axis, then find the distance of the other incident ray from axis.

Sol. We have parabola $y^2 = 4x$.

Let two incident rays AP and BQ , which are parallel to axis of the parabola, strikes the parabola at points A and B , respectively. After reflection, both the rays pass through the focus $S(1, 0)$.



Therefore, AB is focal chord.

Let point A be $(t^2, 2t)$.

Distance of ray AP is 3 units from axis.

$$\therefore 2t = 3$$

$$\text{or } t = \frac{3}{2}$$

Coordinates of point B are $\left(\frac{1}{t^2}, -\frac{2}{t}\right)$ (other end of focal chord).

$$\text{Distance of } B \text{ from the axis} = \frac{2}{t} = \frac{4}{3}$$

CONORMAL POINT AND ITS PROPERTIES

Equation of normal to the parabola $y^2 = 4ax$ at point $P(at^2, 2at)$ is

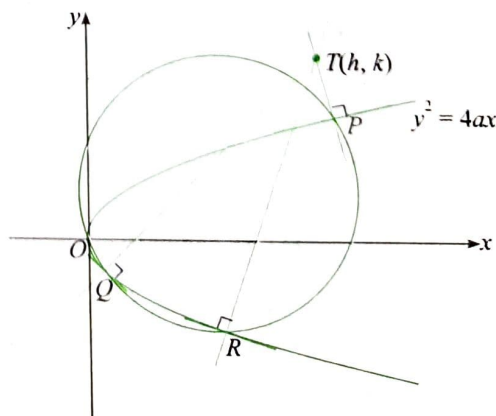
$$y = -tx + 2at + at^3$$

Let this normal pass through $T(h, k)$ on the plane.

$$\therefore k = -th + 2at + at^3$$

$$\text{or } at^3 + (2a - h)t - k = 0$$

This is a cubic equation in t .



So, it has maximum three real roots.

Let equation (1) have three distinct real roots, t_1, t_2 and t_3 .

So, it is possible to draw three normals to parabola through point T .

For three normals, there will be three feet of normals on the parabola, $P(at_1^2, 2at_1)$ ($(at^2, 2at)$ itself), $Q(at_2^2, 2at_2)$ and $R(at_3^2, 2at_3)$.

Points P, Q and R are called conormal points.

From the equation (1),

$$\text{Sum of roots, } t_1 + t_2 + t_3 = 0 \quad (2)$$

Sum of product of roots taken two at a time,

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = \frac{2a - h}{a} \quad (3)$$

We have following properties of the conormal points of the parabola.

- I. The sum of the ordinates of the feet of conormal points is zero.

or

Centroid of the triangle formed by conormal points lie on x -axis or on axis of the parabola in general.

Proof: From (2), we have

$$2at_1 + 2at_2 + 2at_3 = 0$$

$$\therefore \frac{2at_1 + 2at_2 + 2at_3}{3} = 0$$

Thus, ordinate of the centroid of the triangle formed by conormal points is zero.

So, centroid lies on x -axis (axis of the parabola).

- II. Circle through the conormal points passes through origin (vertex of the parabola).

Let the equation of circle passing through three conormal points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

To solve this circle with the parabola, we eliminate x .

$$\text{So, } \left(\frac{y^2}{4a}\right)^2 + y^2 + 2g\frac{y^2}{4a} + 2fy + c = 0$$

$$\therefore y^4 + (16a^2 + 8ag)y^2 + 32a^2 fy + 16a^2 c = 0 \quad (1)$$

Sum of roots, $y_1 + y_2 + y_3 + y_4 = 0$

We have already established that $y_1 + y_2 + y_3 = 0$.

$$\therefore y_4 = 0 \text{ and so, } x_4 = 0.$$

Thus, fourth point where circle meets the parabola is $(0, 0)$, which is vertex of the parabola.

Therefore, $c = 0$ and from equation (1), we have

$$y^4 + (16a^2 + 8ag)y^2 + 32a^2 fy = 0$$

$$\text{or } y^3 + (16a^2 + 8ag)y + 32a^2 f = 0$$

$$\text{or } 8a^3 t^3 + 2a(16a^2 + 8ag)t + 32a^2 f = 0 \quad (\text{Putting } y = 2at)$$

$$\text{or } at^3 + (4a + 2g)t + 4f = 0 \quad (2)$$

Also, normal at $P(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$

It passes through point (h, k) .

$$\therefore at^3 + (2a - h)t - k = 0 \quad (3)$$

So, equations (2) and (3) are identical.

$$\therefore 1 = \frac{4a + 2g}{2a - h} = \frac{4f}{-k}$$

$$\therefore 2g = -(2a + h), \text{ and } 2f = -k/2$$

Therefore, the equation of the circle is

$$x^2 + y^2 - (2a + h)x - \frac{k}{2}y = 0$$

ILLUSTRATION 5.81

If two of the three feet of normals drawn from a point to the parabola $y^2 = 4x$ are $(1, 2)$ and $(1, -2)$, then find the third foot.

Sol. The sum of the ordinates of the feet is $y_1 + y_2 + y_3 = 0$.

Therefore,

$$2 + (-2) + y_3 = 0$$

$$\text{or } y_3 = 0$$

So, the third foot is $(0, 0)$.

ILLUSTRATION 5.82

If the three normals from any point to the parabola $y^2 = 4x$ cut the line $x = 2$ at points whose ordinates are in AP , then prove that the slopes of tangents at the co-normal points are in GP .

Sol. The equation of the normal to the parabola $y^2 = 4x$ is given by

$$y = -tx + 2t + t^3 \quad (1)$$

Since it intersects $x = 2$, we get $y = t^3$.

Let the three ordinates t_1^3, t_2^3 , and t_3^3 be in AP . Then,

$$\begin{aligned} 2t_2^3 &= t_1^3 + t_3^3 \\ &= (t_1 + t_3)^3 - 3t_1 t_3 (t_1 + t_3) \end{aligned} \quad (2)$$

Now, $t_1 + t_2 + t_3 = 0$

$$\text{or } t_1 + t_3 = -t_2$$

Hence, (2) reduces to

$$\begin{aligned} 2t_2^3 &= (-t_2)^3 - 3t_1 t_3 (-t_2) \\ &= -t_2^3 + 3t_1 t_2 t_3 \end{aligned}$$

$$\text{or } 3t_2^3 = 3t_1 t_2 t_3 \text{ or } t_2^2 = t_1 t_3$$

Therefore, t_1, t_2 , and t_3 are in GP .

Hence, the slopes of tangents $1/t_1, 1/t_2$, and $1/t_3$ are in GP .

ILLUSTRATION 5.83

Find the locus of the point of intersection of two normals to a parabola which are at right angles to one another.

Sol. The equation of the normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

It passes through the point (h, k) if

$$k = mh - 2am - am^3$$

$$\text{or } am^3 + m(2a - h) + k = 0 \quad (1)$$

Let the roots of the above equation be m_1, m_2 , and m_3 .
Let normals having slopes m_1 and m_2 be perpendicular.

$$\text{So, } m_1 m_2 = -1$$

$$\text{From (1), } m_1 m_2 m_3 = -k/a.$$

$$\text{Since } m_1 m_2 = -1, m_3 = k/a.$$

Since m_3 is a root of (1), we have

$$a\left(\frac{k}{a}\right)^3 + \frac{k}{a}(2a - h) + k = 0$$

$$k^2 + a(2a - h) + a^2 = 0$$

$$k^2 = a(h - 3a)$$

Hence, the locus of (h, k) is

$$y^2 = a(x - 3a)$$

ILLUSTRATION 5.84

$P(t_1)$ and $Q(t_2)$ are the points t_1 and t_2 on the parabola $y^2 = 4ax$.
The normals at P and Q meet on the parabola. Show that the middle point of PQ lies on the parabola $y^2 = 2a(x + 2a)$.

Sol. The equation of normal to $y^2 = 4ax$ is

$$y = -tx + 2at + at^3$$

It passes through the point (h, k) . Then

$$k = -th + 2at + at^3$$

$$\text{or } at^3 + (2a - h)t - k = 0$$

Since normal at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ passes through the point $R(at_3^2, 2at_3)$ on the parabola,

$$t_1 t_2 t_3 = \frac{k}{a} = \frac{2at_3}{a}$$

$$\therefore t_1 t_2 = 2$$

Also, if (x_1, y_1) is the midpoint of PQ , then

$$x_1 = \frac{1}{2}(at_1^2 + at_2^2) \text{ and } y_1 = \frac{1}{2}(2at_1 + 2at_2) \quad (2)$$

$$\therefore (t_1 + t_2)^2 = \left(\frac{y_1}{a}\right)^2$$

$$\text{or } \left(\frac{y_1}{a}\right)^2 = t_1^2 + t_2^2 + 2t_1 t_2 = \left(\frac{2x_1}{a}\right) + 4$$

$$\text{or } y_1^2 = 2a(x_1 + 2a)$$

Hence, the locus of (x_1, y_1) is $y^2 = 2a(x + 2a)$.

ILLUSTRATION 5.85

Normals are drawn at points A, B , and C on the parabola $y^2 = 4x$ which intersect at $P(h, k)$. Find the locus of the point P if the slope of the line joining the feet of two of them is 2.

Sol. The equation of normal at $(t^2, 2t)$ is

$$y = -tx + 2t + t^3$$

If it passes through $P(h, k)$, then

$$t^3 + t(2 - h) - k = 0 \quad (1)$$

This equation has three roots t_1, t_2 and t_3 which are parameters of three feet of normals on the parabola.

From equation (1), we have

$$t_1 + t_2 + t_3 = 0 \quad (2)$$

Given that slope of the line joining the feet of two of the normals (say $A(t_1)$ and $B(t_2)$) is 2.

$$\therefore \frac{2}{t_1 + t_2} = 2$$

$$\Rightarrow t_1 + t_2 = 1$$

So, from (2), we have $t_3 = -1$, which is one of the roots of equation (1).

So, from (1), we have

$$-1 + h - 2 - k = 0$$

or $x - y - 3 = 0$, which is required locus.

CONCEPT APPLICATION EXERCISE 5.7

1. If the normal to the parabola $y^2 = 4ax$ at point t_1 cuts the parabola again at point t_2 , then prove that $t_2^2 \geq 8$.
2. Find the angle at which normal at point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meets the parabola again at point Q .
3. If normal to parabola $y^2 = 4ax$ at point $P(at^2, 2at)$ intersects the parabola again at Q , such that sum of ordinates of the points P and Q is 3, then find the length of latus rectum in terms of t .
4. If tangents are drawn to $y^2 = 4ax$ from any point P on the parabola $y^2 = a(x + b)$, then show that the normals drawn at their point for contact meet on a fixed line.
5. If line $x - 2y - 1 = 0$ intersects parabola $y^2 = 4x$ at P and Q , then find the point of intersection of normals at P and Q .
6. Find the locus of the point of intersection of the normals at the end of the focal chord of the parabola $y^2 = 4ax$.
7. If incident ray from point $(-2, 4)$ parallel to the axis of the parabola $y^2 = 4x$ strikes the parabola, then find the equation of the reflected ray.
8. Let L_1, L_2 and L_3 be the three normals to the parabola $y^2 = 4ax$ from point P inclined at the angle θ_1, θ_2 and θ_3 with x -axis, respectively. Then find the locus of point P given that $\theta_1 + \theta_2 + \theta_3 = \alpha$ (constant).

ANSWERS

- | | | |
|---|----------------------|--------------|
| 2. $\tan^{-1} \left \frac{t}{2} \right $ | 3. $-3t$ | 5. $(19, 4)$ |
| 6. $y^2 = a(x - 3a)$ | 7. $4x - 3y - 4 = 0$ | |
| 8. $y = \tan \alpha (x - a)$ | | |

Solved Examples

EXAMPLE 5.1

Prove that for a suitable point P on the axis of the parabola $y^2 = 4ax$, a chord AB through the point P can be drawn such that $[(1/AP^2) + (1/BP^2)]$ is the same for all positions of the chord.

Sol. Let the point P be $(p, 0)$ and the equation of the chord through P be

$$\frac{x - p}{\cos \theta} = \frac{y - 0}{\sin \theta} = r \quad (r \in \mathbb{R}) \quad (1)$$

Therefore, $(r \cos \theta + p, r \sin \theta)$ lies on the parabola $y^2 = 4ax$. So,

$$r^2 \sin^2 \theta - 4ar \cos \theta - 4ap = 0 \quad (2)$$

If $AP = r_1$ and $BP = -r_2$, then r_1 and r_2 are the roots of (2). Therefore,

$$r_1 + r_2 = \frac{4a \cos \theta}{\sin^2 \theta}, r_1 r_2 = \frac{-4ap}{\sin^2 \theta}$$

$$\begin{aligned} \text{Now, } \frac{1}{AP^2} + \frac{1}{BP^2} &= \frac{1}{r_1^2} + \frac{1}{r_2^2} \\ &= \frac{(r_1 + r_2)^2 - 2r_1 r_2}{r_1^2 r_2^2} \\ &= \frac{\cos^2 \theta}{p^2} + \frac{\sin^2 \theta}{2ap} \end{aligned}$$

Since $\frac{1}{AP^2} + \frac{1}{BP^2}$ should be independent of θ , we take $p = 2a$. Then,

$$\frac{1}{AP^2} + \frac{1}{BP^2} = \frac{1}{4a^2} (\cos^2 \theta + \sin^2 \theta) = \frac{1}{4a^2}$$

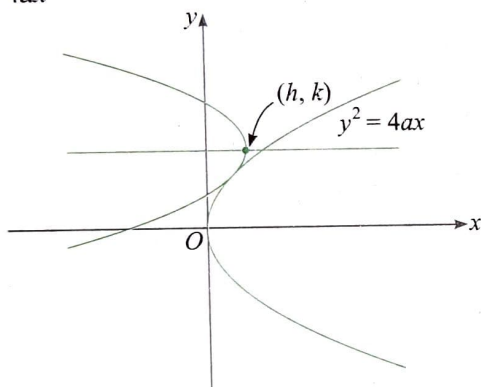
Hence, $\frac{1}{AP^2} + \frac{1}{BP^2}$ is independent of θ for all the positions of the chord if $P \equiv (2a, 0)$.

EXAMPLE 5.2

A parabola of latus-rectum 1 touches a fixed equal parabola. The axes of two parabolas are parallel. Then find the locus of the vertex of the moving parabola

Sol. Let the fixed parabola be

$$y^2 = 4ax \quad (1)$$



Since moving parabola and given parabola are equal, they have same latus rectum length.

So, let moving parabola be

$$(y - k)^2 = -4a(x - h) \quad (2)$$

On solving equations (1) and (2), we get

$$(y - k)^2 = -4a \left(\frac{y^2}{4a} - h \right)$$

$$\Rightarrow y^2 - 2ky + k^2 = -y^2 + 4ah$$

$$\Rightarrow 2y^2 - 2ky + k^2 - 4ah = 0$$

Since the two parabolas touch each other, above equation has two equal roots and therefore, discriminant is zero.

$$\Rightarrow 4k^2 - 8(k^2 - 4ah) = 0$$

$$\Rightarrow -4k^2 + 32ah = 0$$

$$\Rightarrow y^2 = 8ax, \text{ which is locus of the vertex of the moving parabola.}$$

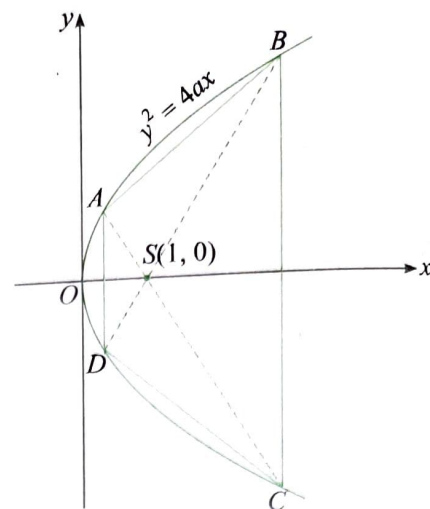
Thus locus is parabola.

EXAMPLE 5.3

Find the area of the trapezium whose vertices lie on the parabola $y^2 = 4x$ and its diagonals pass through $(1, 0)$ and having length

$$\frac{25}{4} \text{ units each.}$$

Sol. Clearly, trapezium $ABCD$ inscribed in the parabola will be isosceles as shown in the following figure.



Let point A be $(t^2, 2t)$.

$$\therefore AS = 1 + t^2 \text{ (focal distance)}$$

$$\therefore CS = AC - AS = \frac{25}{4} - (1 + t^2) = \frac{21}{4} - t^2$$

Also semi latus rectum is harmonic mean of segments AS and CS

$$\therefore \frac{1}{AS} + \frac{1}{CS} = \frac{1}{a}$$

$$\Rightarrow \frac{1}{1 + t^2} + \frac{1}{\frac{21}{4} - t^2} = 1$$

$$\Rightarrow \frac{1}{1 + t^2} + \frac{4}{21 - 4t^2} = 1$$

$$\Rightarrow 21 - 4t^2 + 4 + 4t^2 = 21 + 17t^2 - 4t^4$$

$$\Rightarrow 4t^4 - 17t^2 + 4 = 0$$

$$\Rightarrow (4t^2 - 1)(t^2 - 4) = 0$$

$$\Rightarrow t^2 = 4, \frac{1}{4}$$

$$\Rightarrow t = \pm 2, \pm \frac{1}{2}$$

Thus, $A \equiv (1/4, 1)$, $B \equiv (4, 4)$, $C \equiv (4, -4)$ and $D \equiv (1/4, -1)$

$$\therefore AD = 2$$

And $BC = 8$ and distance between AD and $BC = 4 - \frac{1}{4} = \frac{15}{4}$

$$\therefore \text{Area of trapezium } ABCD = \frac{1}{2} (2 + 8) \times \frac{15}{4} = \frac{75}{4} \text{ sq. units}$$

EXAMPLE 5.4

Find the radius of the largest circle, which passes through the focus of the parabola $y^2 = 4(x + y)$ and contained in it.

Sol. We have parabola

$$y^2 = 4(x + y) \text{ or } (y - 2)^2 = 4(x + 1)$$

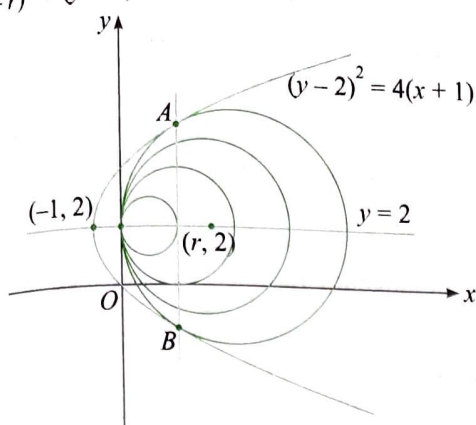
Vertex is $(-1, 2)$ and focus is $(0, 2)$.

For the largest circle passing through $(0, 2)$ contained in the parabola, circle must be symmetrical about the axis of the parabola.

So, centre lies on the line $y = 2$.

Let radius of the circle be r .
Then equation of circle is

$$(x-r)^2 + (y-2)^2 = r^2 \quad (2)$$



Solving (1) and (2), we get

$$(x-r)^2 + 4(x+1) = r^2$$

$$\Rightarrow x^2 + (4-2r)x + 4 = 0$$

Since largest circle touches parabola, abscissa of points A and B must be same.

So, above equation has equal roots.

$$\therefore (4-2r)^2 - 16 = 0$$

$$\Rightarrow 4-2r = \pm 4$$

$$\Rightarrow r = 4$$

EXAMPLE 5.5

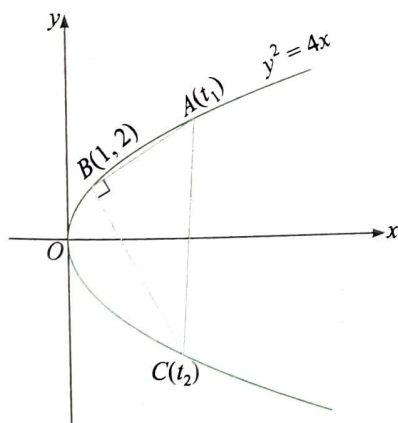
The vertices A , B , and C of a variable right triangle lie on a parabola $y^2 = 4x$. If the vertex B containing the right angle always remains at the point $(1, 2)$, then find the locus of the centroid of triangle ABC .

Sol. Let the points A and C be $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$, respectively. Then,

$$m_{AB} m_{BC} = -1$$

$$\text{or } \frac{2}{(t_1+1)} \cdot \frac{2}{(t_2+1)} = -1$$

$$\text{or } t_1 + t_2 + t_1 t_2 = -5 \quad (1)$$



Let the centroid of $\triangle ABC$ be (h, k) . Then,

$$h = \frac{t_1^2 + t_2^2 + 1}{3}$$

$$\text{and } k = \frac{2t_1 + 2t_2 + 2}{3} \quad (2)$$

From (2), we get

$$t_1 + t_2 = \frac{3k-2}{2} \quad (3)$$

$$\text{or } (t_1 + t_2)^2 - 2t_1 t_2 = 3h - 1$$

$$\text{or } \left(\frac{3k-2}{2}\right)^2 - 2t_1 t_2 = 3h - 1$$

Hence, from (1),

$$\frac{3k-2}{2} + \frac{\{(3k-2)/2\}^2 - (3h-1)}{2} = -5$$

Hence, the locus is

$$3y - 2 + \left(\frac{3y-2}{2}\right)^2 - (3x-1) + 10 = 0$$

EXAMPLE 5.6

Tangents are drawn to the parabola at three distinct points. Prove that the orthocentre of the triangle formed by points of intersection of tangents always lies on the directrix.

Sol. Let the three points on the parabola be $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$, $R(at_3^2, 2at_3)$.

Points of intersection of tangents drawn at these points are

$A(at_2 t_3, a(t_2 + t_3))$, $B(at_1 t_3, a(t_1 + t_3))$ and $C(at_1 t_2, a(t_1 + t_2))$.

To find the orthocentre of the triangle ABC , we need to find equations of altitudes of the triangle.

$$\text{Slope of } BC = \frac{a(t_2 - t_3)}{at_1(t_2 - t_3)} = \frac{1}{t_1}$$

Thus, slope of altitude through vertex A is $-t_1$.

Equation of altitude through A is

$$y - a(t_2 + t_3) = -t_1(x - at_2 t_3)$$

$$\text{or } y - a(t_2 + t_3) = -t_1 x + at_1 t_2 t_3 \quad (1)$$

Similarly, equation of altitude through B is

$$y - a(t_1 + t_3) = -t_2 x + at_1 t_2 t_3 \quad (2)$$

Subtracting (2) from (1), we get

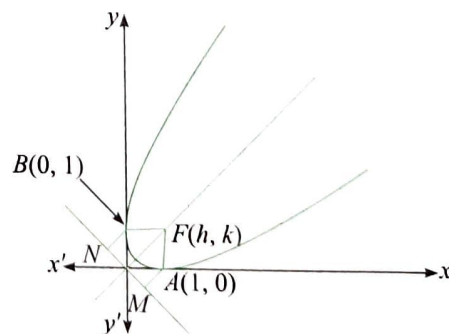
$$a(t_1 - t_2) = (t_2 - t_1)x \text{ or } x = -a.$$

Thus, orthocentre lies on the directrix.

EXAMPLE 5.7

A variable parabola touches the x - and the y -axis at $(1, 0)$ and $(0, 1)$, respectively. Then find the locus of the focus of the parabola.

Sol. Since the x - and the y -axes are two perpendicular tangents to the parabola and both meet at the origin, the directrix passes through the origin.



Let $y = mx$ be the directrix and (h, k) be the focus. Then,

$$FA = AM$$

$$\text{or } \sqrt{(h-1)^2 + k^2} = \left| \frac{m}{\sqrt{1+m^2}} \right| \quad (1)$$

$$\text{and } FB = BN$$

$$\text{or } \sqrt{h^2 + (k-1)^2} = \left| \frac{1}{\sqrt{1+m^2}} \right| \quad (2)$$

Squaring and adding (1) and (2), we get

$$(h-1)^2 + h^2 + k^2 + (k-1)^2 = 1$$

$$\text{or } 2x^2 - 2x + 2y^2 - 2y + 1 = 0,$$

which is the required locus.

EXAMPLE 5.8

Two lines are drawn at right angles, one being a tangent to $y^2 = 4ax$ and the other to $x^2 = 4by$. Then find the locus of their point of intersection.

$$\text{Sol. } y = mx + \frac{a}{m} \quad (1)$$

is a tangent to $y^2 = 4ax$ and

$$x = m_1 y + \frac{b}{m_1} \quad (2)$$

is a tangent to $x^2 = 4by$.

Lines (1) and (2) are perpendicular. Therefore,

$$m \times \frac{1}{m_1} = -1$$

$$\text{or } m_1 = -m$$

Let (h, k) be the point of intersection of (1) and (2). Then,

$$k = mh + \frac{a}{m} \text{ and } h = m_1 k + \frac{b}{m_1}$$

$$\text{or } k = mh + \frac{a}{m} \text{ and } h = -mk - \frac{b}{m}$$

$$\text{or } m^2 h - mk + a = 0 \text{ and } m^2 k + mh + b = 0$$

$$\text{or } \frac{m^2}{-kb - ah} = \frac{m}{ak - bh} = \frac{1}{h^2 + k^2} \quad [\text{By cross-multiplication}]$$

Eliminating m , we have

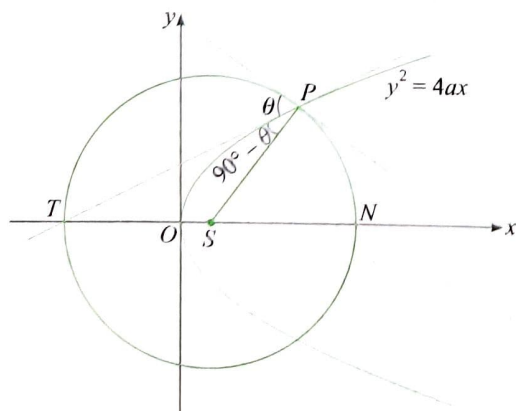
$$-(h^2 + k^2)(kb + ah) = (bh - ak)^2$$

$$\text{or } (x^2 + y^2)(ax + by) + (bx - ay)^2 = 0$$

EXAMPLE 5.9

The tangent and normal at the point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ meet the x -axis in T and N , respectively. Find the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle through P , T and N .

Sol.



Tangent and normal at $P(t)$ on the parabola meet the x -axis at T and N , respectively.

We know that $SP = SN = ST$

So, for circumcircle of triangle NPT , TN is diameter and S is the centre.

SP is normal to the circle. If θ is the angle between tangents at P to the parabola and circle then $(90^\circ - \theta)$ is the angle between PT and SP .

$$\text{Slope of tangent } PT = \frac{1}{t}$$

$$\text{Slope of } SP = \frac{2at}{at^2 - a} = \frac{2t}{t^2 - 1}$$

$$\therefore \tan(90^\circ - \theta) = \frac{\frac{1}{t} - \frac{2t}{t^2 - 1}}{1 + \frac{1}{t} \cdot \frac{2t}{t^2 - 1}} = \frac{1}{t}$$

$$\Rightarrow \cot \theta = \frac{1}{t}$$

$$\Rightarrow \tan \theta = t \Rightarrow \theta = \tan^{-1} t$$

EXAMPLE 5.10

The normals to the parabola $y^2 = 4ax$ at three points P , Q and R meet at A . If S is the focus, then prove that $SP \cdot SQ \cdot SR = aSA^2$.

Sol. Let the coordinates of A be (h, k) .

Equation of normal at $(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$

If this normal passes through point A , then

$$at^3 + (2a - h)t - k = 0 \quad (1)$$

If t_1, t_2 and t_3 are roots of (1), then

$$at^3 + (2a - h)t - k = a(t - t_1)(t - t_2)(t - t_3) \quad (2)$$

For these roots, points on the parabola are $P(t_1)$, $Q(t_2)$ and $R(t_3)$.

Now, put $t = i = \sqrt{-1}$ in equation (2).

$$\therefore -ai + (2a - h)i - k = a(i - t_1)(i - t_2)(i - t_3)$$

$$\Rightarrow |(a - h)i - k| = a|(i - t_1)(i - t_2)(i - t_3)|$$

$$\Rightarrow \sqrt{(a - h)^2 + k^2} = a\sqrt{1 + t_1^2}\sqrt{1 + t_2^2}\sqrt{1 + t_3^2}$$

$$\Rightarrow \sqrt{a}\sqrt{(a - h)^2 + k^2} = \sqrt{a + at_1^2}\sqrt{a + at_2^2}\sqrt{a + at_3^2}$$

Squaring both sides, we get

$$aSA^2 = SP \cdot SQ \cdot SR$$

EXAMPLE 5.11

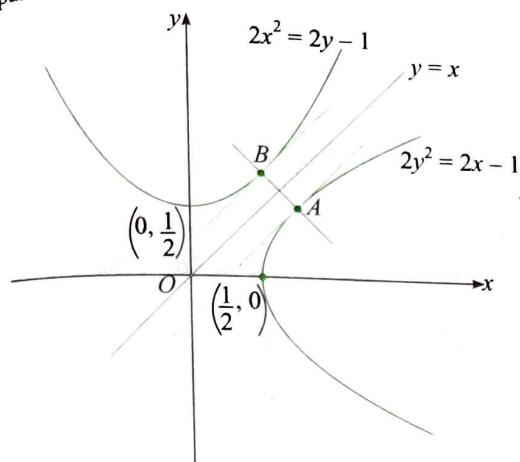
Find the shortest distance between the parabolas $2y^2 = 2x - 1$ and $2x^2 = 2y - 1$.

Sol. We observe that one of the equations of parabola is obtained by interchanging x and y in the other equation of parabola.

So, given parabolas are symmetrical about the line $y = x$.

For parabola, $y^2 = (x - 1/2)$, vertex is $(1/2, 0)$

Now, we know that shortest distance occurs along common normal.
Let common normal line meet parabolas at points A and B .
So, tangents to parabolas at points A and B are parallel and they are also parallel to the line $y = x$.



Differentiating $2y^2 = 2x - 1$ w.r.t. x , we get

$$2y \frac{dy}{dx} = 1$$

or $\frac{dy}{dx} = \frac{1}{2y} = 1$ (since tangent is parallel to $y = x$)

$$\Rightarrow y = \frac{1}{2}$$

So, $x = \frac{3}{4}$

So, $A \equiv \left(\frac{3}{4}, \frac{1}{2}\right)$ and $B \equiv \left(\frac{1}{2}, \frac{3}{4}\right)$

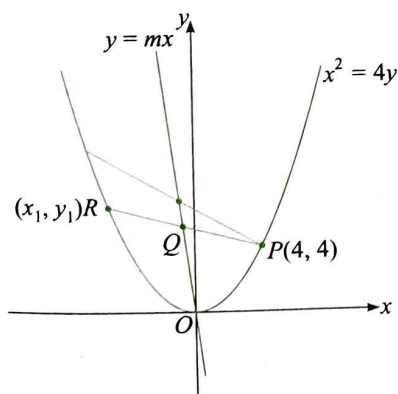
Hence, required shortest distance,

$$\begin{aligned} AB &= \sqrt{\left(\frac{3}{4} - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{3}{4}\right)^2} \\ &= \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2\sqrt{2}} \end{aligned}$$

EXAMPLE 5.12

If two chords drawn from the point $A(4, 4)$ to the parabola $x^2 = 4y$ are bisected by line $y = mx$, then find the range of values of m .

Sol.



Point $(4, 4)$ lies on the given parabola $x^2 = 4y$.

Any point on the line $y = mx$ is $Q(\alpha, m\alpha)$.

Let the chord PR be bisected by Q .

$$\therefore \alpha = \frac{4 + x_1}{2} \Rightarrow x_1 = 2\alpha - 4$$

and $m\alpha = \frac{4 + y_1}{2} \Rightarrow y_1 = 2m\alpha - 4$

Now, Q lies on the curve.

$$\therefore (2\alpha - 4)^2 = 4(2m\alpha - 4)$$

$$\Rightarrow 4\alpha^2 + 16 - 16\alpha = 8(m\alpha - 2)$$

$$\Rightarrow 4\alpha^2 - 8(2 + m)\alpha + 32 = 0$$

Since it is possible to draw two chords which get bisected at point Q , then above equation has two distinct real roots.

$$\therefore \text{Discriminant, } D > 0$$

$$\Rightarrow (8(2 + m))^2 - 4 \times 4 \times 32 > 0$$

$$\Rightarrow (2 + m)^2 - 8 > 0$$

$$\Rightarrow 2 + m > 2\sqrt{2} \text{ or } 2 + m < -2\sqrt{2}$$

$$\Rightarrow m > 2\sqrt{2} - 2 \text{ or } m < -2\sqrt{2} - 2$$

EXAMPLE 5.13

Tangent is drawn at point (p, q) on the parabola $y^2 = 4ax$. Tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$, such that the chords of contact pass through a fixed point (r, s) . Prove that $rq^2 = -4ps^2$.

Sol. Equation of tangent to given parabola at point (p, q) on it is $qy = 2a(x + p)$

Let (x_1, y_1) be a point on this tangent. Then

$$qy_1 = 2a(x_1 + p) \quad (1)$$

Equation of chord of contact to the given circle w.r.t. point (x_1, y_1) is

$$xx_1 + yy_1 = a^2 \quad (2)$$

Given that this passes through point (r, s) .

$$\therefore rx_1 + sy_1 = a^2$$

From (1), putting the value of y_1 in (2), we get

$$xx_1 + y \frac{2a}{q}(x_1 + p) = a^2$$

$$\therefore x_1 \left(x + \frac{2ay}{q} \right) + \left(\frac{2ap}{q} y - a^2 \right) = 0$$

$$\Rightarrow x_1(qx + 2ay) + (2apy - a^2q) = 0$$

This is the variable line which passes through the point of intersection of lines $qx + 2ay = 0$ and $2apy = a^2q$.

$$\Rightarrow x = -\frac{a^2}{p}, y = \frac{aq}{2p}$$

$$\therefore r = -\frac{a^2}{p} \text{ and } s = \frac{aq}{2p}$$

$$\Rightarrow 4ps^2 = -rq^2$$

(Eliminating a)

EXAMPLE 5.14

Find the length of the shortest normal chord of the parabola $y^2 = 4ax$.

Sol. Let AB be a normal chord where $A \equiv (at^2, 2at)$ and $B \equiv (at_1^2, 2at_1)$.

We have $t_1 = -t - \frac{2}{t}$.

$$\begin{aligned} \text{Now, } AB^2 &= [a(t^2 - t_1^2)]^2 + 4a^2(t - t_1)^2 \\ &= a^2(t - t_1)^2 [(t + t_1)^2 + 4] \\ &= a^2 \left(t + t + \frac{2}{t} \right)^2 \left(\frac{4}{t^2} + 4 \right) \\ &= \frac{16a^2(1+t^2)^3}{t^4} \end{aligned}$$

For minimum length of chord AB ,

$$\begin{aligned} \frac{d(AB^2)}{dt} &= 16a^2 \left(\frac{t^4[3(1+t^2)^2 \cdot 2t] - (1+t^2)^3 \cdot 4t^3}{t^8} \right) \\ &= 32a^2(1+t^2)^2 \left(\frac{3t^2 - 2 - 2t^2}{t^5} \right) \\ &= \frac{32a^2(1+t^2)^2}{t^5} (t^2 - 2) \end{aligned}$$

For $\frac{d(AB^2)}{dt} = 0$, $t = \sqrt{2}$ for which AB^2 is minimum.

$$\text{Thus, } AB_{\min} = \frac{4a}{2} (1+2)^{3/2} = 6\sqrt{3}a \text{ units}$$

1. Which one of the following equations represent parametric equation to a parabolic curve?

- (1) $x = 3 \cos t; y = 4 \sin t$
 (2) $x^2 - 2 = 2 \cos t; y = 4 \cos^2 \frac{t}{2}$
 (3) $\sqrt{x} = \tan t; \sqrt{y} = \sec t$
 (4) $x = \sqrt{1 - \sin t}; y = \sin \frac{t}{2} + \cos \frac{t}{2}$

2. A point $P(x, y)$ moves in the xy -plane such that $x = a \cos^2 \theta$ and $y = 2a \sin \theta$, where θ is a parameter. The locus of the point P is a/an

- (1) circle (2) ellipse
 (3) unbounded parabola (4) part of the parabola

3. A line L passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the parabola at two distinct points. If m is the slope of the line L , then

- (1) $-1 < m < 1$ (2) $m < -1$ or $m > 1$
 (3) $m \in \mathbb{R}$ (4) none of these

4. The circle $x^2 + y^2 + 2\lambda x = 0$, $\lambda \in \mathbb{R}$, touches the parabola $y^2 = 4x$ externally. Then,

- (1) $\lambda > 0$ (2) $\lambda < 0$
 (3) $\lambda > 1$ (4) none of these

5. A set of parallel chords of the parabola $y^2 = 4ax$ have their midpoints on

- (1) any straight line through the vertex
 (2) any straight line through the focus
 (3) a straight line parallel to the axis
 (4) another parabola

6. If the points $A(1, 3)$ and $B(5, 5)$ lying on a parabola are equidistant from its focus, then the slope of the directrix is

- (1) $\frac{1}{2}$ (2) $-\frac{1}{2}$ (3) 2 (4) -2

7. The radius of the circle whose centre is $(-4, 0)$ and which cuts the parabola $y^2 = 8x$ at A and B such that the common chord AB subtends a right angle at the vertex of the parabola is equal to

- (1) $4\sqrt{13}$ (2) $3\sqrt{5}$ (3) $3\sqrt{2}$ (4) $2\sqrt{5}$

8. The circle $x^2 + y^2 = 5$ meets the parabola $y^2 = 4x$ at P and Q . Then the length PQ is equal to

- (1) 2 (2) $2\sqrt{2}$
 (3) 4 (4) none of these

9. If y_1, y_2 , and y_3 are the ordinates of the vertices of a triangle inscribed in the parabola $y^2 = 4ax$, then its area is

- (1) $\frac{1}{2a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$
 (2) $\frac{1}{4a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$

(3) $\frac{1}{8a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$

(4) none of these

10. Let P be the point $(1, 0)$ and Q be a point on the locus $y^2 = 8x$. The locus of the midpoint of PQ is

- (1) $y^2 + 4x + 2 = 0$ (2) $y^2 - 4x + 2 = 0$
 (3) $x^2 - 4y + 2 = 0$ (4) $x^2 + 4y + 2 = 0$

11. An equilateral triangle SAB is inscribed in the parabola $y^2 = 4ax$ having its focus at S . If chord AB lies towards the left of S , then the side length of this triangle is

- (1) $2a(2 - \sqrt{3})$ (2) $4a(2 - \sqrt{3})$
 (3) $a(2 - \sqrt{3})$ (4) $8a(2 - \sqrt{3})$

12. $C(0, 1)$ is the center of the circle with radius unity. P is the parabola $y = ax^2$. The set of values of a for which they meet at a point other than the origin is

- (1) $a > 0$ (2) $a \in (0, 1/2)$
 (3) $(1/4, 1/2)$ (4) $(1/2, \infty)$

13. $P(x, y)$ is a variable point on the parabola $y^2 = 4ax$ and $Q(x + c, y + c)$ is another variable point, where c is a constant. The locus of the midpoint of PQ is a/an

- (1) parabola (2) ellipse
 (3) hyperbola (4) circle

14. AB is a chord of the parabola $y^2 = 4ax$ with vertex A . BC is drawn perpendicular to AB meeting the axis at C . The projection of BC on the axis of the parabola is

- (1) a (2) $2a$ (3) $4a$ (4) $8a$

15. Set of values of α for which the point $(\alpha, 1)$ lies inside the circle $x^2 + y^2 - 4 = 0$ and parabola $y^2 = 4x$ is

- (1) $|\alpha| < \sqrt{3}$ (2) $|\alpha| < 2$
 (3) $\frac{1}{4} < \alpha < \sqrt{3}$ (4) none of these

16. If X is the foot of the directrix on axis of the parabola. PP' is a double ordinate of the curve and PX meets the curve again in Q . Then $P'Q$ passes through fixed point which is

- (1) vertex
 (2) focus
 (3) midpoint of vertex and focus
 (4) none of these

17. A water jet from a fountain reaches its maximum height of 4 m at a distance 0.5 m from the vertical passing through the point O of water outlet. The height of the jet above the horizontal OX at a distance of 0.75 m from the point O is

- (1) 5 m (2) 6 m (3) 3 m (4) 7 m

18. Area of the triangle formed by the vertex, focus and one end of latusrectum of the parabola $(x + 2)^2 = -12(y - 1)$ is

- (1) 36 (2) 18
 (3) 9 (4) 6

19. The locus of the vertex of the family of parabolas $y = \frac{a^3 x^2}{3} + \frac{a^2 x}{2} - 2a$ is
 (1) $xy = 105/64$ (2) $xy = 3/4$
 (3) $xy = 35/16$ (4) $xy = 64/105$
20. Two parabolas have the same focus. If their directrices are the x - and the y -axis, respectively, then the slope of their common chord is
 (1) ± 1 (2) $4/3$
 (3) $3/4$ (4) none of these
21. The locus of the point $(\sqrt{3}h, \sqrt{3}k + 2)$ if it lies on the line $x - y - 1 = 0$ is
 (1) a straight line (2) a circle
 (3) a parabola (4) none of these
22. A circle touches the x -axis and also touches the circle with center $(0, 3)$ and radius 2 externally. The locus of the center of the circle is
 (1) a circle (2) an ellipse
 (3) a parabola (4) a hyperbola
23. If parabolas $y^2 = \lambda x$ and $25[(x - 3)^2 + (y + 2)^2] = (3x - 4y - 2)^2$ are equal, then the value of λ is
 (1) 9 (2) 3 (3) 7 (4) 6
24. The length of the latus rectum of the parabola whose focus is $\left(\frac{u^2}{2g} \sin 2\alpha, -\frac{u^2}{2g} \cos 2\alpha\right)$ and directrix is $y = u^2/2g$ is
 (1) $\frac{u^2}{g} \cos^2 \alpha$ (2) $\frac{u^2}{g} \cos 2\alpha$
 (3) $\frac{2u^2}{g} \cos 2\alpha$ (4) $\frac{2u^2}{g} \cos^2 \alpha$
25. The graph of the curve $x^2 + y^2 - 2xy - 8x - 8y + 32 = 0$ falls wholly in the
 (1) first quadrant (2) second quadrant
 (3) third quadrant (4) none of these
26. The vertex of the parabola whose parametric equation is $x = t^2 - t + 1, y = t^2 + t + 1; t \in R$, is
 (1) $(1, 1)$ (2) $(2, 2)$
 (3) $(1/2, 1/2)$ (4) $(3, 3)$
27. If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at P and Q , then $AP \cdot AQ$ is equal to [where $A \equiv (\sqrt{3}, 0)$]
 (1) $\frac{2(\sqrt{3} + 2)}{3}$ (2) $\frac{4\sqrt{3}}{2}$
 (3) $\frac{4(2 - \sqrt{2})}{3}$ (4) $\frac{4(\sqrt{3} + 2)}{3}$
28. A line is drawn from $A(-2, 0)$ to intersect the curve $y^2 = 4x$ at P and Q in the first quadrant such that $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$. Then the slope of the line is always
 (1) $> \sqrt{3}$ (2) $< 1/\sqrt{3}$ (3) $> \sqrt{2}$ (4) $> 1/\sqrt{3}$
29. The length of the chord of the parabola $y^2 = x$ which is bisected at the point $(2, 1)$ is
 (1) $2\sqrt{3}$ (2) $4\sqrt{3}$ (3) $3\sqrt{2}$ (4) $2\sqrt{5}$
30. If a line $y = 3x + 1$ cuts the parabola $x^2 - 4x - 4y + 20 = 0$ at A and B , then the tangent of the angle subtended by line segment AB at the origin is
 (1) $8\sqrt{3}/205$ (2) $8\sqrt{3}/209$
 (3) $8\sqrt{3}/215$ (4) none of these
31. If P is a point on the parabola $y^2 = 3(2x - 3)$ and M is the foot perpendicular drawn from P on the directrix of the parabola, then the length of each side of the equilateral triangle SMP , where S is the focus of the parabola, is
 (1) 2 (2) 4 (3) 6 (4) 8
32. A parabola $y = ax^2 + bx + c$ crosses the x -axis at $(\alpha, 0)$ and $(\beta, 0)$ both to the right of the origin. A circle also passes through these two points. The length of a tangent from the origin to the circle is
 (1) $\sqrt{bc/a}$ (2) ac^2 (3) b/a (4) $\sqrt{c/a}$
33. The number of common chords of the parabolas $x = y^2 - 6y + 11$ and $y = x^2 - 6x + 11$ is
 (1) 1 (2) 2 (3) 4 (4) 6
34. Two parabolas have focus $(3, -2)$. Their directrices are the x -axis and the y -axis, respectively. Then the slope of their common chord is
 (1) -1 (2) $-1/2$
 (3) $-\sqrt{3}/2$ (4) none of these
35. PSQ is a focal chord of a parabola whose focus is S and vertex is A . PA and QA are produced to meet the directrix in R and T , respectively. Then $\angle RST =$
 (1) 30° (2) 90°
 (3) 60° (4) 45°
36. If PSQ is a focal chord of the parabola $y^2 = 8x$ such that $SP = 6$, then the length of SQ is
 (1) 6 (2) 4
 (3) 3 (4) none of these
37. The triangle PQR of area A is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modulus of the difference of the ordinates of the points Q and R is
 (1) $A/2a$ (2) A/a (3) $2A/a$ (4) $4A/a$
38. If A_1B_1 and A_2B_2 are two focal chords of the parabola $y^2 = 4ax$, then the chords A_1A_2 and B_1B_2 intersect on
 (1) directrix (2) axis
 (3) tangent at vertex (4) none of these
39. If a and c are the lengths of segments of any focal chord of the parabola $y^2 = 2bx, (b > 0)$, then the roots of the equation $ax^2 + bx + c = 0$ are
 (1) real and distinct (2) real and equal
 (3) imaginary (4) none of these
40. If $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$, then
 (1) $c = \frac{a}{m}$ (2) $c = am + \frac{a}{m}$
 (3) $c = a + \frac{a}{m}$ (4) none of these

41. The area of the triangle formed by the tangent and the normal to the parabola $y^2 = 4ax$, both drawn at the same end of the latus rectum, and the axis of the parabola is
 (1) $2\sqrt{2}a^2$ (2) $2a^2$
 (3) $4a^2$ (4) none of these
42. Parabolas $y^2 = 4a(x - c_1)$ and $x^2 = 4a(y - c_2)$, where c_1 and c_2 are variable, are such that they touch each other. The locus of their point of contact is
 (1) $xy = 2a^2$ (2) $xy = 4a^2$
 (3) $xy = a^2$ (4) none of these
43. Let $y = f(x)$ be a parabola, having its axis parallel to the y -axis, which is touched by the line $y = x$ at $x = 1$. Then,
 (1) $2f(0) = 1 - f'(0)$ (2) $f(0) + f'(0) + f''(0) = 1$
 (3) $f'(1) = 1$ (4) $f'(0) = f'(1)$
44. If $y = 2x - 3$ is a tangent to the parabola $y^2 = 4a\left(x - \frac{1}{3}\right)$, then a is equal to
 (1) $\frac{22}{3}$ (2) -1 (3) $\frac{14}{3}$ (4) $-\frac{14}{3}$
45. The locus of the center of a circle which cuts orthogonally the parabola $y^2 = 4x$ at $(1, 2)$ will pass through
 (1) $(3, 4)$ (2) $(4, 3)$ (3) $(5, 3)$ (4) $(2, 4)$
46. If the parabola $y = ax^2 - 6x + b$ passes through $(0, 2)$ and has its tangent at $x = 3/2$ parallel to the x -axis, then
 (1) $a = 2, b = -2$ (2) $a = 2, b = 2$
 (3) $a = -2, b = 2$ (4) $a = -2, b = -2$
47. Double ordinate AB of the parabola $y^2 = 4ax$ subtends an angle $\pi/2$ at the focus of the parabola. Then the tangents drawn to the parabola at A and B will intersect at
 (1) $(-4a, 0)$ (2) $(-2a, 0)$
 (3) $(-3a, 0)$ (4) none of these
48. The tangents to $y^2 = 4ax$ make angles θ_1 and θ_2 with the x -axis. If $\cos \theta_1 \cos \theta_2 = \lambda$, then the locus of their point of intersection is
 (1) $x^2 = \lambda^2[(x - a)^2 + 4y^2]$ (2) $x^2 = \lambda^2[(x + a)^2 + y^2]$
 (3) $x^2 = \lambda^2[(x - a)^2 + y^2]$ (4) $4x^2 = \lambda^2[(x + a)^2 + y^2]$
49. A tangent is drawn to the parabola $y^2 = 4ax$ at the point P whose abscissa lies in the interval $(1, 4)$. The maximum possible area of the triangle formed by the tangent at P , the ordinates of the point P , and the x -axis is equal to
 (1) 8 (2) 16 (3) 24 (4) 32
50. The straight lines joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P intersect at R . Then the equation of the locus of R is
 (1) $x^2 + 2y^2 - ax = 0$ (2) $2x^2 + y^2 - 2ax = 0$
 (3) $2x^2 + 2y^2 - ay = 0$ (4) $2x^2 + y^2 - 2ay = 0$
51. Through the vertex O of the parabola $y^2 = 4ax$, two chords OP and OQ are drawn and the circles on OP and OQ as diameters intersect at R . If θ_1, θ_2 , and ϕ are the angles made with the axis by the tangents at P and Q on the parabola and by OR , then the value of $\cot \theta_1 + \cot \theta_2$ is
 (1) $-2 \tan \phi$ (2) $-2 \tan(\pi - \phi)$
 (3) 0 (4) $2 \cot \phi$
52. AB is a double ordinate of the parabola $y^2 = 4ax$. Tangents drawn to the parabola at A and B meet the y -axis at A_1 and B_1 , respectively. If the area of trapezium AA_1B_1B is equal to $24a^2$, then the angle subtended by A_1B_1 at the focus of the parabola is equal to
 (1) $2 \tan^{-1}(3)$ (2) $\tan^{-1}(3)$
 (3) $2 \tan^{-1}(2)$ (4) $\tan^{-1}(2)$
53. If the locus of the middle of point of contact of tangent drawn to the parabola $y^2 = 8x$ and the foot of perpendicular drawn from its focus to the tangents is a conic, then the length of latus rectum of this conic is
 (1) $9/4$ (2) 9 (3) 18 (4) $9/2$
54. If the bisector of angle APB , where PA and PB are the tangents to the parabola $y^2 = 4ax$, is equally inclined to the coordinate axes, then the point P lies on the
 (1) tangent at vertex of the parabola
 (2) directrix of the parabola
 (3) circle with center at the origin and radius a
 (4) the line of latus rectum
55. From a point $A(t)$ on the parabola $y^2 = 4ax$, a focal chord and a tangent are drawn. Two circles are drawn in which one circle is drawn taking focal chord AB as diameter and other is drawn by taking the intercept of tangent between point A and point P on the directrix as diameter. Then the common chord of the circles is
 (1) the line joining focus and P
 (2) the line joining focus and A
 (3) tangent to the parabola at point A
 (4) none of these
56. The point of intersection of the tangents of the parabola $y^2 = 4x$ drawn at the end points of the chord $x + y = 2$ lies on
 (1) $x - 2y = 0$ (2) $x + 2y = 0$
 (3) $y - x = 0$ (4) $x + y = 0$
57. The angle between the tangents to the parabola $y^2 = 4ax$ at the points where it intersects with the line $x - y - a = 0$ is
 (1) $\pi/3$ (2) $\pi/4$ (3) $\pi/6$ (4) $\pi/2$
58. $y = x + 2$ is any tangent to the parabola $y^2 = 8x$. The point P on this tangent is such that the other tangent from it which is perpendicular to it is
 (1) $(2, 4)$ (2) $(-2, 0)$ (3) $(-1, 1)$ (4) $(2, 0)$
59. If $y = m_1x + c$ and $y = m_2x + c$ are two tangents to the parabola $y^2 + 4a(x + a) = 0$, then
 (1) $m_1 + m_2 = 0$ (2) $1 + m_1 + m_2 = 0$
 (3) $m_1m_2 - 1 = 0$ (4) $1 + m_1m_2 = 0$
60. The angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ is
 (1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{4}$

61. Two mutually perpendicular tangents of the parabola $y^2 = 4ax$ meet the axis at P_1 and P_2 . If S is the focus of the parabola, then $\frac{1}{SP_1} + \frac{1}{SP_2}$ is equal to
- (1) $\frac{1}{2a}$ (2) $\frac{1}{a}$ (3) $\frac{2}{a}$ (4) $\frac{4}{a}$
62. Radius of the circle that passes through the origin and touches the parabola $y^2 = 4ax$ at the point $(a, 2a)$ is
- (1) $\frac{5}{\sqrt{2}}a$ (2) $2\sqrt{2}a$ (3) $\sqrt{\frac{5}{2}}a$ (4) $\frac{3}{\sqrt{2}}a$
63. The mirror image of the parabola $y^2 = 4x$ in the tangent to the parabola at the point $(1, 2)$ is
- (1) $(x-1)^2 = 4(y+1)$ (2) $(x+1)^2 = 4(y+1)$
 (3) $(x+1)^2 = 4(y-1)$ (4) $(x-1)^2 = 4(y-1)$
64. Consider the parabola $y^2 = 4x$. Let $A \equiv (4, -4)$ and $B \equiv (9, 6)$ be two fixed points on the parabola. Let C be a moving point on the parabola between A and B such that the area of the triangle ABC is maximum. Then the coordinates of C are
- (1) $(1/4, 1)$ (2) $(4, 4)$
 (3) $(3, 2\sqrt{3})$ (4) $(3, -2\sqrt{3})$
65. A line of slope λ ($0 < \lambda < 1$) touches the parabola $y + 3x^2 = 0$ at P . If S is the focus and M is the foot of the perpendicular of directrix from P , then $\tan \angle MPS$ equals
- (1) 2λ (2) $\frac{2\lambda}{-1 + \lambda^2}$
 (3) $\frac{1 - \lambda^2}{1 + \lambda^2}$ (4) none of these
66. The tangent at any point P on the parabola $y^2 = 4ax$ intersects the y -axis at Q . The tangent to the circumcircle of triangle PQS (S is the focus) at Q is
- (1) a line parallel to x -axis
 (2) y -axis
 (3) a line parallel to y -axis
 (4) none of these
67. If $P(t^2, 2t)$, $t \in [0, 2]$, is an arbitrary point on the parabola $y^2 = 4x$, Q is the foot of perpendicular from focus S on the tangent at P , then the maximum area of $\triangle PQS$ is
- (1) 1 (2) 2 (3) $5/16$ (4) 5
68. The minimum area of circle which touches the parabolas $y = x^2 + 1$ and $y^2 = x - 1$ is
- (1) $\frac{9\pi}{16}$ sq. unit (2) $\frac{9\pi}{32}$ sq. unit
 (3) $\frac{9\pi}{8}$ sq. unit (4) $\frac{9\pi}{4}$ sq. unit
69. If the tangents and normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) , respectively, then
- (1) $x_1 = y_2$ (2) $x_1 = y_1$ (3) $y_1 = y_2$ (4) $x_2 = y_1$
70. At what point on the parabola $y^2 = 4x$ the normal makes equal angle with the axes?
- (1) $(4, 4)$ (2) $(9, 6)$ (3) $(4, -4)$ (4) $(1, \pm 2)$
71. If $2x + y + \lambda = 0$ is a normal to the parabola $y^2 = -8x$, then λ is
- (1) 12 (2) -12 (3) 24 (4) -24
72. If two normals to the parabola $y^2 = 4ax$ intersect at right angles, then the chord joining their feet passes through a fixed point whose coordinates are
- (1) $(-2a, 0)$ (2) $(a, 0)$
 (3) $(2a, 0)$ (4) none of these
73. The equation of the line that passes through $(10, -1)$ and is perpendicular to $y = \frac{x^2}{4} - 2$ is
- (1) $4x + y = 39$ (2) $2x + y = 19$
 (3) $x + y = 9$ (4) $x + 2y = 8$
74. Tangent and normal drawn to a parabola $y^2 = 4ax$ at $A(at^2, 2at)$, $t \neq 0$ meet the x -axis at points B and D , respectively. If the rectangle $ABCD$ is completed, then the locus of C is
- (1) $y = 2a$ (2) $x = 2a - \frac{y^2}{4a}$
 (3) $x = 2a$ (4) none of these
75. The radius of the circle touching the parabola $y^2 = x$ at $(1, 1)$ and having the directrix of $y^2 = x$ as its normal is
- (1) $5\sqrt{5}/8$ (2) $10\sqrt{5}/3$
 (3) $5\sqrt{5}/4$ (4) none of these
76. If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$, then
- (1) $|b| > 1/2\sqrt{2}$ (2) $|b| < 1/2\sqrt{2}$
 (3) $|b| > 1/\sqrt{2}$ (4) $|b| < 1/\sqrt{2}$
77. The maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ is equal to
- (1) 3 (2) 4 (3) 6 (4) 5
78. If line PQ , whose equation is $y = 2x + k$, is a normal to the parabola whose vertex is $(-2, 3)$ and the axis parallel to the x -axis with latus rectum equal to 2, then the value of k is (focus lies to the right of vertex)
- (1) $58/8$ (2) $50/8$ (3) 1 (4) -1
79. $\min[(x_1 - x_2)^2 + (3 + \sqrt{1 - x_1^2} - \sqrt{4x_2^2})^2]$, $\forall x_1, x_2 \in R$, is
- (1) $4\sqrt{5} + 1$ (2) $3 - 2\sqrt{2}$ (3) $\sqrt{5} + 1$ (4) $\sqrt{5} - 1$
80. If the normals to the parabola $y^2 = 4ax$ at three points $(ap^2, 2ap)$, $(aq^2, 2aq)$, and $(ar^2, 2ar)$ are concurrent, then the common root of equations $px^2 + qx + r = 0$ and $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ is
- (1) p (2) q (3) r (4) 1
81. Normals AO , AA_1 , and AA_2 are drawn to the parabola $y^2 = 8x$ from the point $A(h, 0)$. If triangle OA_1A_2 is equilateral, then the possible value of h is
- (1) 26 (2) 24
 (3) 28 (4) none of these
82. If the normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum meet the parabola at Q and Q' , then QQ' is
- (1) $10a$ (2) $4a$ (3) $20a$ (4) $12a$

83. From a point $(\sin \theta, \cos \theta)$, if three normals can be drawn to the parabola $y^2 = 4ax$, then the value of a is

- (1) $(1/2, 1)$ (2) $[-1/2, 0)$
(3) $[1/2, 1]$ (4) $(-1/2, 0) \cup (0, 1/2)$

84. If the normals at points $P(t_1)$ and $Q(t_2)$ on the parabola meet on the same parabola, then

- (1) $t_1 t_2 = -1$ (2) $t_2 = -t_1 - \frac{2}{t_1}$
(3) $t_1 t_2 = 1$ (4) $t_1 t_2 = 2$

85. If the normal to the parabola $y^2 = 4ax$ at P meets the curve again at Q and if PQ and the normal at Q make angle α and β , respectively, with the x -axis, then $\tan \alpha (\tan \alpha + \tan \beta)$ has the value equal to

- (1) 0 (2) -2 (3) -1/2 (4) -1

86. PQ is a normal chord of the parabola $y^2 = 4ax$ at P , A being the vertex of the parabola. Through P , a line is drawn parallel to AQ meeting the x -axis at R . Then the line length of AR is

- (1) equal to the length of the latus rectum
(2) equal to the focal distance of the point P
(3) equal to twice the focal distance of the point P
(4) equal to the distance of the point P from the directrix

87. P, Q , and R are the feet of the normals drawn to a parabola $(y-3)^2 = 8(x-2)$. A circle cuts the above parabola at points P, Q, R , and S . Then this circle always passes through the point

- (1) $(2, 3)$ (2) $(3, 2)$ (3) $(0, 3)$ (4) $(2, 0)$

88. Normals at two points (x_1, y_1) and (x_2, y_2) of the parabola $y^2 = 4x$ meet again on the parabola, where $x_1 + x_2 = 4$. Then $|y_1 + y_2|$ is equal to

- (1) $\sqrt{2}$ (2) $2\sqrt{2}$
(3) $4\sqrt{2}$ (4) none of these

89. The end points of two normal chords of a parabola are concyclic. Then the tangents at the feet of the normals will intersect at

- (1) tangent at vertex of the parabola
(2) axis of the parabola
(3) directrix of the parabola
(4) none of these

90. If normal at point P on the parabola $y^2 = 4ax$, ($a > 0$), meets it again at Q in such a way that OQ is of minimum length, where O is the vertex of parabola, then ΔOPQ is

- (1) a right-angled triangle
(2) an obtuse-angled triangle
(3) an acute-angled triangle
(4) none of these

91. The set of points on the axis of the parabola $(x-1)^2 = 8(y+2)$ from where three distinct normals can be drawn to the parabola is the set (h, k) of points satisfying

- (1) $h > 2$ (2) $h > 1$
(3) $k > 2$ (4) none of these

92. Tangent and normal are drawn at the point $P \equiv (16, 16)$ of the parabola $y^2 = 16x$ which cut the axis of the parabola at the points A and B , respectively. If the center of the circle through P, A , and B is C , then the angle between PC and the axis of x is

- (1) $\tan^{-1} \frac{1}{2}$ (2) $\tan^{-1} 2$
(3) $\tan^{-1} \frac{3}{4}$ (4) $\tan^{-1} \frac{4}{3}$

93. From the point $(15, 12)$, three normals are drawn to the parabola $y^2 = 4x$. Then the centroid of triangle formed by three co-normals points is

- (1) $(16/3, 0)$ (2) $(4, 0)$
(3) $(26/3, 0)$ (4) $(6, 0)$

94. The line $x - y = 1$ intersects the parabola $y^2 = 4x$ at A and B . Normals at A and B intersect at C . If D is the point at which line CD is normal to the parabola, then the coordinates of D are

- (1) $(4, -4)$ (2) $(4, 4)$
(3) $(-4, -4)$ (4) none of these

95. If normals are drawn from a point $P(h, k)$ to the parabola $y^2 = 4ax$, then the sum of the intercepts which the normals cut-off from the axis of the parabola is

- (1) $(h + a)$ (2) $3(h + a)$
(3) $2(h + a)$ (4) none of these

Multiple Correct Answers Type

1. If the focus of the parabola $x^2 - ky + 3 = 0$ is $(0, 2)$, then a values of k is (are)

- (1) 4 (2) 6 (3) 3 (4) 2

2. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then value(s) of k is/are

- (1) -8 (2) 1/8 (3) 1/4 (4) 4

3. The extremities of latus rectum of a parabola are $(1, 1)$ and $(1, -1)$. Then the equation of the parabola can be

- (1) $y^2 = 2x - 1$ (2) $y^2 = 1 - 2x$
(3) $y^2 = 3 - 2x$ (4) $y^2 = 2x - 4$

4. The value(s) of a for which two curves $y = ax^2 + ax + \frac{1}{24}$ and $x = ay^2 + ay + \frac{1}{24}$ touch each other is/are

- (1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) $\frac{3}{2}$ (4) $\frac{1}{2}$

5. In which of the following cases, a unique parabola will be obtained?

- (1) Focus and equation of tangent at vertex are given.
(2) Focus and vertex are given.
(3) Equation of directrix and vertex are given.
(4) Equation of directrix and equation of tangent at vertex are given.

6. A quadrilateral is inscribed in a parabola. Then,

- (1) the quadrilateral may be cyclic
- (2) diagonals of the quadrilateral may be equal
- (3) all possible pairs of adjacent sides may be perpendicular
- (4) none of these

7. The locus of the midpoint of the focal distance of a variable point moving on the parabola $y^2 = 4ax$ is a parabola whose

- (1) latus rectum is half the latus rectum of the original parabola
- (2) vertex is $(a/2, 0)$
- (3) directrix is y -axis
- (4) focus has coordinates $(a, 0)$

8. A square has one vertex at the vertex of the parabola $y^2 = 4ax$ and the diagonal through the vertex lies along the axis of the parabola. If the ends of the other diagonal lie on the parabola, the coordinates of the vertices of the square are

- (1) $(4a, 4a)$ (2) $(4a, -4a)$
- (3) $(0, 0)$ (4) $(8a, 0)$

9. If two distinct chords of a parabola $y^2 = 4ax$ passing through $(a, 2a)$ are bisected on the line $x + y = 1$, then the length of the latus rectum can be

- (1) 2 (2) 1 (3) 4 (4) 3

10. Let PQ be a chord of the parabola $y^2 = 4x$. A circle drawn with PQ as a diameter passes through the vertex V of the parabola. If $\text{ar}(\Delta PVQ) = 20 \text{ unit}^2$, then the coordinates of P are

- (1) $(16, 8)$ (2) $(16, -8)$
- (3) $(9, 6)$ (4) $(9, -6)$

11. The parabola $x^2 = ay$ makes an intercept of length $2\sqrt{10}$ units on the line $y - 2x = 1$, if a is equal to

- (1) -1 (2) -2 (3) 1 (4) 2

12. The equations of the directrix of the parabola with vertex at the origin and having the axis along the x -axis and a common tangent of slope 2 with the circle $x^2 + y^2 = 5$ is (are)

- (1) $x = 10$ (2) $x = 20$ (3) $x = -10$ (4) $x = -20$

13. Tangent is drawn at any point (x_1, y_1) other than the vertex on the parabola $y^2 = 4ax$. If tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2) , then

- (1) x_1, a, x_2 are in GP (2) $\frac{y_1}{2}, a, y_2$ are in GP

- (3) $-4, \frac{y_1}{y_2}, \frac{x_1}{x_2}$ are in GP (4) $x_1x_2 + y_1y_2 = a^2$

14. The parabola $y^2 = 4x$ and the circle having its center at $(6, 5)$ intersect at right angle. The possible point of intersection of these curves can be

- (1) $(9, 6)$ (2) $(2, \sqrt{8})$
- (3) $(4, 4)$ (4) $(3, 2\sqrt{3})$

15. Which of the following line can be tangent to the parabola $y^2 = 8x$?

- (1) $x - y + 2 = 0$ (2) $9x - 3y + 2 = 0$
- (3) $x + 2y + 8 = 0$ (4) $x + 3y + 12 = 0$

16. If the line $k^2(x - 1) + k(y - 2) + 1 = 0$ touches the parabola $y^2 - 4x - 4y + 8 = 0$, then k can be

- (1) -3 (2) $-\sqrt{5}$ (3) $\frac{7}{19}$ (4) 1000

17. The equation of the line that touches the curves $y = x|x|$ and $x^2 + (y - 2)^2 = 4$, where $x \neq 0$, is

- (1) $y = 4\sqrt{5}x + 20$ (2) $y = 4\sqrt{3}x - 12$
- (3) $y = 0$ (4) $y = -4\sqrt{5}x - 20$

18. The equations of the common tangents to the parabola $y = x^2$ and $y = -(x - 2)^2$ is/are

- (1) $y = 4(x - 1)$ (2) $y = 0$
- (3) $y = -4(x - 1)$ (4) $y = -30x - 50$

19. The line $x + y + 2 = 0$ is a tangent to a parabola at point A . This line intersects the directrix at B and tangent at vertex at C , respectively. If the focus of parabola is $S(2, 0)$, then

- (1) CS is perpendicular to AB
- (2) $AC \cdot BC = CS^2$
- (3) $AC \cdot BC = 8$
- (4) $AC = BC$

20. Which of the following line can be normal to parabola $y^2 = 12x$?

- (1) $x + y - 9 = 0$ (2) $2x - y - 32 = 0$
- (3) $2x + y - 36 = 0$ (4) $3x - y - 99 = 0$

21. A normal drawn to the parabola $y^2 = 4ax$ at P meets the curve again at Q such that the angle subtended by PQ at the vertex is 90° . Then the coordinates of P can be

- (1) $(8a, 4\sqrt{2}a)$ (2) $(8a, 4a)$
- (3) $(2a, -2\sqrt{2}a)$ (4) $(2a, 2\sqrt{2}a)$

22. A circle is drawn having centre at $C(0, 2)$ and passing through focus S of the parabola $y^2 = 8x$. If CS intersects the parabola at point P , then

- (1) distance of point P from directrix is $(8 - 4\sqrt{2})$
- (2) distance of point C from point P is $(6\sqrt{2} - 8)$
- (3) angle subtended by intercept made by circle on directrix at its centre is $\frac{\pi}{2}$
- (4) point P is the midpoint of C and S

23. From any point P on the parabola $y^2 = 4ax$, perpendicular PN is drawn on the axis meeting it at N . Normal at P meets the axis in G . For what value/values of t , the point N divides SG internally in the ratio $1 : 3$, where S is the focus?

- (1) $\sqrt{\frac{3}{5}}$ (2) $-\sqrt{\frac{5}{3}}$ (3) $-\sqrt{\frac{3}{5}}$ (4) $\sqrt{\frac{5}{3}}$

24. Let C_1 and C_2 be, respectively, the parabola $x^2 = y - 1$ and $y^2 = x - 1$. Also, let P be any point on C_1 and Q be any point on C_2 . If P_1 and Q_1 are the reflections of P and Q , respectively, with respect to the line $y = x$, then
- (1) P_1 lies on C_2 and Q_1 lies on C_1
 - (2) $PQ \geq \min \{PP_1, QQ_1\}$
 - (3) point P_0 on C_1 such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) is $\left(\frac{1}{3}, \frac{10}{9}\right)$
 - (4) point Q_0 on C_2 such that $P_0Q_0 \leq PQ$ for all pairs of points (P, Q) is $\left(\frac{10}{9}, \frac{1}{3}\right)$

Linked Comprehension Type

For Problems 1–3

A tangent is drawn at any point $P(t)$ on the parabola $y^2 = 8x$ and from it is taken a point $Q(\alpha, \beta)$ from which a pair of tangents QA and QB are drawn to the circle $x^2 + y^2 = 8$. Using this information, answer the following questions:

1. The locus of the point of concurrency of the chord of contact AB of the circle $x^2 + y^2 = 4$ is
 - (1) $y^2 - 2x = 0$
 - (2) $y^2 - x^2 = 4$
 - (3) $y^2 + 4x = 0$
 - (4) $y^2 - 2x^2 = 4$
2. The point from which perpendicular tangents can be drawn both to the given circle and the parabola is
 - (1) $(4, \pm\sqrt{3})$
 - (2) $(-1, \sqrt{2})$
 - (3) $(-\sqrt{2}, -\sqrt{2})$
 - (4) $(-2, \pm 2\sqrt{3})$
3. The locus of circumcenter of $\triangle AQB$ if $t = 2$ is
 - (1) $x - 2y + 2 = 0$
 - (2) $x + 2y - 4 = 0$
 - (3) $x - 2y - 4 = 0$
 - (4) $x + 2y + 4 = 0$

For Problems 4–6

A tangent to the parabola $y = x^2 + ax + 1$ at the point of intersection of the y -axis also touches the circle $x^2 + y^2 = r^2$. Also, no point of the parabola is below the x -axis.

4. The radius of circle when a attains its maximum value is
 - (1) $1/\sqrt{10}$
 - (2) $1/\sqrt{5}$
 - (3) 1
 - (4) $\sqrt{5}$
5. The slope of the tangent when the radius of the circle is maximum is
 - (1) -1
 - (2) 1
 - (3) 0
 - (4) 2
6. The minimum area bounded by the tangent and the coordinate axes is
 - (1) 1
 - (2) $1/3$
 - (3) $1/2$
 - (4) $1/4$

For Problems 7–9

The locus of the circumcenter of a variable triangle having sides on the y -axis, $y = 2$, and $lx + my = 1$, where (l, m) lies on the parabola $y^2 = 4x$, is a curve C .

7. The coordinates of the vertex of this curve C is
 - (1) $(-2, 3/2)$
 - (2) $(-2, -3/2)$
 - (3) $(2, 3/2)$
 - (4) $(2, -3/2)$

8. The length of the smallest focal chord of this curve C is
 - (1) $1/4$
 - (2) $1/12$
 - (3) $1/8$
 - (4) $1/16$
9. The curve C is symmetric about the line
 - (1) $x = 3/2$
 - (2) $y = -3/2$
 - (3) $x = -3/2$
 - (4) $y = 3/2$

For Problems 10–12

$y = x$ is tangent to the parabola $y = ax^2 + c$.

10. If $a = 2$, then the value of c is
 - (1) 1
 - (2) $-1/2$
 - (3) $1/2$
 - (4) $1/8$
11. If $(1, 1)$ is the point of contact, then a is
 - (1) $1/4$
 - (2) $1/3$
 - (3) $1/2$
 - (4) $1/6$
12. If $c = 2$, then the point of contact is
 - (1) $(3, 3)$
 - (2) $(2, 2)$
 - (3) $(6, 6)$
 - (4) $(4, 4)$

For Problems 13–15

If l and m are variable real numbers such that $5l^2 + 6m^2 - 4lm + 3l = 0$, then the variable line $lx + my = 1$ always touches a fixed parabola, whose axes is parallel to the x -axis.

13. The vertex of the parabola is
 - (1) $(-5/3, 4/3)$
 - (2) $(-7/4, 3/4)$
 - (3) $(5/6, -7/6)$
 - (4) $(1/2, -3/4)$
14. The focus of the parabola is
 - (1) $(1/6, -7/6)$
 - (2) $(1/3, 4/3)$
 - (3) $(3/2, -3/2)$
 - (4) $(-3/4, 3/4)$
15. The directrix of the parabola is
 - (1) $6x + 7 = 0$
 - (2) $4x + 11 = 0$
 - (3) $3x + 11 = 0$
 - (4) none of these

For Problems 16–18

Consider the parabola whose focus is at $(0, 0)$ and tangent at vertex is $x - y + 1 = 0$.

16. The length of latus rectum is
 - (1) $4\sqrt{2}$
 - (2) $2\sqrt{2}$
 - (3) $8\sqrt{2}$
 - (4) $3\sqrt{2}$
17. The length of the chord of parabola on the x -axis is
 - (1) $4\sqrt{2}$
 - (2) $2\sqrt{2}$
 - (3) $8\sqrt{2}$
 - (4) $3\sqrt{2}$
18. Tangents drawn to the parabola at the extremities of the chord $3x + 2y = 0$ intersect at an angle
 - (1) $\pi/6$
 - (2) $\pi/3$
 - (3) $\pi/2$
 - (4) none of these

For Problems 19–21

Two tangents on a parabola are $x - y = 0$ and $x + y = 0$. $S(2, 3)$ is the focus of the parabola.

19. The equation of tangent at vertex is
 - (1) $4x - 6y + 5 = 0$
 - (2) $4x - 6y + 3 = 0$
 - (3) $4x - 6y + 1 = 0$
 - (4) $4x - 6y + 3/2 = 0$
20. The length of latus rectum of the parabola is
 - (1) $6/\sqrt{3}$
 - (2) $10/\sqrt{13}$
 - (3) $2/\sqrt{13}$
 - (4) none of these

21. If P and Q are the ends of the focal chord of the parabola,

$$\text{then } \frac{1}{SP} + \frac{1}{SQ} =$$

- (1) $2\sqrt{13}/3$ (2) $2\sqrt{13}$
(3) $2\sqrt{13}/5$ (4) none of these

For Problems 22–24

$y^2 = 4x$ and $y^2 = -8(x - a)$ intersect at points A and C . Points $O(0, 0)$, A , $B(a, 0)$, and C are concyclic.

22. The length of the common chord of the parabolas is

- (1) $2\sqrt{6}$ (2) $4\sqrt{3}$ (3) $6\sqrt{5}$ (4) $8\sqrt{2}$

23. The area of cyclic quadrilateral $OABC$ is

- (1) $24\sqrt{3}$ (2) $48\sqrt{2}$ (3) $12\sqrt{6}$ (4) $18\sqrt{5}$

24. Tangents to the parabola $y^2 = 4x$ at A and C intersect at point D and tangents to the parabola $y^2 = -8(x - a)$ intersect at point E . Then the area of quadrilateral $DAEC$ is

- (1) $96\sqrt{2}$ (2) $48\sqrt{3}$ (3) $54\sqrt{5}$ (4) $36\sqrt{6}$

For Problems 25–27

PQ is the double ordinate of the parabola $y^2 = 4x$ which passes through the focus S . ΔPQA is an isosceles right angle triangle, where A is on the axis of the parabola to the right of focus. Line PA meets the parabola at C and QA meets the parabola at B .

25. The area of trapezium $PBCQ$ is

- (1) 96 sq. units (2) 64 sq. units
(3) 72 sq. units (4) 48 sq. units

26. The circumradius of trapezium $PBCQ$ is

- (1) $6\sqrt{5}$ (2) $3\sqrt{6}$ (3) $2\sqrt{10}$ (4) $5\sqrt{3}$

27. The ratio of the inradius of ΔABC and that of ΔPAQ is

- (1) 2 : 1 (2) 3 : 2 (3) 4 : 3 (4) 3 : 1

For Problems 28–30

Consider the inequality $9^x - a \cdot 3^x - a + 3 \leq 0$, where a is a real parameter.

28. The given inequality has at least one negative solution for $a \in$

- (1) $(-\infty, 2)$ (2) $(3, \infty)$ (3) $(-2, \infty)$ (4) $(2, 3)$

29. The given inequality has at least one positive solution for $a \in$

- (1) $(-\infty, -2)$ (2) $[3, \infty)$ (3) $(2, \infty)$ (4) $[-2, \infty)$

30. The given inequality has at least one real solution for $a \in$

- (1) $(-\infty, 3)$ (2) $[2, \infty)$ (3) $(3, \infty)$ (4) $[-2, \infty)$

For Problems 31 and 32

Consider one side AB of a square $ABCD$ in order on line $y = 2x - 17$, and other two vertices C, D on $y = x^2$.

31. The minimum intercept of line CD on the y -axis is

- (1) 3 (2) 4 (3) 2 (4) 6

32. The maximum possible area of square $ABCD$ is

- (1) 1180 (2) 1250 (3) 1280 (4) none

Matrix Match Type

1. Consider the parabola $(x - 1)^2 + (y - 2)^2 = \frac{(12x - 5y + 3)^2}{169}$ and match the following lists:

List I	List II
a. Locus of point of intersection of perpendicular tangent	p. $12x - 5y - 2 = 0$
b. Locus of foot of perpendicular from focus upon any tangent	q. $5x + 12y - 29 = 0$
c. Line along which minimum length of focal chord occurs	r. $12x - 5y + 3 = 0$
d. Line about which parabola is symmetrical	s. $24x - 10y + 1 = 0$

2. Consider the parabola $y^2 = 12x$ and match the following lists:

List I	List II
a. Equation of tangent can be	p. $2x + y - 6 = 0$
b. Equation of normal can be	q. $3x - y + 1 = 0$
c. Equation of chord of contact w.r.t. any point on the directrix can be	r. $x - 2y - 12 = 0$
d. Equation of chord which subtends right angle at the vertex can be	s. $2x - y - 36 = 0$

3. Match the following lists:

List I	List II
a. Tangents are drawn from point $(2, 3)$ to the parabola $y^2 = 4x$. Then the points of contact are	p. $(9, -6)$
b. From a point P on the circle $x^2 + y^2 = 5$, the equation of chord of contact to the parabola $y^2 = 4x$ is $y = 2(x - 2)$. Then the coordinate of point P will be	q. $(1, 2)$
c. $P(4, -4)$ and Q are points on the parabola $y^2 = 4x$ such that the area of ΔPOQ is 6 sq. units, where O is the vertex. Then the coordinates of Q may be	r. $(-2, 1)$
d. The common chord of the circle $x^2 + y^2 = 5$ and the parabola $6y = 5x^2 + 7x$ will pass through point(s)	s. $(4, 4)$

4. Match the following lists and then choose the correct code

List I	List II
a. Common normals to the parabola $y^2 = 4ax$ and $x^2 = 4ay$ is/are	p. $x = a$
b. The locus of point P , if tangents from P to the parabola $y^2 = 4ax$ intersect the coordinate axes in concyclic points, is	q. $x + y = 3a$
c. The locus of P , if tangents from it to the parabolas $y^2 = 4a(x + a)$ and $y^2 = 8a(x + 2a)$ are perpendicular, is	r. $x + 3a = 0$

d. The chord of contact of a point P w.r.t. the parabola $y^2 + 4ax = 0$ subtends right angle at the vertex. Then the locus of point of intersection of tangents at the end points of such chords is

$$s. x = 4a$$

Codes:

	a	b	c	d
(1)	p	r	q	q
(2)	q	p	r	s
(3)	s	p	q	r
(4)	r	s	q	p

Numerical Value Type

- If the length of the latus rectum of the parabola $169\{(x-1)^2 + (y-3)^2\} = (5x-12y+17)^2$ is L , then the value of $13L/4$ is _____.
- A circle is drawn through the point of intersection of the parabola $y = x^2 - 5x + 4$ and the x -axis such that origin lies outside it. The length of a tangent to the circle from the origin is _____.
- The focal chord of $y^2 = 16x$ is tangent to $(x-6)^2 + y^2 = 2$. Then the possible value of the square of slope of this chord is _____.
- Two tangents are drawn from the point $(-2, -1)$ to the parabola $y^2 = 4x$. If θ is the angle between these tangents, then $\tan \theta =$ _____.
- The equation of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is $ax + by + c = 0$. Then the value of $a + b + c$ is _____.
- If the point $P(4, -2)$ is one end of the focal chord PQ of $y^2 = x$, then the slope of the tangent at Q is _____.
- If the line $x + y = 6$ is a normal to the parabola $y^2 = 8x$ at point (a, b) , then the value of $a + b$ is _____.
- The locus of the midpoints of the portion of the normal to the parabola $y^2 = 16x$ intercepted between the curve and the axis is another parabola whose latus rectum is _____.
- Consider the locus of center of the circle which touches the circle $x^2 + y^2 = 4$ externally and the line $x = 4$. The distance of the vertex of the locus from the origin is _____.
- If on a given base BC [$B(0, 0)$ and $C(2, 0)$], a triangle is described such that the sum of the tangents of the base angles is 4, then the equation of the locus of the opposite vertex A is a parabola whose directrix is $y = k$. The value of k is _____.

- PQ is any focal chord of the parabola $y^2 = 8x$. Then the length of PQ can never be less than _____.
- The length of focal chord to the parabola $y^2 = 12x$ drawn from the point $(3, 6)$ on it is _____.
- From the point $(-1, 2)$, tangent lines are drawn to the parabola $y^2 = 4x$. If the area of the triangle formed by the chord of contact and the tangents is A , then the value of $A/\sqrt{2}$ is _____.
- Line $y = 2x - b$ cuts the parabola $y = x^2 - 4x$ at points A and B . Then the value of b for which $\angle AOB$ is a right angle is (where O is origin) _____.
- A line through the origin intersects the parabola $5y = 2x^2 - 9x + 10$ at two points whose x -coordinates add up to 17. Then the slope of the line is _____.
- If the circle $(x-6)^2 + y^2 = r^2$ and the parabola $y^2 = 4x$ have maximum number of common chords, then the least integral value of r is _____.
- The slope of the line which belongs to the family of lines $(1+\lambda)x + (\lambda-1)y + 2(1-\lambda) = 0$ and makes shortest intercept on $x^2 = 4y - 4$ is _____.
- If $3x + 4y + k = 0$ represents the equation of tangent at the vertex of the parabola $16x^2 - 24xy + 9y^2 + 14x + 2y + 7 = 0$, then the value of k is _____.
- Normals at (x_1, y_1) , (x_2, y_2) and (x_3, y_3) to the parabola $y^2 = 4x$ are concurrent at point P . If $y_1 y_2 + y_2 y_3 + y_3 y_1 = x_1 x_2 x_3$, then locus of point P is part of a parabola, length of whose latus rectum is _____.
- Foot of perpendicular from point P on the parabola $y^2 = 4ax$ to the axis is N . A straight line is drawn parallel to the axis which bisects PN and cuts the curve at Q . If NQ meets the tangent at the vertex A at a point T , then $\frac{PN}{AT} =$ _____.
- Points A, B and C lie on the parabola $y^2 = 4ax$. The tangents to the parabola at A, B and C , taken in pairs, intersect at points P, Q and R . Then the ratio of the areas of the triangles ABC and PQR is _____.
- Normals are drawn from the point P with slopes m_1, m_2 and m_3 to that parabola $y^2 = 4x$. If the locus of P with $m_1 m_2 = \alpha$ is a part of the parabola itself then the value of α is _____.

Archives

JEE MAIN

Single Correct Answer Type

- If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is
 - $2x - 1 = 0$
 - $x = 1$
 - $2x + 1 = 0$
 - $x = -1$
 (AIEEE 2010)

- Given:** A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.
Statement 1: An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement 2: If the line, $y = mx + \frac{\sqrt{5}}{m}$ ($m \neq 0$) is their common tangent, then ' m ' satisfies $m^4 - 3m^2 + 2 = 0$

- (1) Statement 1 is true; statement 2 is true; statement 2 is a correct explanation for statement 1.
 (2) Statement 1 is true; statement 2 is true; statement 2 is not a correct explanation for statement 1.
 (3) Statement 1 is true; statement 2 is false.
 (4) Statement 1 is false; statement 2 is true.

(JEE Main 2013)

3. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

- (1) $1/2$ (2) $3/2$ (3) $1/8$ (4) $2/3$

(JEE Main 2014)

4. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio $1 : 3$, then the locus of P is

- (1) $x^2 = y$ (2) $y^2 = x$
 (3) $y^2 = 2x$ (4) $x^2 = 2y$ (JEE Main 2015)

5. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is

- (1) $x^2 + y^2 - x + 4y - 12 = 0$
 (2) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

(3) $x^3 + y^2 - 4x + 9y + 18 = 0$

(4) $x^2 + y^2 - 4x + 8y + 12 = 0$ (JEE Main 2016)

6. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is

- (1) $4(\sqrt{2} + 1)$ (2) $2(\sqrt{2} + 1)$
 (3) $2(\sqrt{2} - 1)$ (4) $4(\sqrt{2} - 1)$

(JEE Main 2017)

7. If the tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is

- (1) 95 (2) 195 (3) 185 (4) 85

(JEE Main 2018)

8. Tangent and normal are drawn at $P(16, 16)$ on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B , respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan \theta$ is

- (1) $4/3$ (2) $1/2$ (3) 2 (4) 3

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio $1 : 3$. Then the locus of P is

- (1) $x^2 = y$ (2) $y^2 = 2x$
 (3) $y^2 = x$ (4) $x^2 = 2y$ (IIT-JEE 2011)

2. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S . Then the area of the quadrilateral $PQRS$ is

- (1) 3 (2) 6 (3) 9 (4) 15

(JEE Advanced 2014)

Multiple Correct Answers Type

1. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N , respectively. The locus of the centroid of triangle PTN is a parabola whose

- (1) vertex is $(2a/3, 0)$ (2) directrix is $x = 0$
 (3) latus rectum is $2a/3$ (4) focus is $(a, 0)$

(IIT-JEE 2009)

2. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be

- (1) $-1/r$ (2) $1/r$
 (3) $2/r$ (4) $-2/r$ (IIT-JEE 2010)

3. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is given by

- (1) $y - x + 3 = 0$ (2) $y + 3x - 33 = 0$
 (3) $y + x - 15 = 0$ (4) $y - 2x + 12 = 0$

(IIT-JEE 2011)

4. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P ?

- (1) $(4, 2\sqrt{2})$ (2) $(9, 3\sqrt{2})$
 (3) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (4) $(1, \sqrt{2})$

(JEE Advanced 2015)

5. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then

- (1) $SP = 2\sqrt{5}$
 (2) $SQ : QP = (\sqrt{5} + 1) : 2$
 (3) the x -intercept of the normal to the parabola at P is 0
 (4) the slope of the tangent to the circle at Q is $\frac{1}{2}$

(JEE Advanced 2016)

6. The circle $C_1 : x^2 + y^2 = 3$, with centre at O , intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3 , respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3 , respectively. If Q_2 and Q_3 lie on the y -axis, then

- (1) $Q_2Q_3 = 12$
 (2) $R_2R_3 = 4\sqrt{6}$
 (3) area of the triangle OR_2R_3 is $6\sqrt{2}$
 (4) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

(JEE Advanced 2016)

7. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k) , then which of the following is (are) possible value(s) of p, h and k ?

- (1) $p = 5, h = 4, k = -3$ (2) $p = -1, h = 1, k = -3$
(3) $p = -2, h = 2, k = -4$ (4) $p = 2, h = 3, k = -4$

(JEE Advanced 2017)

Linked Comprehension Type

For Problems 1 and 2

Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a, a > 0$.

(JEE Advanced 2013)

1. The length of chord PQ is

- (1) $7a$ (2) $5a$
(3) $2a$ (4) $3a$

2. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan \theta =$

- (1) $2\sqrt{7}/3$ (2) $-2\sqrt{7}/3$
(3) $2\sqrt{5}/3$ (4) $-2\sqrt{5}/3$

For Problems 3 and 4

Let a, r, s, t be non-zero real numbers. Let $P(at^2, 2at)$, $Q(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$.

(JEE Advanced 2014)

3. The value of r is

- (1) $-\frac{1}{t}$ (2) $\frac{t^2 + 1}{t}$ (3) $\frac{1}{t}$ (4) $\frac{t^2 - 1}{t}$

4. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

- (1) $\frac{(t^2 + 1)^2}{2t^3}$ (2) $\frac{a(t^2 + 1)^2}{2t^3}$
(3) $\frac{a(t^2 + 1)^2}{t^3}$ (4) $\frac{a(t^2 + 2)^2}{t^3}$

Matrix Match Type

1. A line $L: y = mx + 3$ meets the y -axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x, 0 \leq y \leq 6$, at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y -axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of triangle EFG has a local maximum.

List I	List II
a. $m =$	p. $1/2$
b. Maximum area of $DEFG$ is	q. 4
c. $y_0 =$	r. 2
d. $y_1 =$	s. 1

(JEE Advanced 2013)

Numerical Value Type

1. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P(1/2, 2)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then Δ_1/Δ_2 is _____.

(IIT-JEE 2011)

2. Let S be the focus of the parabola $y^2 = 8x$ and PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of triangle PQS is _____.

(IIT-JEE 2012)

3. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is _____.

(JEE Advanced 2015)

4. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$ then the value of r^2 is _____.

(JEE Advanced 2015)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (2) | 2. (4) | 3. (4) | 4. (1) | 5. (3) |
| 6. (1) | 7. (1) | 8. (3) | 9. (3) | 10. (2) |
| 11. (2) | 12. (4) | 13. (1) | 14. (3) | 15. (3) |
| 16. (2) | 17. (3) | 18. (3) | 19. (1) | 20. (1) |
| 21. (3) | 22. (3) | 23. (4) | 24. (4) | 25. (1) |
| 26. (1) | 27. (4) | 28. (1) | 29. (4) | 30. (2) |
| 31. (3) | 32. (4) | 33. (4) | 34. (1) | 35. (2) |
| 36. (3) | 37. (3) | 38. (1) | 39. (3) | 40. (2) |
| 41. (3) | 42. (2) | 43. (1) | 44. (4) | 45. (1) |
| 46. (2) | 47. (1) | 48. (3) | 49. (2) | 50. (2) |
| 51. (1) | 52. (3) | 53. (2) | 54. (4) | 55. (3) |
| 56. (3) | 57. (4) | 58. (2) | 59. (4) | 60. (1) |
| 61. (2) | 62. (1) | 63. (3) | 64. (1) | 65. (2) |

- | | | | | |
|---------|---------|---------|---------|---------|
| 66. (2) | 67. (4) | 68. (2) | 69. (3) | 70. (4) |
| 71. (3) | 72. (2) | 73. (4) | 74. (2) | 75. (3) |
| 76. (2) | 77. (4) | 78. (3) | 79. (2) | 80. (4) |
| 81. (3) | 82. (4) | 83. (4) | 84. (4) | 85. (2) |
| 86. (3) | 87. (1) | 88. (3) | 89. (2) | 90. (1) |
| 91. (3) | 92. (4) | 93. (3) | 94. (2) | 95. (3) |

Multiple Correct Answers Type

- | | |
|-----------------------|------------------------|
| 1. (2), (4) | 2. (1), (4) |
| 3. (1), (3) | 4. (1), (3) |
| 5. (1), (2), (3) | 6. (1), (2) |
| 7. (1), (2), (3), (4) | 8. (1), (2), (3), (4) |
| 9. (1), (2), (4) | 10. (1), (2) |
| 11. (2), (3) | 12. (1), (3) |
| 13. (2), (3), (4) | 14. (1), (3) |
| 15. (1), (2), (3) | 16. (1), (2), (3), (4) |

ARCHIVES**JEE MAIN**

17. (1), (2), (3) 18. (1), (2)
 19. (1), (2), (3) 20. (1), (3), (4)
 21. (3), (4) 22. (1), (2), (3)
 23. (2), (4) 24. (1), (2)

- Single Correct Answer Type
 1. (4) 2. (2) 3. (1) 4. (4) 5. (4)
 6. (None) 7. (1) 8. (3)

JEE ADVANCED**Single Correct Answer Type**

1. (3) 2. (4)

Multiple Correct Answers Type

1. (1), (4) 2. (3), (4)
 3. (1), (2), (4) 4. (1), (4)
 5. (1), (3), (4) 6. (1), (2), (3)
 7. (4)

Matrix Match Type

1. $a \rightarrow r; b \rightarrow s; c \rightarrow p; d \rightarrow q$
 2. $a \rightarrow q; b \rightarrow s; c \rightarrow p; d \rightarrow r$
 3. $a \rightarrow q, s; b \rightarrow r; c \rightarrow p, q; d \rightarrow q, r$
 4. (2)

Linked Comprehension Type

1. (2) 2. (4) 3. (4) 4. (2)

Numerical Value Type

1. (7) 2. (2) 3. (1) 4. (3) 5. (3)
 6. (4) 7. (6) 8. (4) 9. (3) 10. (2.125)
 11. (8) 12. (12) 13. (8) 14. (7) 15. (5)
 16. (5) 17. (0) 18. (3) 19. (4) 20. (1.5)
 21. (2) 22. (2)

Matrix Match Type

1. $a \rightarrow s; b \rightarrow p; c \rightarrow q; d \rightarrow r$

Numerical Value Type

1. (2) 2. (4) 3. (4) 4. (2)

6

Ellipse

DEFINITION

In the previous chapter, we have learned that ellipse is one of the conic sections.

We have also learned that ellipse is locus of a point such that ratio of its distance from a fixed point (focus) and a fixed line (directrix) is constant, the value of which is always less than 1. The constant ratio is called eccentricity e . Thus, for ellipse, $e \in (0, 1)$.

For ellipse, let focus be $S(a, \beta)$ and directrix be $lx + my + n = 0$.

The locus of variable point $P(x, y)$ will be ellipse if $\frac{SP}{PM} = e$ (constant), where $e \in (0, 1)$.

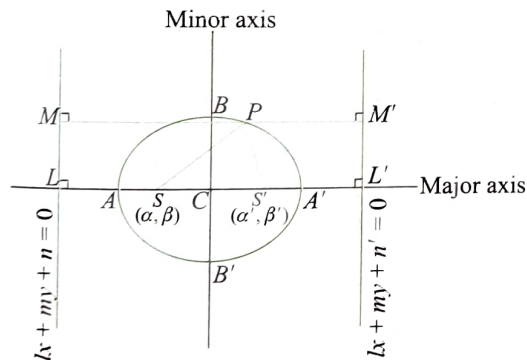
$$\therefore SP = e \cdot PM$$

Therefore, equation of ellipse is

$$\sqrt{(x - \alpha)^2 + (y - \beta)^2} = e \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

The above equation takes the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, which represents an ellipse if $h^2 - ab < 0$ and $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$.

The graph of the ellipse is as shown in the following figure.



We observe that graph of the ellipse is a closed curve.

The line which passes through the focus and is perpendicular to the directrix is called the major axis (focal axis) of the ellipse.

In the figure, major axis intersects the ellipse at points A and A' which are two vertices of the ellipse. Here, AA' is the major axis (or length of major axis). It is the longest chord of the ellipse. Graph of ellipse is symmetric about major axis.

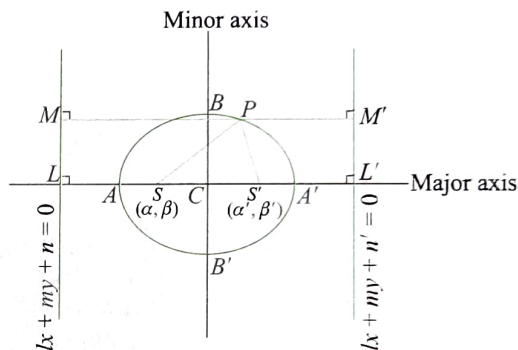
We have another axis of the ellipse called minor axis which is perpendicular bisector of AA' . Graph of the ellipse is symmetrical about the minor axis also. Minor axis meets the ellipse at points B and B' . BB' is the minor axis (or length of minor axis). It is the shortest chord of the ellipse.

Both the axes intersect at point C , which is called the centre of ellipse. Centre bisects every chord of the ellipse passing through it.

Because of the symmetry of the ellipse about its axes, it is possible to define ellipse using another focus $S'(a', \beta')$ and directrix $lx + my + n' = 0$, which are symmetrical about minor axis. In that case, $\frac{S'P}{PM'} = e$.

In the figure, $CS = CS'$ and $CL = CL'$.

SP and $S'P$ are called focal radii of point P .



We have

$$\begin{aligned} SP + S'P &= e \cdot PM + e \cdot PM' \\ &= e (PM + PM') \\ &= e \cdot MM' \\ &= \frac{e}{2} \times 2LL' \\ &= \frac{e}{2} \times (LA + AL' + LA' + A'L') \\ &= \frac{e}{2} \times \left(\frac{AS}{e} + \frac{AS'}{e} + \frac{A'S}{e} + \frac{A'S'}{e} \right) \\ &= \frac{1}{2} \times ((AS + A'S) + (AS' + A'S')) \\ &= \frac{1}{2} (AA' + AA') = AA' \\ &= \text{major axis} \end{aligned}$$

Thus, sum of two focal radii of variable point on the ellipse is constant which is equal to length of the major axis.

Further, in triangle PSS' , SS' is fixed and sum of two variable sides SP and $S'P$ is constant. Thus, perimeter of triangle PSS' is constant. This leads to another definition of ellipse as follows:

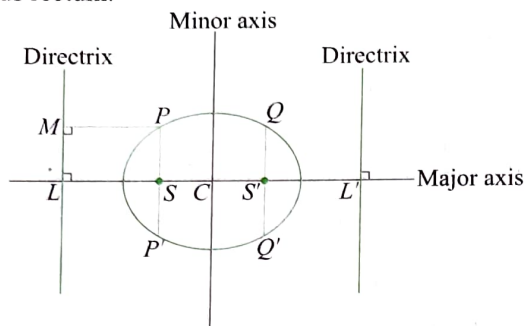
In a triangle, if one side (SS') is fixed and variable vertex (P) moves in a such a way that perimeter of the triangle remains constant then locus of variable vertex is ellipse.

This gives the idea to draw the ellipse on the plane of paper. Insert two drawing pins into a board and slip the loop of thread

(non-stretchable) over them. Insert a pencil point in the loop and position it so that the thread is tight. Move the pencil around the pins, always keeping the thread tight and, thus, ellipse is traced out.

Latus Rectum of Ellipse

We know that double ordinate through focus terminated by conic is its latus rectum.



In the figure, double ordinate through foci S and S' are PP' and QQ' , respectively.

$$\begin{aligned}\text{Latus rectum} &= PP' (= QQ') \\ &= 2 \times SP \\ &= 2 \times e \times PM \text{ (from definition of ellipse)} \\ &= 2 \times e \times (\text{distance of focus from directrix})\end{aligned}$$

ILLUSTRATION 6.1

Two circles are given such that one is completely lying inside the other without touching it. Then prove that the locus of centre of the variable circle which touches smaller circle externally and bigger circle internally is an ellipse.

Sol. Let the circle S_2 having centre at C_2 and radius r_2 lie completely inside the circle S_1 having centre at C_1 and radius r_1 .

Variable circle S touches S_1 internally and S_2 externally.

Centre C of circle S is variable point and its radius ' r ' is also variable.

From the figure, $CC_2 = r + r_2$ and $CC_1 = r_1 - r$

So, $CC_1 + CC_2 = r_1 + r_2$ (constant)

Thus, locus of C is ellipse whose foci are at C_1 and C_2 .

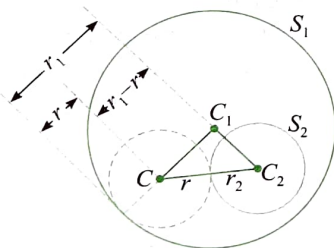


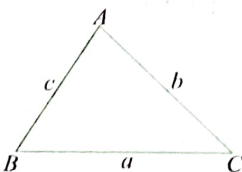
ILLUSTRATION 6.2

Base BC of triangle ABC is fixed and opposite vertex A moves in such a way that $\tan \frac{B}{2} \tan \frac{C}{2} = \text{constant}$. Prove that locus of vertex A is ellipse.

Sol. Given that in triangle ABC , ' a ' is constant.

And $\tan \frac{B}{2} \tan \frac{C}{2} = \lambda$ (constant)

$$\Rightarrow \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \lambda$$



$$\Rightarrow \frac{s-a}{s} = \lambda$$

$$\Rightarrow \frac{2s-a}{a} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow b+c = a \frac{\lambda+1}{1-\lambda} \text{ (constant)}$$

Thus, sum of distance of variable point A from two given fixed points B and C is constant, therefore, locus of A is ellipse.

ILLUSTRATION 6.3

If the equation $(5x-1)^2 + (5y-2)^2 = (\lambda^2 - 2\lambda + 1)(3x+4y-1)^2$ represents an ellipse, then find the values of λ .

Sol. Given equation is

$$\left(x - \frac{1}{5}\right)^2 + \left(y - \frac{2}{5}\right)^2 = (\lambda - 1)^2 \left[\frac{3x+4y-1}{\sqrt{3^2+4^2}}\right]^2$$

$$\text{or } \underbrace{\sqrt{\left(x - \frac{1}{5}\right)^2 + \left(y - \frac{2}{5}\right)^2}}_{\text{Distance between } P(x, y) \text{ and } S\left(\frac{1}{5}, \frac{2}{5}\right)} = \underbrace{|\lambda - 1|}_{e} \underbrace{\left|\frac{3x+4y-1}{\sqrt{3^2+4^2}}\right|}_{\text{Distance of } P(x, y) \text{ from line } 3x+4y-1=0}$$

Thus, we have

$$SP = e \cdot PM, \text{ where } 0 < e < 1$$

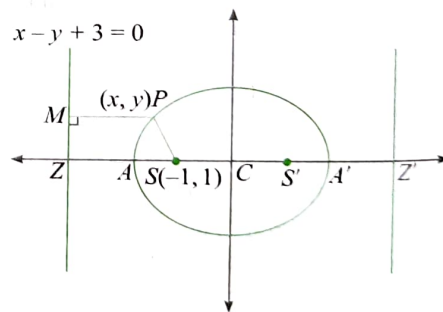
$$\therefore 0 < |\lambda - 1| < 1$$

$$\Rightarrow \lambda \in (0, 2) - \{1\}$$

ILLUSTRATION 6.4

Find the equation to the ellipse whose focus is $S(-1, 1)$, the corresponding directrix is $x - y + 3 = 0$ and the eccentricity is $1/2$. Also, find its centre, the second focus, the equation of the second directrix and the length of latus rectum.

Sol.



Let $P(x, y)$ be any point on the ellipse and PM be perpendicular to the directrix, then $PS = e \cdot PM$ gives

$$\sqrt{(x+1)^2 + (y-1)^2} = \frac{1}{2} \frac{|x-y+3|}{\sqrt{2}} \quad (1)$$

$$\Rightarrow 7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$$

The major axis passes through $S(-1, 1)$ and is perpendicular to directrix.

So, the equation of the major axis is $x + y = 0$ (2)

Axis meets the directrix in Z, then coordinates of Z are $(-3/2, 3/2)$.

A divides ZS internally in the ratio $1 : e$ or $1 : 1/2$ or $2 : 1$.

$$\therefore A \equiv (-7/6, 7/6).$$

A' divides ZS externally in the ratio $2 : 1$.

$$\therefore A' \equiv (-1/2, 1/2)$$

Centre is midpoint of AA'.

$$\therefore C \equiv (-5/6, 5/6) \quad (3)$$

Let other focus S' be (h, k) .

$$\text{Then } \frac{1}{2}(h-1) = -\frac{5}{6} \text{ and } \frac{1}{2}(k+1) = \frac{5}{6} \text{ (as C is midpoint of SS')}$$

$$\therefore S' \equiv (-2/3, 2/3) \quad (4)$$

If major axis meets the other directrix at Z'(a, β), then

$$\frac{1}{2}(\alpha - 3/2) = -\frac{5}{6} \text{ and } \frac{1}{2}(\beta + 3/2) = \frac{5}{6}$$

$$\therefore Z' \equiv (-1/6, 1/6)$$

The second directrix passes through Z'(-1/6, 1/6) and is perpendicular to the major axis.

Thus, the equation of other directrix is

$$x - y + 1/3 = 0 \quad (5)$$

Also, length of L.R.

$$= 2 \times e \times \text{distance of } (-1, 1) \text{ from the line } x - y + 3 = 0$$

$$= 2 \times \frac{1}{2} \frac{|-1-1+3|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

CONCEPT APPLICATION EXERCISE 6.1

- Find the equation of ellipse having focus at $(1, 2)$, corresponding directrix $x - y + 2 = 0$ and eccentricity 0.5.
- If the locus of the moving point $P(x, y)$ satisfying $\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + (y-\sqrt{12})^2} = a$ is an ellipse, then find the values of a .
- Find the eccentricity, one of the foci, the directrix, and the length of the latus rectum for the conic $(3x - 12)^2 + (3y + 15)^2 = \frac{(3x - 4y + 5)^2}{25}$.

ANSWERS

- $7x^2 + 7y^2 + 2xy - 20x - 28y + 36 = 0$
- $a > 4$
- $e = 1/3$; focus $\equiv (4, 5)$; directrix: $3x - 4y + 5 = 0$; L.R. $= 2/5$

EQUATION OF ELLIPSE IN DIFFERENT FORMS

STANDARD EQUATION OF ELLIPSE

For standard equation of ellipse, we consider coordinate axes as axes of the ellipse. So, we have following two cases:

Major Axis is x-axis and Minor Axis is y-axis

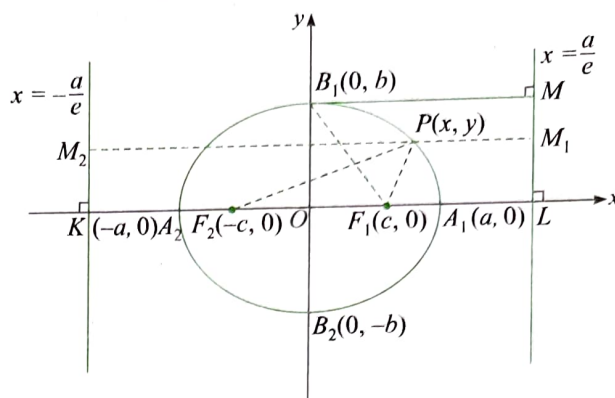
Let major axis be x-axis and minor axis be y-axis. So, centre will be origin.

Let foci be $F_1(c, 0)$ and $F_2(-c, 0)$.

Let ellipse intersect x-axis at $A_1(a, 0)$ and $A_2(-a, 0)$.

According to the definition of the ellipse for any point $P(x, y)$ on the ellipse, we have

$$PF_1 + PF_2 = 2a$$



$$\therefore \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \quad (1)$$

$$\text{Now, } [(x-c)^2 + y^2] - [(x+c)^2 + y^2] = -4cx \quad (2)$$

Dividing (2) by (1), we get

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = -\frac{2cx}{a} \quad (3)$$

Adding (1) and (3), we get

$$\sqrt{(x-c)^2 + y^2} = a - \frac{cx}{a}$$

Squaring both sides and simplifying, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a^2 - c^2 = b^2 \text{ and } a > b.$$

Ellipse intersects y-axis at $B_1(0, b)$ and $B_2(0, -b)$.

Here, length of major axis is $2a$ and length of minor axis is $2b$.

Also, 'a' is called semi-major axis and 'b' is semi-minor axis.

Now, in triangle F_1OB_1 ,

$$F_1O^2 + OB_1^2 = F_1B_1^2$$

$$\therefore c^2 + b^2 = F_1B_1^2 = a^2$$

$$\text{For point } B_1, \frac{F_1B_1}{B_1M} = e$$

$$\therefore B_1M = \frac{a}{e}$$

$$\text{So, equation of directrix is } x = \frac{a}{e}.$$

$$\text{Hence, equation of another directrix is } x = -\frac{a}{e}$$

$$\text{Thus, } F_1P = e \cdot PM_1 = e \left(\frac{a}{e} - x \right) = a - ex$$

$$\text{and } F_2P = e \cdot PM_2 = e \left(\frac{a}{e} + x \right) = a + ex$$

Also, for point A_1 ,

$$F_1A_1 = e \cdot A_1L \therefore a - c = e \left(\frac{a}{e} - a \right) = a - ea$$

$$\therefore c = ae$$

$$\text{or } e = \frac{c}{a} = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}}$$

Thus, e is also measure of degree of flatness of the ellipse.

If e approaches to 0, then foci move towards centre and ellipse gets thicker, tends to become circle.

If e approaches to 1, then foci move towards two vertices and ellipse gets thinner, tends to become a line segment.

Also, from $a^2 - c^2 = b^2$, we have

$$a^2 - a^2 e^2 = b^2$$

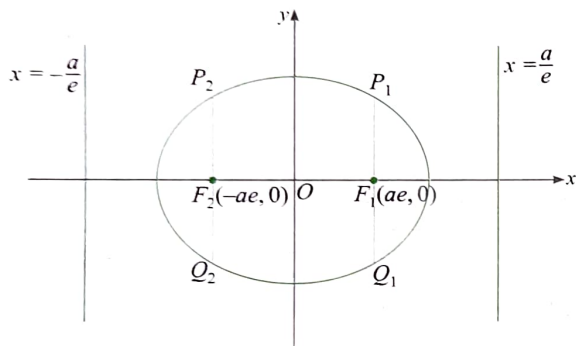
$$\text{or } b^2 = a^2(1 - e^2)$$

$$\text{or } e^2 = 1 - \frac{b^2}{a^2}$$

$$\text{Also, } e = \frac{2ae}{2a} = \frac{F_1 F_2}{2a}$$

$$= \frac{\text{Distance between foci}}{\text{Major axis}}$$

Latus rectum



In the figure, latus rectum = $P_1 Q_1$ ($= P_2 Q_2$)

Equation of line passing through foci is given by $x = \pm ae$.

Solving with ellipse, we get

$$\frac{a^2 e^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore y^2 = b^2(1 - e^2) = b^2 \frac{b^2}{a^2} = \frac{b^4}{a^2}$$

$$\therefore y = \pm \frac{b^2}{a}$$

So, endpoints of latus rectum are $P_1\left(ae, \frac{b^2}{a}\right)$, $Q_1\left(ae, -\frac{b^2}{a}\right)$ and $P_2\left(-ae, \frac{b^2}{a}\right)$, $Q_2\left(-ae, -\frac{b^2}{a}\right)$.

$$\text{Length of latus rectum} = P_1 Q_1 = P_2 Q_2 = \frac{2b^2}{a}$$

Also, length of latus rectum

$$= 2 \times e \times \text{Distance of focus from directrix}$$

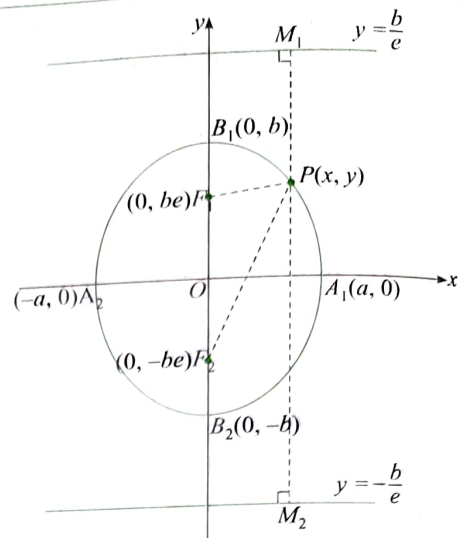
$$= 2 \times e \times \left(\frac{a}{e} - ae\right)$$

$$= 2a(1 - e^2)$$

$$= 2a \frac{b^2}{a^2} = \frac{2b^2}{a}$$

Major Axis is y-axis and Minor Axis is x-axis

In this case also, we consider equation of ellipse as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a < b$.



Foci of the ellipse are $F_1(0, be)$ and $F_2(0, -be)$.

Equations of directrix is given by $y = \pm \frac{b}{e}$.

Major axis = $B_1 B_2 = 2b$

Minor axis = $A_1 A_2 = 2a$

For any point $P(x, y)$ on the ellipse, we have

$$PF_1 = e \cdot PM_1 = e \left(\frac{b}{e} - y\right) = b - ey$$

$$\text{and } PF_2 = e \cdot PM_2 = e \left(\frac{b}{e} + y\right) = b + ey$$

From $PF_1 = b - ey$, we have

$$\sqrt{x^2 + (y - be)^2} = b - ey$$

$$\Rightarrow x^2 + y^2 - 2be + b^2 e^2 = b^2 - 2bey + e^2 y^2$$

$$\Rightarrow x^2 + (1 - e^2)y^2 = b^2(1 - e^2)$$

$$\Rightarrow \frac{x^2}{b^2(1 - e^2)} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a^2 = b^2(1 - e^2), b > a$$

Ellipse intersects y-axis at $(0, \pm b)$, which are vertices of ellipse.

Extremities of latus rectum are $\left(\pm \frac{a^2}{b}, \pm be\right)$ and length of latus rectum is $\frac{2a^2}{b}$.

ILLUSTRATION 6.5

Find the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose latus rectum is half of its major axis.

Sol. Let $a > b$.

Then the latus rectum of ellipse is $\frac{2b^2}{a}$.

$$\text{Given that } \frac{2b^2}{a} = a$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{1}{2}$$

$$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$e = \frac{1}{\sqrt{2}}$$

ILLUSTRATION 6.6

Find the standard equation of the ellipse whose foci are $(\pm 2, 0)$ and eccentricity $1/2$.

Sol. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Given $e = \frac{1}{2}$

Also, foci of ellipse are $(\pm ae, 0) \equiv (\pm 2, 0)$

$$\Rightarrow ae = 2 \Rightarrow a = 4$$

Now, $b^2 = a^2(1 - e^2) \Rightarrow b^2 = 12$

Therefore, equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

ILLUSTRATION 6.7

If the focal distance of an end of the minor axis of standard ellipse is k and distance between its foci is $2h$ ($k > h$), then find its equation.

Sol. Equation of standard equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Distance between foci $= 2ae = 2h$ (given)

$$\Rightarrow ae = h \quad (1)$$

Also, given that focal distance of the one end of minor axis is k .

$$\therefore a = k$$

$$\Rightarrow b^2 = a^2 - a^2e^2 = k^2 - h^2$$

So, the equation of the ellipse is $\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$.

ILLUSTRATION 6.8

Find the equation of the ellipse having minor axis of length 1 and foci $(0, 1)$, $(0, -1)$. Also find its latus rectum.

Sol. Since foci are on y -axis, we consider equation of ellipse

as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a < b$.

Given foci are $(0, \pm be) \equiv (0, \pm 1)$

$$\therefore be = 1$$

Length of minor axis is $2a$.

So, $2a = 1 \Rightarrow a = \frac{1}{2}$

Now, $a^2 = b^2(1 - e^2)$

$$\Rightarrow \frac{1}{4} = b^2 - b^2e^2 = b^2 - 1$$

$$\Rightarrow b^2 = \frac{5}{4}$$

So, the equation of ellipse is $\frac{x^2}{1/4} + \frac{y^2}{5/4} = 1$ or $4x^2 + \frac{4y^2}{5} = 1$.

ILLUSTRATION 6.9

$P(x, y)$ is any point on ellipse $16x^2 + 25y^2 = 400$. If $F_1 \equiv (3, 0)$ and $F_2 \equiv (-3, 0)$, then prove that $PF_1 + PF_2$ is constant and find its value.

Sol. We have, ellipse

$$16x^2 + 25y^2 = 400$$

or $\frac{x^2}{25} + \frac{y^2}{16} = 1$

$$\therefore a^2e^2 = a^2 - b^2 = 25 - 16 = 9$$

$$\therefore ae = 3$$

So, foci are $(\pm ae, 0) \equiv (\pm 3, 0)$

Thus, F_1 and F_2 are the foci of the ellipse.

Since the sum of the focal distances of a variable point P on an ellipse is equal to its major axis,

$$PF_1 + PF_2 = 2a = 10$$

ILLUSTRATION 6.10

A rod of length l cm moves with its ends A (on x -axis) and B (on y -axis) always touching the coordinate axes. Prove that the point P on the rod which divides AB in the ratio λ ($\neq 1$) is ellipse. Also, find the eccentricity of the ellipse.

Sol. From the figure,

In $\triangle PQB$, $\cos \theta = \frac{PQ}{PB} = \frac{h}{PB}$

In $\triangle ARP$, $\sin \theta = \frac{PR}{PA} = \frac{k}{PA}$

Now, $\frac{AP}{PB} = \frac{\lambda}{1}$

$$\Rightarrow \frac{AP}{\lambda} = \frac{PB}{1} = \frac{AP + PB}{\lambda + 1} = \frac{AB}{\lambda + 1} = \frac{l}{\lambda + 1}$$

$$\therefore AP = \frac{\lambda l}{\lambda + 1} \text{ and } PB = \frac{l}{\lambda + 1}$$

$$\therefore \cos \theta = \frac{h(\lambda + 1)}{l} \text{ and } \sin \theta = \frac{k(\lambda + 1)}{\lambda l}$$

Squaring and adding,

$$\frac{h^2(\lambda + 1)^2}{l^2} + \frac{k^2(\lambda + 1)^2}{\lambda^2 l^2} = 1$$

or $\frac{x^2}{l^2} + \frac{y^2}{\lambda^2 l^2} = 1$

This is the equation of required locus.

Clearly, this is the equation of ellipse.

$$\text{Let } \lambda < 1, \text{ then } e^2 = 1 - \frac{(\lambda + 1)^2}{l^2} = 1 - \lambda^2$$

$$\therefore e = \sqrt{1 - \lambda^2}$$

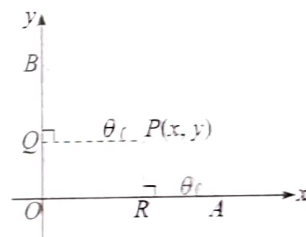
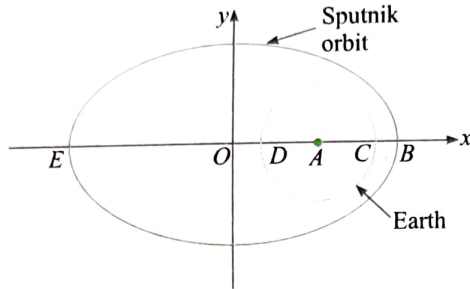


ILLUSTRATION 6.11

The first artificial satellite to orbit the earth was Sputnik I. Its highest point above earth's surface was 947 km, and its lowest point was 228 km. The center of the earth was at one of the foci of the elliptical orbit. The radius of the earth is 6378 km. Find the eccentricity of the orbit.

Sol.

As shown in the figure, Sputnik satellite is travelling on the elliptical orbit having equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with the center of the earth A at the focus.

$$AC = AD = \text{Radius of the earth} = 6378 \text{ km}$$

According to the question, the highest point of Sputnik above earth's surface is E and the lowest point is B. Therefore,

$$BC = 228 \text{ km and } ED = 947 \text{ km}$$

From the figure,

$$AB = a - ae = AC + BC = 6378 + 228 = 6606$$

$$\text{and } AE = a + ae = AD + ED = 6378 + 947 = 7325$$

$$\therefore \frac{a + ae}{a - ae} = \frac{7325}{6606}$$

$$\therefore \frac{1 + e}{1 - e} = \frac{7325}{6606}$$

$$\therefore e = \frac{7325 - 6606}{7325 + 6606} = \frac{719}{13931} \approx 0.05161$$

ILLUSTRATION 6.12

If (5, 12) and (24, 7) are the foci of an ellipse passing through the origin, then find the eccentricity of the ellipse.

Sol. If two foci are S(5, 12) and S'(24, 7) and the ellipse passes through the origin O, then

$$SO = \sqrt{25 + 144} = 13$$

$$S'O = \sqrt{576 + 49} = 25$$

$$\text{and } SS' = \sqrt{386}$$

If the conic is an ellipse, then

$$SO + S'O = 2a$$

$$\text{and } SS' = 2ae$$

$$\therefore e = \frac{SS'}{S'O + SO} = \frac{\sqrt{386}}{38}$$

ILLUSTRATION 6.13

Variable complex number z satisfies the equation $|z - 1 + 2i| + |z + 3 - i| = 10$. Prove that locus of complex number z is ellipse. Also, find centre, foci and eccentricity of the ellipse.

Sol. Given equation is $|z - (1 - 2i)| + |z - (-3 + i)| = 10$

$$|z - (1 - 2i)| = \text{distance between } z \text{ and } 1 - 2i (= z_1)$$

$$|z - (-3 + i)| = \text{distance between } z \text{ and } -3 + i (= z_2)$$

Thus, sum of distances of z from points $(1 - 2i)$ and $(-3 + i)$ is constant 10. So, locus of z is ellipse with foci at z_1 and z_2 .

Centre is midpoint of z_1 and z_2 , which is $-1 - \frac{i}{2}$.

$$\text{Distance between foci} = |z_1 - z_2| = |1 - 2i - (-3 + i)| = |4 - 3i| = 5 = 2ae$$

$$\text{Major axis} = 10 = 2a$$

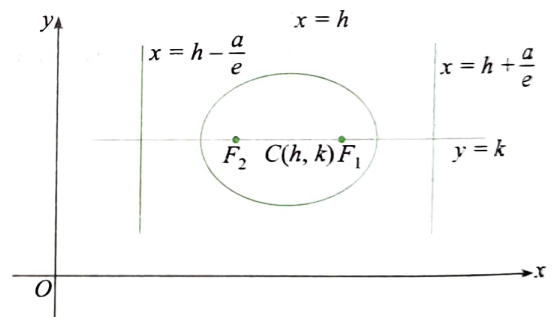
$$\therefore \text{Eccentricity} = e = \frac{5}{10} = \frac{1}{2}$$

EQUATION OF ELLIPSE HAVING AXES PARALLEL TO COORDINATE AXES

If the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, is shifted (translated) such that its centre is at (h, k) , then equation of ellipse becomes

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

Since $a > b$, major axis is along $y = k$ and minor axis along $x = h$.

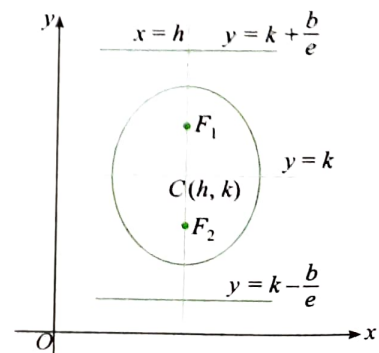


Eccentricity is given by relation $b^2 = a^2(1 - e^2)$.

Foci are $F_1(h + ae, k)$ and $F_2(h - ae, k)$.

Equation of two directrices are $x = h \pm \frac{a}{e}$.

If equation of ellipse is $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$, $a < b$, then $a^2 = b^2(1 - e^2)$.



Foci are $F_1(h, k + ae)$ and $F_2(h, k - ae)$.

Equation of two directrices are $y = k \pm \frac{b}{e}$.

ILLUSTRATION 6.14

Find the centre, foci, eccentricity and equations of directrix of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$.

Sol. The given equation is

$$2(x^2 - 2x) + 3(y^2 - 4y) + 13 = 0$$

$$\Rightarrow 2(x - 1)^2 + 3(y - 2)^2 = 1$$

$$\Rightarrow \frac{(x - 1)^2}{1/2} + \frac{(y - 2)^2}{1/3} = 1$$

Centre of the ellipse is $(1, 2)$.

$$a^2 = 1/2 \text{ and } b^2 = 1/3$$

Since $a > b$, equation of major axis is $y = 2$ and equation of minor axis is $x = 1$.

$$\text{Length of major axis} = 2a = \sqrt{2}$$

$$\text{Length of minor axis} = 2b = \frac{2}{\sqrt{3}}$$

$$\text{and } e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

$$a^2 e^2 = a^2 - b^2 = \frac{1}{6}$$

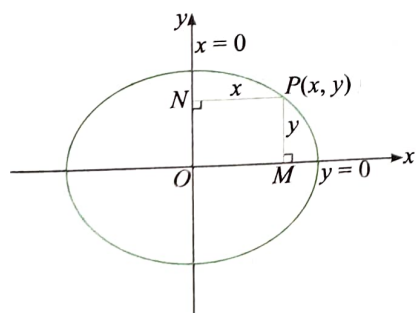
$$\therefore ae = \frac{1}{\sqrt{6}}$$

$$\text{Then foci are } \left(1 \pm \frac{1}{\sqrt{6}}, 2\right).$$

$$\text{Equations of directrix are } x - 1 = \pm \frac{a}{e} \text{ or } x = 1 \pm \frac{\sqrt{3}}{\sqrt{2}}.$$

EQUATION OF AN ELLIPSE REFERRED TO TWO PERPENDICULAR LINES AS AXES

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

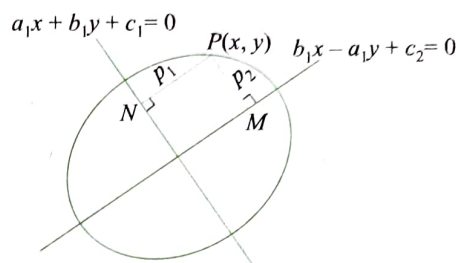


Let $P(x, y)$ be any point on the ellipse. Then, $PM = y$ and $PN = x$.

$$\therefore \frac{PN^2}{a^2} + \frac{PM^2}{b^2} = 1$$

In general, we can say that if perpendicular distances p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv b_1x - a_1y + c_2 = 0$, respectively, satisfy the equation $\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1$ or

$$\frac{\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} + \frac{\left(\frac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1, \text{ then the locus of point } P \text{ is ellipse.}$$



The centre of the ellipse is the point of intersection of the lines L_1 and L_2 .

The major axis lies along line L_2 and the minor axis lies along line L_1 if $a > b$.

ILLUSTRATION 6.15

Find the equation to the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ having equations as $x - 3y + 3 = 0$ and $3x + y - 1 = 0$, respectively.

Sol. Length of major axis is 6 which is along the line $x - 3y + 3 = 0$ and length of minor axis is $2\sqrt{6}$ which is along the line $3x + y - 1 = 0$.

$$\text{Thus, equation of ellipse is } \left(\frac{3x + y - 1}{\sqrt{9 + 1}}\right)^2 + \left(\frac{x - 3y + 3}{\sqrt{1 + 9}}\right)^2 = 1.$$

$$\text{or } \frac{(3x + y - 1)^2}{90} + \frac{(x - 3y + 3)^2}{60} = 1$$

$$\Rightarrow 2(3x + y - 1)^2 + 3(x - 3y + 3)^2 = 180$$

$$\Rightarrow 21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$$

AREA OF ELLIPSE

Area of an ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab .

This area can be obtained using integration.

$$\text{We have } \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\therefore y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

The part of the ellipse above x -axis, represents the function

$$y = \frac{b}{a} \sqrt{a^2 - x^2}.$$

Since ellipse is symmetrical about both the axes,
Area of ellipse = $4 \times$ Area of ellipse in first quadrant

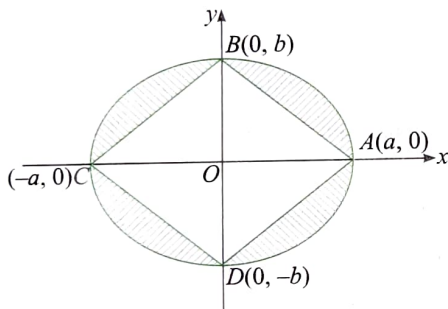
$$\begin{aligned} &= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx \\ &= \frac{b}{a} \times 4 \int_0^a \sqrt{a^2 - x^2} dx \\ &= \frac{b}{a} \times \text{Area of circle having radius } a \\ &= \frac{b}{a} \times \pi a^2 = \pi ab \end{aligned}$$

ILLUSTRATION 6.16

Find the area between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and lines $\frac{|x|}{a} + \frac{|y|}{b} = 1$.

Sol. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Given lines $\frac{|x|}{a} + \frac{|y|}{b} = 1$ form a parallelogram whose diagonals are major and minor axes of the given ellipse.



Required region is shaded region in the figure.

$\therefore$ Required area = Area of ellipse - Area of parallelogram

$$= \pi ab - 4 \times \frac{1}{2} ab = (\pi - 2)ab$$

POSITION OF A POINT W.R.T. ELLIPSE

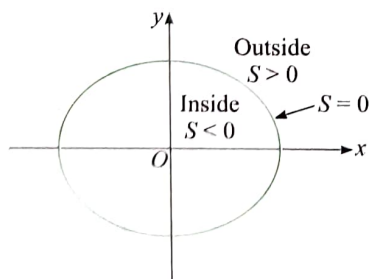
Consider standard equation of ellipse $S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

One of the foci is $F_1(ae, 0)$ and corresponding directrix is $x = \frac{a}{e}$.

Let M be foot of the perpendicular from point $P(x_1, y_1)$ on the directrix.

Graph of the ellipse divides the plane into two parts.

If point $P(x_1, y_1)$ is on the ellipse, we have $F_1P = e \cdot PM$, for which $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0$.



If point $P(x_1, y_1)$ is inside the ellipse, we have $F_1P < e \cdot PM$, for which $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0$.

If point $P(x_1, y_1)$ is outside the ellipse, we have $F_1P > e \cdot PM$, for which $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0$.

Thus, $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0$ represents interior region and $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 > 0$ represents exterior region of ellipse.

IMPORTANT PROPERTY OF FOCAL CHORD

A chord of ellipse passing through focus is called focal chord.

Consider focal chord PQ of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) through one of its foci S . This chord has two segments SP and SQ . The harmonic mean of SP and SQ is equal to semi latus rectum of the ellipse, i.e., $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$.

Proof:

Let slope of the focal chord PQ is $\tan \theta$.

Thus, the equation of chord is $\frac{x - ae}{\cos \theta} = \frac{y - 0}{\sin \theta} = r$.

$\therefore x = ae + r \cos \theta$ and $y = r \sin \theta$

Solving with ellipse, we get

$$\begin{aligned} &\frac{(ae + r \cos \theta)^2}{a^2} + \frac{(r \sin \theta)^2}{b^2} = 1 \\ \therefore &\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) r^2 + \frac{2e \cos \theta}{a} r + e^2 - 1 = 0 \end{aligned}$$

Since chord intersects ellipse at two points, this equation has two roots r_1 and r_2 such that $r_1 = SP$ and $r_2 = -SQ$. It should be noted that roots are of opposite signs as points P and Q lie on the opposite sides of focus S .

$$\begin{aligned} \frac{1}{SP} + \frac{1}{SQ} &= \left| \frac{1}{r_1} - \frac{1}{r_2} \right| = \left| \frac{r_1 - r_2}{r_1 r_2} \right| \\ &= \frac{\sqrt{(r_1 + r_2)^2 - 4r_1 r_2}}{|r_1 r_2|} \\ &= \frac{\sqrt{\frac{4e^2 \cos^2 \theta}{a^2} - \frac{4(e^2 - 1)}{\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)^2}}}{\frac{1 - e^2}{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \\ &= \frac{2}{1 - e^2} \sqrt{\frac{e^2 \cos^2 \theta}{a^2} + (1 - e^2) \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)} \\ &= \frac{2a^2}{b^2} \sqrt{\frac{(a^2 - b^2) \cos^2 \theta}{a^4} + \frac{b^2 \cos^2 \theta + \sin^2 \theta}{a^4}} \\ &= \frac{2a^2}{b^2} \cdot \frac{1}{a} = \frac{2a}{b^2} \end{aligned}$$

ILLUSTRATION 6.17

If S and S' are two foci of the ellipse $16x^2 + 25y^2 = 400$ and PSQ is a focal chord such that $SP = 16$, then find SQ and $S'Q$.

Sol. Given ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$.
Here, $a = 5$ and $b = 4$.

Now, PSQ is focal chord, where S is focus.

We know that, $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$

Then for given ellipse, we have

$$\frac{1}{16} + \frac{1}{SQ} = 2 \times \frac{5}{16}$$

$$\Rightarrow \frac{1}{16} + \frac{1}{SQ} = \frac{5}{8}$$

$$\Rightarrow SQ = \frac{16}{9}$$

$$\text{Now, } SQ + S'Q = 2a = 10$$

$$\therefore S'Q = 10 - \frac{16}{9} = \frac{74}{9}$$

EQUATION OF CHORD OF ELLIPSE BISECTED AT GIVEN POINT

Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let QR be the chord joining two points $Q(x_2, y_2)$ and $R(x_3, y_3)$.

Let midpoint of QR be $P(x_1, y_1)$ and also slope of the chord be $\tan \theta$.

Then equation of chord is

$$\frac{x - x_1}{a \cos \theta} = \frac{y - y_1}{b \sin \theta} = r$$

Solving this line with ellipse, we get

$$\frac{(x_1 + r \cos \theta)^2}{a^2} + \frac{(y_1 + r \sin \theta)^2}{b^2} = 1 \quad (1)$$

Since line cuts the ellipse in two points Q and R , this is quadratic in r , whose roots are $r_1 = PQ$ and $r_2 = -QR$.

Also, $PQ = QR$.

$$\therefore \text{Sum of roots} = r_1 + r_2 = 0$$

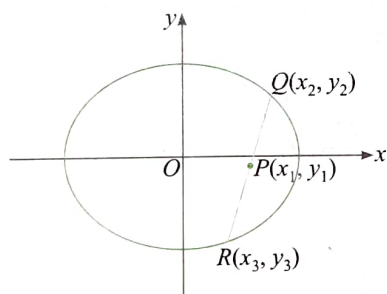
Therefore, in equation (1), coefficient of r is zero.

$$\therefore \frac{2x_1 \cos \theta}{a^2} + \frac{2y_1 \sin \theta}{b^2} = 0$$

$$\Rightarrow \tan \theta = -\frac{b^2 x_1}{a^2 y_1}$$

Hence, equation of chord is

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1} (x - x_1)$$



$$\text{or } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

$$\text{or } T = S_1, \text{ where } T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 \text{ and } S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

ILLUSTRATION 6.18

Find the length of the chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ whose midpoint is $(1/2, 2/5)$.

Sol. Equation of the chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ whose midpoint is $(1/2, 2/5)$ is

$$\frac{(1/2)x}{25} + \frac{(2/5)y}{16} - 1 = \frac{1/4}{25} + \frac{4/25}{16} - 1 \quad (\text{Using } T = S_1)$$

$$\text{or } 4x + 5y = 4$$

$$\text{or } 5y = 4(1 - x) \quad (1)$$

Solving with ellipse, we get

$$16x^2 + 16(1 - x)^2 = 400$$

$$\Rightarrow x^2 - x - 12 = 0$$

$$\Rightarrow x = 4, -3$$

$$\text{For } x = 4, y = -12/5;$$

$$\text{For } x = -3, y = 16/5$$

So, extremities of the chord are $P(4, -12/5)$ and $Q(-3, 16/5)$.

$$\therefore \text{Length of the chord, } PQ = \sqrt{7^2 + \left(\frac{28}{5}\right)^2} = \frac{7}{5} \sqrt{41}.$$

ILLUSTRATION 6.19

Find the locus of the midpoints of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Let the midpoint of the focal chord of the given ellipse be (h, k) . Then its equation is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad [\text{Using } T = S_1]$$

Since this passes through $(ae, 0)$, we have

$$\frac{hae}{a^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

$$\text{or } \frac{he}{a} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

Therefore, the locus of (h, k) is $\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

CONCEPT APPLICATION EXERCISE 6.2

- Find the standard equation of ellipse whose eccentricity is $\frac{2}{3}$, latus rectum is 5.
- An ellipse has OB as the semi-minor axis, F and F' as its foci, and $\angle FBF'$ a right angle. Then, find the eccentricity of the ellipse.
- An ellipse is inscribed in a rectangle and the angle between the diagonals of the rectangle is $\tan^{-1}(2\sqrt{2})$, then find the eccentricity of the ellipse.

- Find the equation of an ellipse whose axes are the x - and y -axis and whose one focus is at $(4, 0)$ and eccentricity is $4/5$.
- An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm, then find the length of the string and distance between the pins.
- An arc of a bridge is semi-elliptical with the major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal, then prove that the best approximation of the height of the arc 2 m from the center of the base is $8/3$ m.
- If the foci of an ellipse are $(0, \pm 1)$ and the minor axis is of unit length, then find the equation of the ellipse. The axes of ellipse are the coordinate axes.
- A point P lies on the ellipse $\frac{(y-1)^2}{64} + \frac{(x+2)^2}{49} = 1$. If the distance of P from one focus is 10 units, then find its distance from other focus.
- An ellipse circumscribes a quadrilateral whose sides are given by $x = \pm 2$ and $y = \pm 4$. If the distance between foci is $4\sqrt{6}$ and major axis is along y -axis then find the equation of ellipse.
- Find the eccentricity of an ellipse distance between whose foci is 10 and that between focus and corresponding directrix is 15.
- A circle whose diameter is major axis of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b > 0$) meets minor axis at point P . If the orthocentre of $\triangle PF_1F_2$ lies on ellipse where F_1 and F_2 are foci of ellipse, then find the eccentricity of the ellipse.
- What is the ratio of the greatest and least focal distances of a point on the ellipse $4x^2 + 9y^2 = 36$?
- An ellipse is drawn with the major and minor axes of lengths 10 and 8, respectively. Using one focus as center a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. Then find the radius of the circle.
- The moon travels an elliptical path with Earth as one focus. The maximum distance from the moon to the earth is 405,500 km and the minimum distance is 363,300 km. What is the eccentricity of the orbit?
- Find the foci of the ellipse $25(x+1)^2 + 9(y+2)^2 = 225$.
- Prove that the area bounded by the circle $x^2 + y^2 = a^2$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the area of another ellipse having semi-axis $a-b$ and b , $a > b$.
- Find the lengths of the major and minor axes and the eccentricity of the ellipse $\frac{(3x-4y+2)^2}{16} + \frac{(4x+3y-5)^2}{9} = 1$.
- Find the locus of the middle points of all chords of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ which are at a distance of 2 units from the vertex of parabola $y^2 = -8ax$.

- $\frac{4x^2}{45} + \frac{4y^2}{81} = 1$ or $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$
- $\frac{1}{\sqrt{2}}$
- $\frac{1}{\sqrt{2}}$
- $\frac{x^2}{25} + \frac{y^2}{9} = 1$
- Length of string = $6 + 2\sqrt{5}$, Distance between pins = $2\sqrt{5}$
- $20x^2 + 4y^2 = 5$
- 6
- $\frac{x^2}{8} + \frac{y^2}{32} = 1$
- $\frac{1}{2}$
- $\frac{\sqrt{5}-1}{2}$
- $\frac{7+3\sqrt{5}}{2}$
- 2
- 0.05489
- $(-1, 2)$ $(-1, -6)$
- Major axis = $8/5$, Minor axis = $6/5$, $e = \frac{\sqrt{7}}{4}$
- $4\left(\frac{x^2}{16} + \frac{y^2}{81}\right) = \left(\frac{x^2}{4} + \frac{y^2}{9}\right)^2$

PARAMETRIC FORM OF ELLIPSE

We know that parametric form of circle having equation $x^2 + y^2 = a^2$ is given by $x = a \cos \theta$, $y = a \sin \theta$.

Parametric form of ellipse having equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is given by

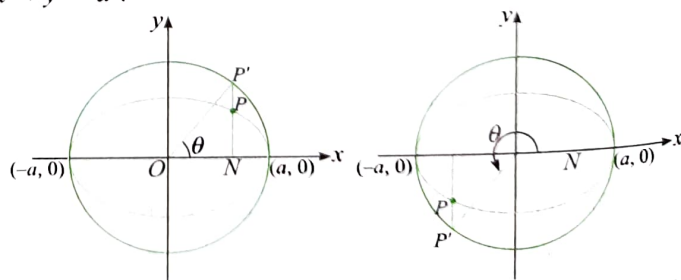
$$\frac{x}{a} = \cos \theta, \quad \frac{y}{b} = \sin \theta$$

$$\text{or } x = a \cos \theta, y = b \sin \theta$$

Here, θ is parameter and called eccentric angle of point $(a \cos \theta, b \sin \theta)$ on the ellipse. This point is also denoted as $P(\theta)$. To get the eccentric angle of point on the ellipse, we define auxiliary circle.

Auxiliary circle is the circle described on major axis of the ellipse as a diameter.

For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, equation of auxiliary circle is $x^2 + y^2 = a^2$.



From point $P(x, y)$ on the ellipse, we draw a line perpendicular to major axis intersecting major axis in N and auxiliary circle in P' . The points P and P' are called corresponding points on the ellipse and on auxiliary circle, respectively, which lie on the same side of major axis.

From the figure, $P' \equiv (a \cos \theta, a \sin \theta)$.

Since PP' is vertical line, for point P , $x = a \cos \theta$.

Putting the value of x in the equation of ellipse, we get

$$\cos^2 \theta + \frac{y^2}{b^2} = 1$$

$$\therefore y^2 = b^2 \sin^2 \theta$$

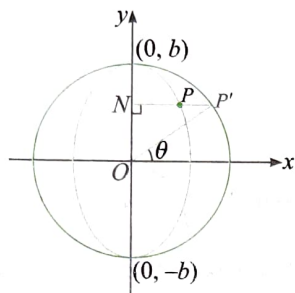
or $y = b \sin \theta$ (considering positive sign, as quadrant of the point is decided by θ)

$$\therefore P \equiv (a \cos \theta, b \sin \theta), \theta \in [0, 2\pi]$$

$$\text{Also, we have } \frac{PN}{PP'} = \frac{b \sin \theta}{a \sin \theta - b \sin \theta} = \frac{b}{a - b} = \text{Constant}$$

Thus, if from each point on a circle, a perpendicular is drawn on a fixed diameter, then the locus of a point P dividing these perpendiculars in a constant ratio is an ellipse whose auxiliary circle is the original circle.

For ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a < b$, equation of auxiliary circle is $x^2 + y^2 = b^2$. Corresponding points on ellipse and circle are $P(a \cos \theta, b \sin \theta)$ and $P'(b \cos \theta, b \sin \theta)$, respectively.



For ellipse having equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, parametric form is $x = h + a \cos \theta, y = k + b \sin \theta$.

ILLUSTRATION 6.20

Find the eccentric angle of a point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ whose distance from the centre of the ellipse is $\sqrt{5}$.

Sol. Let variable point on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ be $P(\sqrt{6} \cos \theta, \sqrt{2} \sin \theta)$.

Given that

$$CP = \sqrt{5} \quad (\text{where } C \text{ is centre})$$

$$\Rightarrow \sqrt{6 \cos^2 \theta + 2 \sin^2 \theta} = \sqrt{5}$$

$$\Rightarrow 6(1 - \sin^2 \theta) + 2 \sin^2 \theta = 5$$

$$\Rightarrow 4 \sin^2 \theta = 1$$

$$\Rightarrow \sin \theta = \pm \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

ILLUSTRATION 6.21

Find the number of rational points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

$$\text{Sol. } \frac{x^2}{9} + \frac{y^2}{4} = 1$$

Let any point on the ellipse be $(3 \cos \theta, 2 \sin \theta)$.

Since $\sin \theta$ and $\cos \theta$ can be rational for infinitely many values of $\theta \in [0, 2\pi]$, there exist infinite points.

ILLUSTRATION 6.22

Find the equation of the curve whose parametric equations are $x = 1 + 4 \cos \theta, y = 2 + 3 \sin \theta, \theta \in R$.

Sol. We have $x = 1 + 4 \cos \theta, y = 2 + 3 \sin \theta$. Therefore,

$$\left(\frac{x-1}{4}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$$

$$\text{or } \frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

which is an ellipse.

ILLUSTRATION 6.23

A line passing through origin $O(0, 0)$ intersects the two concentric circles of radii a and b at P and Q , respectively. If lines parallel to x -axis and y -axis through Q and P , respectively, meet at point R , then find the locus of R .

Sol.

Let $R \equiv (h, k)$.

From the figure,

$$h = a \cos \theta$$

$$\text{and } k = b \sin \theta$$

Squaring and adding, we get

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$$

So, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is required locus.

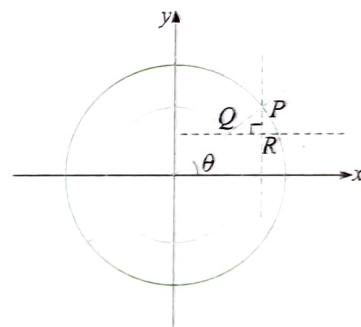


ILLUSTRATION 6.24

The auxiliary circle of a family of ellipses passes through the origin and makes intercepts of 8 units and 6 units on the x - and y -axis, respectively. If the eccentricity of all such ellipses is $1/2$, then find the locus of the focus.

Sol. The auxiliary circle makes intercepts of 8 units and 6 units on the x - and y -axes, respectively. Therefore, the center of the circle is $(4, 3)$ and the radius is 5.

$(4, 3)$ is the center of inscribed ellipse also.

Now, the distance of the focus from the center of the ellipse is ae .

Since the radius of the auxiliary circle is 5, the length of the semi-major axis, a will be 5.

$$\text{Given } e = \frac{1}{2}$$

$$\therefore ae = \frac{5}{2}$$

Therefore, the locus of the focus is

$$(x-4)^2 + (y-3)^2 = \frac{25}{4}$$

ILLUSTRATION 6.25

If the line $lx + my + n = 0$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in points whose eccentric angles differ by $\frac{\pi}{2}$, then find the value of $\frac{a^2l^2 + b^2m^2}{n^2}$.

Sol. Let the line $lx + my + n = 0$ cut the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at

$$P(a \cos \theta, b \sin \theta) \text{ and } Q\left(a \cos\left(\frac{\pi}{2} + \theta\right), b \sin\left(\frac{\pi}{2} + \theta\right)\right) \\ \equiv Q(-a \sin \theta, b \cos \theta).$$

Since points P and Q lie on the line, we have

$$la \cos \theta + mb \sin \theta = -n$$

$$\text{and } -la \sin \theta + mb \cos \theta = -n$$

Squaring and adding, we get

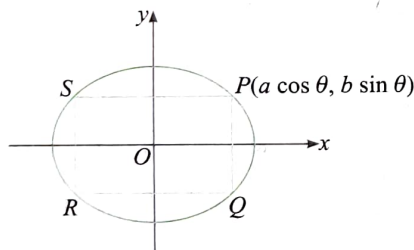
$$a^2l^2 + b^2m^2 = 2n^2$$

$$\Rightarrow \frac{a^2l^2 + b^2m^2}{n^2} = 2$$

ILLUSTRATION 6.26

Find the greatest area of the rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Rectangle can be inscribed in the ellipse with its sides parallel to axes of the ellipse as shown in the following figure.



$$\text{Let } P \equiv (a \cos \theta, b \sin \theta)$$

$$\therefore \text{Area of rectangle PQRS}$$

$$= (2a \cos \theta)(2b \sin \theta)$$

$$= 2ab \sin 2\theta$$

This is maximum when $\sin 2\theta = 1$.

Hence, maximum area $= 2ab(1) = 2ab$

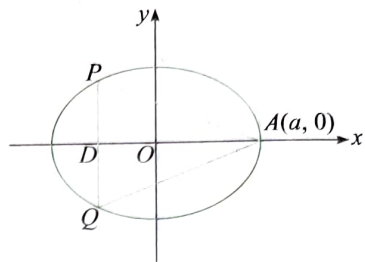
ILLUSTRATION 6.27

Find the area of the greatest isosceles triangle that can be inscribed in the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ having its vertex coincident with one extremity of the major axis.

Sol. Let APQ be the isosceles triangle inscribed in the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ with one vertex A at the extremity $(a, 0)$ of the major axis. If the coordinates of P are $(a \cos \theta, b \sin \theta)$, then those of Q will be $(a \cos \theta, -b \sin \theta)$.

So, the area of ΔAPQ is given by

$$\begin{aligned} \Delta &= \frac{1}{2} PQ \cdot AD \\ &= PD \cdot AD \\ &= PD (OA + OD) \\ &= b \sin \theta (a - a \cos \theta) \\ &= \frac{1}{2} ab (2 \sin \theta - \sin 2\theta), \quad 0 < \theta < \pi \end{aligned}$$



$$\therefore \frac{d\Delta}{d\theta} = ab (\cos \theta - \cos 2\theta),$$

$$\text{and } \frac{d^2\Delta}{d\theta^2} = ab(-\sin \theta + 2 \sin 2\theta)$$

For maximum or minimum of Δ ,

$$\frac{d\Delta}{d\theta} = 0$$

$$\text{or } \cos \theta - \cos 2\theta = 0$$

$$\text{or } 2 \cos^2 \theta - \cos \theta - 1 = 0$$

$$\text{or } (2 \cos \theta + 1)(\cos \theta - 1) = 0$$

$$\text{or } \cos \theta = -\frac{1}{2} \quad (\because \cos \theta \neq 1 \text{ as } \theta \neq 0)$$

$$\text{or } \theta = \frac{2\pi}{3}$$

When $\theta = 2\pi/3$, we have

$$\frac{d^2\Delta}{d\theta^2} = -\frac{1}{2}(3\sqrt{3})ab, \quad (-ve)$$

Therefore, the area is maximum when $\theta = 2\pi/3$.

Therefore, maximum area

$$\begin{aligned} \Delta &= \frac{1}{2} ab \left[2 \sin \frac{2\pi}{3} - \sin \frac{4\pi}{3} \right] \\ &= \frac{1}{4} (3\sqrt{3}) ab \end{aligned}$$

ILLUSTRATION 6.28

Prove that the sum of eccentric angles of four concyclic points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $2n\pi$, where $n \in \mathbb{Z}$.

Sol. Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cut the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in four points P, Q, R and S .

Solving circle and ellipse ($x = a \cos \theta, y = b \sin \theta$), we have $a^2 \cos^2 \theta + b^2 \sin^2 \theta + 2ag \cos \theta + 2bf \sin \theta + c = 0$

$$\begin{aligned} \Rightarrow a^2 \left(\frac{1-t^2}{1+t^2} \right)^2 + b^2 \left(\frac{2t}{1+t^2} \right)^2 + 2ag \left(\frac{1-t^2}{1+t^2} \right) \\ + 2bf \left(\frac{2t}{1+t^2} \right) + c = 0, \text{ where } t = \tan \frac{\theta}{2} \end{aligned}$$

$$\Rightarrow a^2(1-t^2)^2 + 4b^2t^2 + 2ag(1-t^2)(1+t^2) + 4bft(1+t^2) + c(1+t^2)^2 = 0$$

$$\Rightarrow (a^2 - 2ag + c)t^4 + 4bft^3 + (-2a^2 + 4b^2 + 2c)t^2 + 4bft + (a^2 + 2ag + c) = 0 \quad (1)$$

Roots of equation are $\tan \frac{\alpha}{2}$, $\tan \frac{\beta}{2}$, $\tan \frac{\gamma}{2}$ and $\tan \frac{\delta}{2}$, where α, β, γ and δ are eccentric angles of P, Q, R and S , respectively.

$$\text{Now, } \tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = \frac{s_1 - s_3}{1 - s_2 + s_4}$$

Where s_i = sum of products of tangents of half angles taken 'i' at a time

Clearly, from equation (1),

$$s_1 = s_3 = \frac{-4bf}{a^2 - 2ag + c}$$

$$\Rightarrow \tan\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2}\right) = 0$$

$$\Rightarrow \frac{\alpha + \beta + \gamma + \delta}{2} = n\pi, n \in \mathbb{Z}$$

$$\Rightarrow \alpha + \beta + \gamma + \delta = 2n\pi, n \in \mathbb{Z}$$

SOME IMPORTANT PROPERTIES OF ELLIPSE

- I. The ratio of area of any triangle PQR inscribed in an ellipse to the triangle formed by corresponding points on the auxiliary circle is equal to the ratio of semi minor axis to semi major axis.

Proof: Consider the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$.

Let the three points on the ellipse be $P(a \cos \alpha, b \sin \alpha)$, $Q(a \cos \beta, b \sin \beta)$ and $R(a \cos \gamma, b \sin \gamma)$.

Then corresponding points on the auxiliary circle are $P'(a \cos \alpha, a \sin \alpha)$, $Q'(a \cos \beta, a \sin \beta)$ and $R'(a \cos \gamma, a \sin \gamma)$.

$$\text{Now, } \frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta P'Q'R'} = \frac{\frac{1}{2} \begin{vmatrix} a \cos \alpha & b \sin \alpha & 1 \\ a \cos \beta & b \sin \beta & 1 \\ a \cos \gamma & b \sin \gamma & 1 \end{vmatrix}}{\frac{1}{2} \begin{vmatrix} a \cos \alpha & a \sin \alpha & 1 \\ a \cos \beta & a \sin \beta & 1 \\ a \cos \gamma & a \sin \gamma & 1 \end{vmatrix}}$$

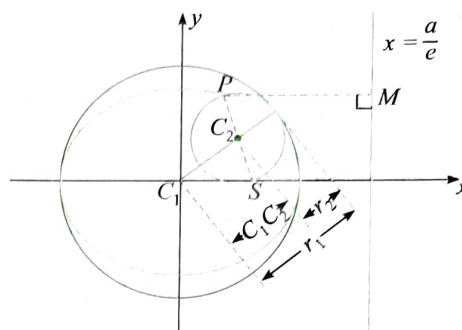
$$= \frac{ab \begin{vmatrix} \cos \alpha & \sin \alpha & 1 \\ \cos \beta & \sin \beta & 1 \\ \cos \gamma & \sin \gamma & 1 \end{vmatrix}}{a^2 \begin{vmatrix} \cos \alpha & \sin \alpha & 1 \\ \cos \beta & \sin \beta & 1 \\ \cos \gamma & \sin \gamma & 1 \end{vmatrix}}$$

$$= \frac{b}{a} = \text{constant}$$

- II. Circle described on focal length as diameter always touches auxiliary circle.

Proof: Consider the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$.

Let point P on the ellipse be $(a \cos \theta, b \sin \theta)$.



From the definition of the ellipse,

$$SP = e \cdot PM = e \left(\frac{a}{e} - a \cos \theta \right) = a(1 - e \cos \theta)$$

Equation of auxiliary circle is $x^2 + y^2 = a^2$, having centre $C_1(0, 0)$ and radius $r_1 = a$.

Circle described on SP as a diameter has centre $C_2\left(\frac{ae + a \cos \theta}{2}, \frac{b \sin \theta}{2}\right)$ and radius $r_2 = \frac{a(1 - e \cos \theta)}{2}$.

$$\text{Now, } r_1 - r_2 = \frac{a(1 + e \cos \theta)}{2}$$

$$\begin{aligned} \text{and } C_1C_2 &= \sqrt{\frac{a^2(e + \cos \theta)^2}{4} + \frac{b^2 \sin^2 \theta}{4}} \\ &= \frac{a}{2} \sqrt{e^2 + 2e \cos \theta + \cos^2 \theta + \frac{b^2}{a^2} \sin^2 \theta} \\ &= \frac{a}{2} \sqrt{e^2 + 2e \cos \theta + \cos^2 \theta + (1 - e^2) \sin^2 \theta} \\ &= \frac{a}{2} \sqrt{e^2 \cos^2 \theta + 2e \cos \theta + 1} \\ &= \frac{a}{2} (e \cos \theta + 1) \\ &= r_1 - r_2 \end{aligned}$$

Hence, the circle on SP as a diameter touches the auxiliary circle internally.

ILLUSTRATION 6.29

The ratio of the area of triangle inscribed in ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to that of triangle formed by the corresponding points on the auxiliary circle is 0.5. Then, find the eccentricity of the ellipse.

Sol. The given ratio is

$$\frac{b}{a} = \frac{1}{2}$$

$$\text{Now, } e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{4} = \frac{3}{4} \text{ or } e = \frac{\sqrt{3}}{2}$$

EQUATION OF CHORD OF ELLIPSE

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Equation of the chord passing through the points $P(a \cos \alpha, b \sin \alpha)$ and $Q(a \cos \beta, b \sin \beta)$ is

$$y - b \sin \alpha = \frac{b \sin \beta - b \sin \alpha}{a \cos \beta - a \cos \alpha} (x - a \cos \alpha)$$

$$\Rightarrow b(\sin \alpha - \sin \beta)x + a(\cos \beta - \cos \alpha)y = ab \sin(\alpha - \beta)$$

$$\Rightarrow \frac{x}{a} \cos \frac{\alpha + \beta}{2} + \frac{y}{b} \sin \frac{\alpha + \beta}{2} = \cos \frac{\alpha - \beta}{2}$$

ILLUSTRATION 6.30

If α and β are the eccentric angles of the extremities of a focal chord through $S(ae, 0)$ of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then prove that the eccentricity of the ellipse is $\frac{\sin \alpha + \sin \beta}{\sin(\alpha + \beta)}$.

Sol. Equation of chord through points $P(a \cos \alpha, b \sin \alpha)$, $Q(a \cos \beta, b \sin \beta)$ is

$$\frac{x}{a} \cos \frac{\alpha + \beta}{2} + \frac{y}{b} \sin \frac{\alpha + \beta}{2} = \cos \frac{\alpha - \beta}{2}$$

If it passes through focus $S(ae, 0)$ then

$$e \cos \frac{\alpha + \beta}{2} = \cos \frac{\alpha - \beta}{2}$$

$$\Rightarrow e = \frac{\cos \frac{\alpha - \beta}{2}}{\cos \frac{\alpha + \beta}{2}}$$

$$= \frac{2 \sin \left(\frac{\alpha + \beta}{2} \right) \cdot \cos \left(\frac{\alpha - \beta}{2} \right)}{2 \sin \left(\frac{\alpha + \beta}{2} \right) \cdot \cos \left(\frac{\alpha + \beta}{2} \right)} = \frac{\sin \alpha + \sin \beta}{\sin(\alpha + \beta)}$$

ILLUSTRATION 6.31

Find the equation of chord of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ joining two points $P\left(\frac{\pi}{3}\right)$ and $Q\left(\frac{\pi}{6}\right)$.

Sol. Equation of chord PQ is

$$\frac{x}{5} \cos \left(\frac{\frac{\pi}{6} + \frac{\pi}{3}}{2} \right) + \frac{y}{4} \sin \left(\frac{\frac{\pi}{6} + \frac{\pi}{3}}{2} \right) = \cos \left(\frac{\frac{\pi}{6} - \frac{\pi}{3}}{2} \right)$$

$$\Rightarrow \frac{x}{5} \cos \frac{\pi}{4} + \frac{y}{4} \sin \frac{\pi}{4} = \cos \frac{\pi}{12}$$

$$\Rightarrow \frac{x}{5} \frac{1}{\sqrt{2}} + \frac{y}{4} \frac{1}{\sqrt{2}} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\Rightarrow \frac{x}{5} + \frac{y}{4} = \frac{\sqrt{3} + 1}{2}$$

CONCEPT APPLICATION EXERCISE 6.3

1. Prove that any point on the ellipse whose foci are $(-1, 0)$ and $(7, 0)$ and eccentricity is $1/2$ is $(3 + 8 \cos \theta, 4\sqrt{3} \sin \theta)$, $\theta \in \mathbb{R}$.
2. Find the eccentric angles of the extremities of the latus recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
3. If the chord joining points $P(\alpha)$ and $Q(\beta)$ on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ subtends a right angle at the vertex $A(a, 0)$, then prove that $\tan(\alpha/2) \tan(\beta/2) = -b^2/a^2$.
4. If the area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is 4π , then find the maximum area of rectangle inscribed in the ellipse.
5. Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$, and the x -axis is maximum.
6. Find the maximum length of chord of the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$, such that eccentric angles of its extremities differ by $\frac{\pi}{2}$.
7. If $P(\theta)$ and $Q\left(\frac{\pi}{2} + \theta\right)$ are two points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, then find the locus of midpoint of PQ .
8. Let P be any point on ellipse $3x^2 + 4y^2 = 12$. If S and S' are its foci then find the locus of the centroid of triangle PSS' .
9. The line $x + 2y = 1$ cuts the ellipse $x^2 + 4y^2 = 1$ at two distinct points A and B . Point C is on the ellipse such that area of triangle ABC is maximum, then find point C .

ANSWERS

2. $\tan^{-1}\left(\frac{b}{ae}\right), \pi - \tan^{-1}\left(\frac{b}{ae}\right), \pi + \tan^{-1}\left(\frac{b}{ae}\right), 2\pi - \tan^{-1}\left(\frac{b}{ae}\right)$
4. 8 sq. units
5. $(3/2, 1)$
6. 4
7. $\frac{4x^2}{9} + y^2 = 2$
8. $\frac{9x^2}{4} + 3y^2 = 1$
9. $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$

TANGENT TO ELLIPSE

A LINE AND AN ELLIPSE

A line may meet the ellipse in one point or two distinct points or it may not meet ellipse at all.

Consider the standard equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

and a line having equation

$$y = mx + c \quad (2)$$

Solving line and ellipse, we get

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

$$\text{or } (a^2m^2 + b^2)x^2 + 2a^2cmx + (a^2c^2 - a^2b^2) = 0 \quad (3)$$

Discriminant of this equation is

$$D = 4a^4c^2m^2 - 4(a^2c^2 - a^2b^2)(a^2m^2 + b^2) \\ = 4a^2b^2(a^2m^2 + b^2 - c^2)$$

Now, this equation has either two distinct real roots or two equal roots or has non-real roots.

If line meets the ellipse in two distinct points (A and B), then equation (3) has two distinct real roots.

$$\therefore D > 0 \text{ or } c^2 < a^2m^2 + b^2$$

If line meets the ellipse in one point P (i.e., it is tangent to the ellipse), then equation (3) has two equal roots.

$$\therefore D = 0 \text{ or } c^2 = a^2m^2 + b^2$$

If line doesn't meet the ellipse, then equation (3) has imaginary roots.

$$\therefore D < 0 \text{ or } c^2 > a^2m^2 + b^2$$

EQUATION OF TANGENT TO ELLIPSE HAVING GIVEN SLOPE

We have already learned that if line $y = mx + c$ touches the ellipse

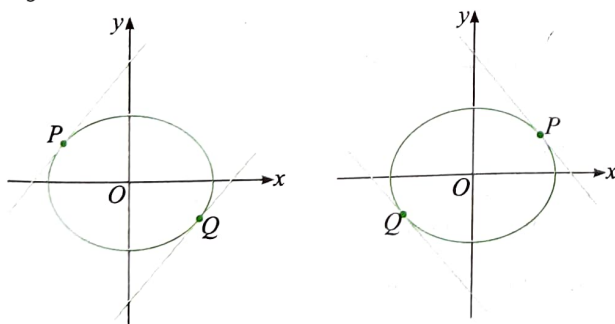
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ then } c^2 = a^2m^2 + b^2.$$

In that case, equation of line is

$$y = mx \pm \sqrt{a^2m^2 + b^2}, m \in R \quad (4)$$

The equation (4) represents two parallel tangents to ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ having slope } m.$$



From equation (3), if $D = 0$, then we have

$$x = \frac{-a^2cm}{a^2m^2 + b^2} = \frac{-a^2cm}{c^2} = \frac{-a^2m}{c}$$

$$\therefore y = m\left(\frac{-a^2m}{c}\right) + c = \frac{-a^2m^2 + c^2}{c} = \frac{b^2}{c}$$

Thus, the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$.

For tangent $y = mx + \sqrt{a^2m^2 + b^2}$, point of contact is

$$P\left(\frac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right).$$

For tangent $y = mx - \sqrt{a^2m^2 + b^2}$, point of contact is

$$Q\left(\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{-b^2}{\sqrt{a^2m^2 + b^2}}\right).$$

ILLUSTRATION 6.32

Find the points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the tangents at each of them make equal angles with the axes.

Sol. Since tangents make equal angles with the axes, their slope i.e., m should be ± 1 .

Points of contact of tangent having slope ' m ' are given by

$$\left(\frac{\pm a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2 + b^2}}\right).$$

For $m = 1$, points of contact are $\left(\frac{a^2}{\sqrt{a^2 + b^2}}, \frac{-b^2}{\sqrt{a^2 + b^2}}\right)$ and

$$\left(\frac{-a^2}{\sqrt{a^2 + b^2}}, \frac{b^2}{\sqrt{a^2 + b^2}}\right).$$

For $m = -1$, points of contact are $\left(\frac{a^2}{\sqrt{a^2 + b^2}}, \frac{b^2}{\sqrt{a^2 + b^2}}\right)$ and

$$\left(\frac{-a^2}{\sqrt{a^2 + b^2}}, \frac{-b^2}{\sqrt{a^2 + b^2}}\right).$$

ILLUSTRATION 6.33

An ellipse passes through the point (4, -1) and touches the line $x + 4y - 10 = 0$. Find its equation if its axes coincide with coordinate axes.

Sol. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

Since it passes through the point (4, -1), we have

$$a^2 + 16b^2 = a^2b^2 \quad (2)$$

Equation of given tangent is

$$x + 4y - 10 = 0$$

$$\text{or } y = -\frac{1}{4}x + \frac{5}{2}$$

Comparing with $y = mx + c$, we have $m = -\frac{1}{4}$, $c = \frac{5}{2}$.

Now, $c^2 = a^2m^2 + b^2$

$$\Rightarrow \frac{25}{4} = \frac{a^2}{16} + b^2$$

$$\Rightarrow a^2 + 16b^2 = 100 \quad (3)$$

From (2) and (3), we get

$$a^2b^2 = 100 \text{ or } ab = 10 \quad (4)$$

Solving (3) and (4), we get

$$a = 4\sqrt{5} \text{ and } b = \frac{\sqrt{5}}{2} \text{ or } a = 2\sqrt{5}, b = \sqrt{5}$$

Hence, the equations of ellipses are

$$\frac{x^2}{80} + \frac{4y^2}{5} = 1 \text{ and } \frac{x^2}{20} + \frac{y^2}{5} = 1$$

ILLUSTRATION 6.34

Find the maximum area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which touches the line $y = 3x + 2$.

Sol. Area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab .

Line $y = 3x + 2$ touches the ellipse.

Here, $m = 3$ and $c = 2$

Using relation, $c^2 = a^2m^2 + b^2$, we have

$$4 = 9a^2 + b^2$$

Now, $(3a - b)^2 \geq 0$

$$\Rightarrow 9a^2 + b^2 \geq 6ab$$

$$\Rightarrow 6ab \leq 4$$

$$\Rightarrow \pi ab \leq \frac{2\pi}{3}$$

So, maximum area of the ellipse is $\frac{2\pi}{3}$ sq. units.

ILLUSTRATION 6.35

Find the locus of the foot of the perpendicular drawn from the centre upon any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Let the foot of perpendicular from $O(0, 0)$ on the tangent be $P(h, k)$.

Thus, the slope of OP is $\frac{k}{h}$.

Therefore, the slope of tangent is $-\frac{h}{k}$.

So, equation of tangent is

$$y - k = -\frac{h}{k}(x - h)$$

$$\text{or } y = -\frac{h}{k}x + \frac{h^2 + k^2}{k}$$

Using $c^2 = a^2m^2 + b^2$, we have

$$\left(\frac{h^2 + k^2}{k}\right)^2 = a^2\left(-\frac{h}{k}\right)^2 + b^2$$

$$\text{or } a^2x^2 + b^2y^2 = (x^2 + y^2)^2$$

This is the required locus.

EQUATION OF TANGENT TO ELLIPSE AT A GIVEN POINT ON IT

We know that the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$.

$$\text{Let } \frac{-a^2m}{c} = x_1 \text{ and } \frac{b^2}{c} = y_1.$$

So, equation of tangent $y = mx + c$ becomes

$$y = -\frac{b^2x_1}{a^2y_1}x + \frac{b^2}{y_1}$$

$$\text{Or } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

$$\text{or } T = 0, \text{ where } T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

This is equation of tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point (x_1, y_1) on it.

The slope of the tangent can also be obtained using differentiation.

Differentiating equation of ellipse w.r.t. x , we get

$$\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y} = \text{Slope of tangent to ellipse at any point on it}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{b^2x_1}{a^2y_1}$$

So, equation of tangent at point (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1)$$

Simplifying, we get the equation as $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$.

If $(x_1, y_1) \equiv (a \cos \theta, b \sin \theta)$, then equation of tangent becomes

$$\frac{x(a \cos \theta)}{a^2} + \frac{y(b \sin \theta)}{b^2} - 1 = 0$$

$$\text{or } \frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta - 1 = 0$$

ILLUSTRATION 3.36

If the line $3x + 4y = \sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$, then find the point of contact.

Sol. Equation of given tangent is

$$3x + 4y = \sqrt{7} \quad (1)$$

Let this line touch the given ellipse at point $P(x_1, y_1)$.

Thus, equation of tangent to ellipse at point P is

$$3xx_1 + 4yy_1 = 1 \quad (2)$$

Equations (1) and (2) represent the same straight line.

$$\therefore \frac{3x_1}{3} = \frac{4y_1}{4} = \frac{1}{\sqrt{7}}$$

$$\therefore P(x_1, y_1) \equiv \left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$$

ILLUSTRATION 6.37

If $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then find its eccentric angle θ of point of contact.

Sol. Equation of tangent to ellipse at $P(a \cos \theta, b \sin \theta)$ is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

Also, $\frac{x}{a\sqrt{2}} + \frac{y}{b\sqrt{2}} = 1$ is the tangent. Therefore,

$$\frac{\cos \theta}{1/\sqrt{2}} + \frac{\sin \theta}{1/\sqrt{2}} = 1$$

$$\text{or } \cos \theta = \frac{1}{\sqrt{2}}; \sin \theta = \frac{1}{\sqrt{2}}$$

$$\text{or } \theta = 45^\circ$$

ILLUSTRATION 3.38

Tangent is drawn to ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3} \cos \theta, \sin \theta)$, $\theta \in (0, \frac{\pi}{2})$. Then find the value of θ such that sum of intercepts on axes made by this tangent is minimum.

Sol. The equation of tangent at a given point on the ellipse is

$$\frac{x \cdot 3\sqrt{3} \cos \theta}{27} + \frac{y \cdot \sin \theta}{1} = 1$$

$$\text{or } \frac{x}{3\sqrt{3} \sec \theta} + \frac{y}{\csc \theta} = 1$$

Sum of the intercepts on axes is given by

$$S = 3\sqrt{3} \sec \theta + \csc \theta; \theta \in (0, \frac{\pi}{2})$$

To find the minimum value of S , we differentiate S w.r.t. θ .

$$\therefore \frac{dS}{d\theta} = 3\sqrt{3} \sec \theta \tan \theta - \csc \theta \cot \theta$$

$$\text{If } \frac{dS}{d\theta} = 0, \text{ then } \tan^3 \theta = \frac{1}{3\sqrt{3}} \text{ or } \tan \theta = \frac{1}{\sqrt{3}}.$$

Therefore, $\theta = \frac{\pi}{6}$, for which value of S is minimum as

$$\lim_{\theta \rightarrow 0, \frac{\pi}{2}} (3\sqrt{3} \sec \theta + \csc \theta) \rightarrow \infty.$$

ILLUSTRATION 6.39

The tangent at a point P on an ellipse intersects the major axis at T and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

Sol. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $P(a \cos \theta, b \sin \theta)$ be a point on the ellipse.

The equation of the tangent at P is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

It meets the major axis at $T(a \sec \theta, 0)$.

The coordinates of N are $(a \cos \theta, 0)$.

The equation of the circle with NT as its diameter is

$$(x - a \sec \theta)(x - a \cos \theta) + y^2 = 0$$

$$\text{or } x^2 + y^2 - ax(\sec \theta + \cos \theta) + a^2 = 0$$

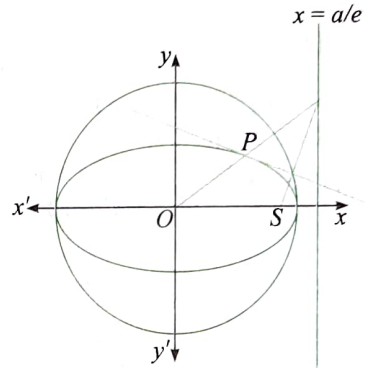
It cuts the auxiliary circle $x^2 + y^2 - a^2 = 0$ orthogonally as

$$2\left(\frac{-a \sec \theta - a \cos \theta}{2}\right)(0) + 2(0)(0) = a^2 - a^2 = 0.$$

ILLUSTRATION 6.40

Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the center of the ellipse to the point of contact meet at the corresponding directrix.

Sol.



Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and O be the center.

The tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

whose slope is $-b^2 x_1 / a^2 y_1$.

The focus of the ellipse is $S(ae, 0)$.

The equation of the line through $S(ae, 0)$ perpendicular to the tangent at P is

$$y = \frac{a^2 y_1}{b^2 x_1} (x - ae) \quad (1)$$

The equation of OP is

$$y = \frac{y_1}{x_1} x \quad (2)$$

Solving (1) and (2), we get

$$\frac{y_1}{x_1} x = \frac{a^2 y_1}{b^2 x_1} (x - ae)$$

$$\text{or } x(a^2 - b^2) = a^3 e$$

$$\text{or } x a^2 e^2 = a^3 e$$

$$\text{or } x = \frac{a}{e}$$

This is the corresponding directrix.

ILLUSTRATION 6.41

Find the point on the ellipse $16x^2 + 11y^2 = 256$ where the common tangent to it and the circle $x^2 + y^2 - 2x = 15$ touch.

Sol. The given ellipse is

$$\frac{x^2}{16} + \frac{y^2}{(256/11)} = 1$$

Tangent to it at point $P(4 \cos \theta, (16/\sqrt{11}) \sin \theta)$ is

$$\frac{x}{4} \cos \theta + \frac{y}{(16/\sqrt{11})} \sin \theta = 1$$

It also touches the circle $(x-1)^2 + y^2 = 16$. Therefore,

$$\frac{\left| \frac{1}{4} \cos \theta - 1 \right|}{\sqrt{\frac{\cos^2 \theta}{16} + \frac{11 \sin^2 \theta}{256}}} = 4$$

$$\text{or } \frac{|\cos \theta - 4|}{\sqrt{16 \cos^2 \theta + 11 \sin^2 \theta}} = 1$$

$$\text{or } \cos^2 \theta - 8 \cos \theta + 16 = 11 + 5 \cos^2 \theta$$

$$\text{or } 4 \cos^2 \theta + 8 \cos \theta - 5 = 0$$

$$\text{or } (2 \cos \theta - 1)(2 \cos \theta + 5) = 0$$

$$\text{or } \cos \theta = \frac{1}{2} \text{ or } \theta = \pm \frac{\pi}{3}$$

Therefore, points are $(2, \pm 8\sqrt{3}/\sqrt{11})$.

ILLUSTRATION 6.42

Consider an ellipse $\frac{x^2}{4} + y^2 = \alpha$ (α is parameter > 0) and a parabola $y^2 = 8x$. If a common tangent to the ellipse and the parabola meets the coordinate axes at A and B , respectively, then find the locus of the midpoint of AB .

Sol. The equation of tangent to $y^2 = 8x$ at $(2t^2, 4t)$ is

$$yt - x - 2t^2 = 0 \quad (1)$$

The equation of tangent to the ellipse

$$\frac{x^2}{4\alpha} + \frac{y^2}{\alpha} = 1$$

$$\text{at } (2\sqrt{\alpha} \cos \theta, \sqrt{\alpha} \sin \theta)$$

$$\text{is } \frac{x \cos \theta}{2\sqrt{\alpha}} + \frac{y \sin \theta}{\sqrt{\alpha}} = 1 \quad (2)$$

Comparing (1) and (2), we get

$$\frac{\sqrt{\alpha}}{\cos \theta} = -t^2, \quad \frac{\sqrt{\alpha}}{\sin \theta} = 2t \quad (3)$$

If the tangent meets the coordinate axes at A and B , then

$$A \equiv \left(\frac{2\sqrt{\alpha}}{\cos \theta}, 0 \right), B \equiv \left(0, \frac{\sqrt{\alpha}}{\sin \theta} \right)$$

Let the midpoint of AB be (h, k) . Then,

$$h = \frac{\sqrt{\alpha}}{\cos \theta}, k = \frac{\sqrt{\alpha}}{2 \sin \theta}$$

$$\therefore h = -t^2, k = t \text{ or } k^2 = -h$$

$$\text{or } y^2 = -x$$

[From (3)]

EQUATION OF TANGENT TO ELLIPSE FROM EXTERNAL POINT

Consider the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let the tangents be drawn to the ellipse from point $P(x_1, y_1)$ lying outside the ellipse.

To get the slope of the tangents, we consider one of the equations of tangents to ellipse having slope m .

Thus, tangent having equation $y = mx + \sqrt{a^2 m^2 + b^2}$ passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 + \sqrt{a^2 m^2 + b^2}$$

$$\text{or } (y_1 - mx_1)^2 = a^2 m^2 + b^2$$

Solving this quadratic equation, we get two values of slopes of tangents which pass through point P .

ILLUSTRATION 6.43

Find the equations of the tangents drawn from the point $(2, 3)$ to the ellipse $9x^2 + 16y^2 = 144$.

Sol. Given ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Let the equation of tangent be $y = mx + \sqrt{16m^2 + 9}$.

It passes through the point $(2, 3)$.

$$\therefore 3 = 2m + \sqrt{16m^2 + 9}$$

$$\Rightarrow (3 - 2m)^2 = 16m^2 + 9$$

$$\Rightarrow 12m^2 + 12m = 0$$

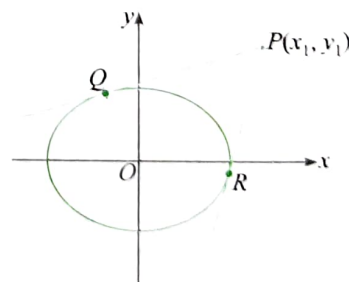
$$\Rightarrow m = 0, -1$$

Therefore, equations of tangents are

$$y - 3 = -1(x - 2) \text{ and } y - 3 = 0$$

$$\text{or } y = -x + 5 \text{ and } y = 3$$

EQUATION OF PAIR OF TANGENTS FROM EXTERNAL POINT



Equation of pair of tangents from external point $P(x_1, y_1)$ to the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is

$$\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 \right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \right)$$

$$\text{or } T^2 = SS_1$$

ILLUSTRATION 6.44

Find the angle between the pair of tangents from the point $(1, 2)$ to the ellipse $3x^2 + 2y^2 = 5$.

Sol. The combined equation of the pair of tangents drawn from $(1, 2)$ to the ellipse $3x^2 + 2y^2 = 5$ is

$$(3x + 4y - 5)^2 = (3x^2 + 2y^2 - 5)(3 + 8 - 5)$$

[Using $T^2 = SS_1$]

$$\text{or } 9x^2 - 24xy - 4y^2 + \dots = 0$$

If angle between pair of lines is θ , then

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}, \text{ where } a = 9, h = -12, b = -4$$

$$\therefore \tan \theta = \frac{12}{\sqrt{5}}$$

$$\Rightarrow \theta = \tan^{-1} \frac{12}{\sqrt{5}}$$

DIRECTOR CIRCLE

In parabola, locus of point of intersection of perpendicular tangents is directrix. But in ellipse, locus of point of intersection of perpendicular tangents is circle which is called director circle.

Consider the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

Let perpendicular tangents to ellipse intersect at point $P(h, k)$. So, the equation of pair of tangents through point P is

$$\left(\frac{hx}{a^2} + \frac{ky}{b^2} - 1\right)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)$$

Since tangents are perpendicular, in above equation sum of coefficients of x^2 and y^2 is zero.

$$\therefore \frac{1}{a^2} \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) + \frac{1}{b^2} \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) - \frac{h^2}{a^4} - \frac{k^2}{b^4} = 0$$

$$\Rightarrow \frac{k^2}{a^2 b^2} - \frac{1}{a^2} + \frac{h^2}{a^2 b^2} - \frac{1}{b^2} = 0$$

$$\Rightarrow h^2 + k^2 = a^2 + b^2$$

Thus, locus of point is P is $x^2 + y^2 = a^2 + b^2$, which is circle.

This is called director circle of ellipse.

Alternative method:

One of the equations of tangents to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ having slope ' m ' is

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

If it passes through the point $P(h, k)$ then

$$k = mh + \sqrt{a^2 m^2 + b^2}$$

$$\text{or } (a^2 m^2 + b^2) = (k - mh)^2$$

$$\text{or } (h^2 - a^2)m^2 - 2hkm + k^2 - b^2 = 0$$

This equation has two roots m_1 and m_2 , which are slopes of two tangents to ellipse drawn from point P .

Since tangents are perpendicular,

$$m_1 m_2 = -1$$

$$\Rightarrow \frac{k^2 - b^2}{h^2 - a^2} = -1$$

$$\text{or } h^2 + k^2 = a^2 + b^2,$$

So, locus of point is P is $x^2 + y^2 = a^2 + b^2$.

ILLUSTRATION 6.45

An ellipse having fixed length of major and minor axes slides between two perpendicular straight lines. Identify the locus of centre of the ellipse.

Sol. Ellipse slides between two perpendicular lines.

So, these lines are perpendicular tangents and their point of intersection P lies on the director circle.

If centre of the ellipse is fixed, then all the points of intersection of perpendicular tangents lie at fixed distance which is equal to radius of the director circle.

So, if point of intersection of perpendicular tangents is fixed (point P), then centre of variable ellipse also lies at fixed distance from point P .

Therefore, locus of centre of the ellipse is a circle.

ILLUSTRATION 6.46

If there exists exactly one point on the line $3x + 4y + 25 = 0$, from which perpendicular tangents can be drawn to ellipse

$\frac{x^2}{a^2} + y^2 = 1$ ($a > 1$), then find the eccentricity of the ellipse.

Sol. Locus of point of intersection of perpendicular tangents is director circle.

If there exists exactly one such point on the line $3x + 4y + 25 = 0$, then line must touch the director circle whose equation is $x^2 + y^2 = a^2 + 1$.

Distance of centre of the circle from the line is 5.

So, we must have

$$25 = a^2 + 1$$

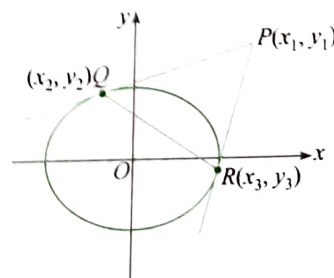
$$\Rightarrow a^2 = 24 \quad \therefore a = \sqrt{24}$$

$$\therefore \text{Eccentricity} = \sqrt{1 - \frac{1}{24}} = \sqrt{\frac{23}{24}}$$

CHORD OF CONTACT

Let the tangents drawn from point $P(x_1, y_1)$ to the ellipse touch the ellipse at points $Q(x_2, y_2)$ and $R(x_3, y_3)$. Then the line QR is called chord of contact.

Consider ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Equation of chord of contact w.r.t. point P is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0 \text{ or } T = 0$$

This equation can be derived the same way as we derived it for circle and parabola.

ILLUSTRATION 6.47

If from a point P , tangents PQ and PR are drawn to the ellipse $\frac{x^2}{2} + y^2 = 1$ so that equation of chord of contact QR is $x + 3y = 1$, then find the coordinates of P .

Sol. Let coordinates of P be (x_1, y_1) .
Then equation of QR is

$$\frac{x_1 x}{2} + y_1 y = 1 \quad (1)$$

But given equation of QR is

$$x + 3y = 1 \quad (2)$$

Equation (1) and (2) must be identical.

$$\begin{aligned} \therefore \frac{x_1}{2} &= \frac{y_1}{3} = 1 \\ \Rightarrow x_1 &= 2 \text{ and } y_1 = 3 \\ \therefore P &\equiv (2, 3) \end{aligned}$$

ILLUSTRATION 6.48

Prove that chord of contact of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ w.r.t. any point on the directrix is focal chord.

Sol. Variable point on the directrix can be taken as $P(a/e, k)$, $k \in R$.

Equation of chord of contact of the ellipse w.r.t. this point is

$$\frac{(a/e)x}{a^2} + \frac{ky}{b^2} = 1$$

Clearly, focus $S(ae, 0)$ satisfies the above line.
Hence, proved.

ILLUSTRATION 6.49

Tangents are drawn from the points on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$. Prove that all the chords of contact pass through a fixed point.

Sol. Variable point on the line $x - y - 5 = 0$ can be taken as $(t, t - 5)$, $t \in R$.

Chord of contact of the ellipse $x^2 + 4y^2 = 4$ w.r.t. this point is

$$tx + 4(t - 5)y - 4 = 0$$

$$\text{or } (-20y - 4) + t(x + 4y) = 0$$

This is the equation of family of straight lines.

Each member of this family passes through the point of intersection of straight lines $-20y - 4 = 0$ and $x + 4y = 0$ which is $\left(\frac{1}{5}, -\frac{4}{5}\right)$.

ILLUSTRATION 6.50

Find the locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ forms a triangle of constant area with the coordinate axes.

Sol. The chord of contact of tangents from $P(h, k)$ is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1$$

It meets the axes at the points $A\left(\frac{a^2}{h}, 0\right)$ and $B\left(0, \frac{b^2}{k}\right)$.

Area of the triangle, OAB is

$$\frac{1}{2} \cdot \frac{a^2}{h} \cdot \frac{b^2}{k} = c \text{ (constant)}$$

$$\Rightarrow hk = \frac{a^2 b^2}{2c} \text{ (constant)}$$

$$\Rightarrow xy = \frac{a^2 b^2}{2c}$$

This is the required equation of locus.

ILLUSTRATION 6.51

Prove that the chords of contact of perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touch another fixed ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{(a^2 + b^2)}$$

Sol. We know that the locus of the point of intersection of perpendicular tangents to the given ellipse is $x^2 + y^2 = a^2 + b^2$. Any point on this circle can be taken as

$$P \equiv \left(\sqrt{a^2 + b^2} \cos \theta, \sqrt{a^2 + b^2} \sin \theta\right)$$

The equation of the chord of contact of tangents from P is

$$\frac{x}{a^2} \sqrt{a^2 + b^2} \cos \theta + \frac{y}{b^2} \sqrt{a^2 + b^2} \sin \theta = 1.$$

Let this line be a tangent to the fixed ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

$$\therefore \frac{x}{A} \cos \theta + \frac{y}{B} \sin \theta = 1$$

$$\text{where } A = \frac{a^2}{\sqrt{a^2 + b^2}}, B = \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$\text{Hence, the ellipse is } \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{(a^2 + b^2)}.$$

ILLUSTRATION 6.52

Tangents are drawn from the point $(3, 2)$ to the ellipse $x^2 + 4y^2 = 9$. Find the equation to their chord of contact and the middle point of this chord of contact.

Sol. $x^2 + 4y^2 = 9$

The equation of the chord of contact of the pair of tangents from (3, 2) is

$$3x + 8y = 9 \quad (1)$$

This must be the same as the chord whose middle point is (h, k).

$$T = S_1$$

$$\frac{hx}{9} + \frac{ky}{9/4} = \frac{h^2}{9} + \frac{k^2}{9/4}$$

$$\text{or } hx + 4ky = h^2 + 4k^2 \quad (2)$$

Equations (1) and (2) represent the same straight lines.

Comparing the coefficients of (1) and (2), we get

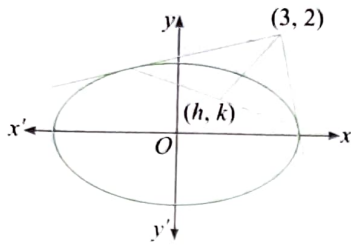
$$\frac{h}{3} = \frac{4k}{8} = \frac{h^2 + 4k^2}{9}$$

$$\text{or } 2h = 3k \text{ and } 3h = h^2 + 4k^2$$

$$\text{or } 3h = h^2 + 4 \times \frac{4h^2}{9}$$

$$\text{or } \frac{25h^2}{9} = 3h$$

$$\text{or } h = \frac{27}{25} \text{ and } k = \frac{18}{25}$$



$$\begin{aligned} &= a^2 \left[\frac{b^2 + a^2 \sin^2 \theta - b^2 \sin^2 \theta}{a^2 b^2} \right] [\because a^2 e^2 = a^2 - b^2] \\ &= a^2 \left[\frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{a^2 b^2} \right] \\ &= a^2 \left[\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right] \end{aligned}$$

Therefore, locus of point of intersection of line given by equations (1) and (2) is $x^2 + y^2 = a^2$, which is the auxiliary circle.

Thus, feet of perpendiculars (M_1 and M_2 in the figure) from foci upon any tangent lie on the auxiliary circle.

II. Product of the lengths of perpendiculars from foci upon any tangent of ellipse is equal to the square of the minor axis.

Proof: One of the equations of tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \text{ having slope 'm' is}$$

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

$$\text{or } mx - y + \sqrt{a^2 m^2 + b^2} = 0$$

Product of perpendiculars from foci upon this tangent is

$$\begin{aligned} &\left| \frac{mae + \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}} \cdot \frac{-mae + \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}} \right| \\ &= \left| \frac{(a^2 m^2 + b^2) - m^2 a^2 e^2}{1 + m^2} \right| \\ &= \left| \frac{a^2 m^2 + b^2 - m^2 (a^2 - b^2)}{1 + m^2} \right| \\ &= \left| \frac{b^2 (1 + m^2)}{1 + m^2} \right| = b^2 = \text{square of minor axis} \end{aligned}$$

III. Length of tangent between the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.

Proof: In the figure given in property 1, we have to prove that $\angle PF_1 Q_1 = \angle PF_2 Q_2 = 90^\circ$.

$$\text{Slope of } PF_1, m_1 = \frac{b \sin \theta}{a \cos \theta - ae}$$

Solving equation of tangent at P and directrix $x = ae$, we get

$$\begin{aligned} Q_1 \left(\frac{a}{e}, \frac{b}{\sin \theta} \left(1 - \frac{\cos \theta}{e} \right) \right) \\ \therefore \text{Slope of } Q_1 F_1, m_2 = \frac{\frac{b}{\sin \theta} \left(1 - \frac{\cos \theta}{e} \right)}{\frac{a}{e} - ae} \\ = \frac{b(e - \cos \theta)}{a \sin \theta (1 - e^2)} \end{aligned}$$

PROPERTIES OF TANGENT

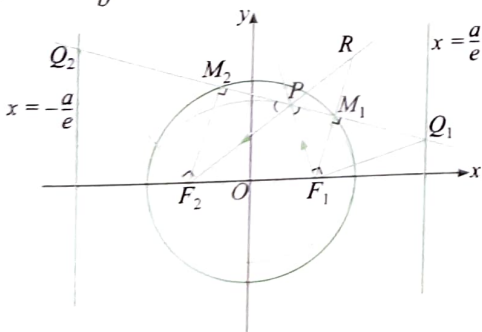
There are many interesting properties of tangent to ellipse. Let us study these properties using standard equation of ellipse i.e.,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b.$$

I. Feet of perpendiculars from foci upon any tangent lies on the auxiliary circle.

Proof: The equation of tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $P(a \cos \theta, b \sin \theta)$ on it is given by

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad (1)$$



Equations of perpendicular lines from foci $F_1(ae, 0)$ and $F_2(-ae, 0)$ to tangent are

$$\frac{x \sin \theta}{b} - \frac{y \cos \theta}{a} = \pm \frac{ae \sin \theta}{b} \quad (2)$$

Squaring and adding equations (1) and (2), we get

$$\begin{aligned} x^2 \left[\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right] + y^2 \left[\frac{\sin^2 \theta}{b^2} + \frac{\cos^2 \theta}{a^2} \right] \\ = 1 + \frac{a^2 e^2 \sin^2 \theta}{b^2} \end{aligned}$$

$$\begin{aligned}\text{Now, } m_1 m_2 &= \frac{b \sin \theta}{a \cos \theta - ae} \times \frac{b(e - \cos \theta)}{a \sin \theta (1 - e^2)} \\ &= -\frac{b^2}{a^2(1 - e^2)} = -1\end{aligned}$$

Thus, $\angle PF_1 Q_1 = 90^\circ$

Similarly, we can prove that $\angle PF_2 Q_2 = 90^\circ$.

- IV. Tangents at the extremities of latus rectum pass through the corresponding foot of directrix on major axis.

Proof: Equations of tangents at endpoints of latus rectum

$\left(ae, \pm \frac{b^2}{a}\right)$ are given by

$$\frac{(ae)x}{a^2} \pm \frac{\left(\frac{b^2}{a}\right)y}{b^2} = 1$$

$$\text{or } \frac{ex}{a} \pm \frac{y}{a} = 1$$

Clearly, these equations satisfy point $\left(\frac{a}{e}, 0\right)$.

Similarly, tangents at the other endpoints of latus rectum

$\left(-ae, \pm \frac{b^2}{a}\right)$ pass through $\left(-\frac{a}{e}, 0\right)$.

- V. Chord of contact w.r.t. any point on the directrix passes through the corresponding focus.

Proof: Let $P(a/e, y_1)$ be a point on the directrix $x = a/e$.
Equation of chord of contact of ellipse w.r.t. P is

$$\frac{x(a/e)}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\text{or } \frac{x}{ae} + \frac{yy_1}{b^2} = 1$$

Clearly, this line satisfies focus $(ae, 0)$.

- VI. Tangent at point P on the ellipse bisects the external angle between two focal radii of point P .

Proof: In the figure,

$$\text{Slope of } PF_1, m_1 = \frac{b \sin \theta}{a \cos \theta - ae}$$

$$\text{Slope of } PF_2, m_2 = \frac{b \sin \theta}{a \cos \theta + ae}$$

$$\text{Slope of tangent at point } P, m = -\frac{b \cos \theta}{a \sin \theta}$$

$$\therefore \left| \frac{m_1 - m}{1 + m_1 m} \right| = \left| \frac{m_2 - m}{1 + m_2 m} \right| = \left| \frac{b}{ae \sin \theta} \right|$$

So, tangent makes same angle with PF_1 and PF_2 .

Therefore, tangent bisects the angle between PF_1 and PF_2 .

- VII. Image of one of the foci in the tangent lies on the line joining other focus and point of contact.

(In the figure, image R of focus F_1 in the tangent at point P lies on PF_2 .)

- VIII. If incident ray emanating from one of the foci gets reflected by the ellipse, then reflected ray passes through the other focus.

(In the figure, incident ray is PF_1 . It strikes the surface of the ellipse at point P and reflected ray passes through other focus F_2 . This is because tangent is angle bisector (external) of angle $F_1 P F_2$.)

ILLUSTRATION 6.53

If F_1 and F_2 are the feet of the perpendiculars from the foci S_1 and S_2 , respectively, of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ on the tangent at any point P on the ellipse, then prove that $S_1 F_1 + S_2 F_2 \geq 8$.

Sol. We know that the product of perpendiculars from two foci of an ellipse upon any tangent is equal to the square of the semi-minor axis.

$$\text{Then } (S_1 F_1) \cdot (S_2 F_2) = 16$$

Now, $A.M. \geq G.M.$

$$\Rightarrow \frac{S_1 F_1 + S_2 F_2}{2} \geq \sqrt{S_1 F_1 \times S_2 F_2}$$

$$\Rightarrow S_1 F_1 + S_2 F_2 \geq 8$$

ILLUSTRATION 6.54

An ellipse has the points $(1, -1)$ and $(2, -1)$ as its foci and $x + y - 5 = 0$ as one of its tangents. Find the point where this line touches the ellipse.

Sol.

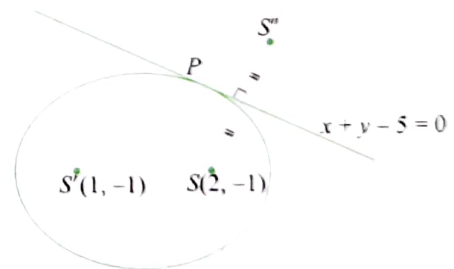


Image of focus $S(2, -1)$ in the tangent at P lies on the line SS' .

Let image $S' \equiv (h, k)$

$$\therefore \frac{h - 2}{1} = \frac{k + 1}{1} = \frac{-2(2 - 1 - 5)}{2} = 4$$

$$\Rightarrow S' = (6, 3)$$

Equation of line SS' is

$$4x - 5y = 9$$

Solving tangent and SS' , we get $P \equiv \left(\frac{34}{9}, \frac{11}{9}\right)$.

CONCEPT APPLICATION EXERCISE 6.4

1. If the line $x \cos \alpha + y \sin \alpha = p$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.
2. A tangent having slope of $-4/3$ to the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$ intersects the major and minor axes at points A and B , respectively. If C is the center of the ellipse, then find area of triangle ABC .
3. Find the slope of a common tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a concentric circle of radius r .
4. If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepts equal lengths l on the axes, then find l .
5. If two points are taken on the minor axis of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the same distance from the center as the foci, then prove that the sum of the squares of the perpendicular distances from these points on any tangent to the ellipse is $2a^2$.
6. If the tangent at any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact, then show that the eccentricity of the ellipse is given by $e = \cos \beta / \cos \alpha$.
7. From any point on any directrix of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, a pair of tangents is drawn to the auxiliary circle. Show that the chord of contact will pass through the corresponding focus of the ellipse.
8. If the tangent to the ellipse $x^2 + 2y^2 = 1$ at point $P(1/\sqrt{2}, 1/2)$ meets the auxiliary circle at points R and Q , then find the points of intersection of tangents to the circle at Q and R .
9. If the line $2px + y\sqrt{5 - 6p^2} = 1$, $p \in \left[-\sqrt{\frac{5}{6}}, \sqrt{\frac{5}{6}}\right]$, always touches the standard ellipse. Then find the eccentricity of the ellipse.
10. If a quadrilateral formed by four tangents to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is a square, then find the area of the square.
11. A variable tangent to the circle $x^2 + y^2 = 1$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at points P and Q . The locus of the point of intersection of tangents to the ellipse at P and Q is another ellipse. Then find its eccentricity.
12. If the chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then find the value of $x_1 x_2 / y_1 y_2$.

13. From the point $A(4, 3)$, tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ to touch the ellipse at B and C . EF is a tangent to the ellipse parallel to line BC and towards point A . Then find the distance of A from EF .
14. Tangents TQ and TR are drawn at the extremities of a chord QR of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$. If the midpoint of the chord is $P(1, 1)$, then find the point of intersection of tangents.
15. Let $S(3, 4)$ and $S'(9, 12)$ be two foci of an ellipse. If foot of the perpendicular from focus S to a tangent of the ellipse is $(1, -4)$, then find the eccentricity of the ellipse.

ANSWERS

- | | | |
|---|---|---------------------------|
| 2. 24 sq. units | 3. $\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$ | 4. $\sqrt{a^2 + b^2}$ |
| 8. $(1/\sqrt{2}, 1)$ | 9. $1/\sqrt{3}$ | 10. 26. sq. units |
| 11. $\sqrt{3}/2$ | 12. $-a^4/b^4$ | 13. $ 24 - 4\sqrt{18} /5$ |
| 14. $\left(\frac{144}{25}, \frac{144}{25}\right)$ | 15. $5/13$ | |

NORMAL TO ELLIPSE

EQUATION OF NORMAL TO ELLIPSE AT A GIVEN POINT ON IT

Equation of tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $P(x_1, y_1)$ on it is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$.

$$\text{Slope of tangent is } -\frac{b^2 x_1}{a^2 y_1}.$$

Therefore, slope of normal to ellipse at point P is $\frac{a^2 y_1}{b^2 x_1}$.

Hence, equation of normal at point P is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1} (x - x_1)$$

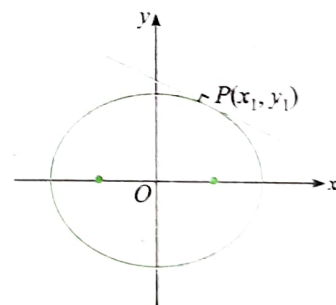
$$\text{or } \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2 \quad (1)$$

If $(x_1, y_1) \equiv (a \cos \theta, b \sin \theta)$ then equation of normal is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad (2)$$

Clearly, major axis and minor axis are two normals to ellipse.

Normal other than major axis, never passes through focus, which can be verified by putting x and y values of the foci in the equation (2) of normal.



EQUATION OF NORMAL TO ELLIPSE WITH GIVEN SLOPE

The equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $P(a \cos \theta, b \sin \theta)$ is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

$$\text{or } y = \left(\frac{a}{b} \tan \theta \right) x - \frac{(a^2 - b^2)}{b} \sin \theta$$

$$\text{Let } \frac{a}{b} \tan \theta = m$$

$$\Rightarrow \sin \theta = \pm \frac{bm}{\sqrt{a^2 + b^2 m^2}}$$

Hence, the equation of the normal becomes

$$y = mx \pm \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2 m^2}}, \text{ where } m \in \mathbb{R}, \text{ as } \frac{a}{b} \tan \theta \in \mathbb{R}$$

ILLUSTRATION 6.55

Find the normal to the ellipse $\frac{x^2}{18} + \frac{y^2}{8} = 1$ at point $(3, 2)$.

Sol. Equation of normal to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $P(x_1, y_1)$ is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

So, equation of normal to ellipse $\frac{x^2}{18} + \frac{y^2}{8} = 1$ at point $P(3, 2)$ is

$$\frac{18x}{3} - \frac{8y}{2} = 18 - 8$$

$$\text{or } 3x - 2y = 5$$

ILLUSTRATION 6.56

Find the points on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, on which the normals are parallel to the line $2x - y = 1$.

Sol. The normal at $P(2 \cos \theta, 3 \sin \theta)$ is

$$\frac{2x}{\cos \theta} - \frac{3y}{\sin \theta} = -5$$

Now, this normal is parallel to $2x - y = 1$. Then,

$$\frac{2/\cos \theta}{3/\sin \theta} = 2$$

$$\text{or } \tan \theta = \frac{3}{1}$$

$$\text{or } \cos \theta = \pm \frac{1}{\sqrt{10}} \text{ and } \sin \theta = \pm \frac{3}{\sqrt{10}}$$

Hence, the points are $(2/\sqrt{10}, 9/\sqrt{10}), (-2/\sqrt{10}, -9/\sqrt{10})$.

ILLUSTRATION 6.57

Find the values of λ for which the line $2x - \frac{8}{3}\lambda y = -3$ is a normal to the ellipse $x^2 + \frac{y^2}{4} = 1$.

Sol. Given normal to ellipse is

$$2x - \frac{8}{3}\lambda y = -3$$

$$\text{or } y = \left(\frac{3}{4\lambda} \right) x + \left(\frac{9}{8\lambda} \right)$$

$$\text{Here, } m = \frac{3}{4\lambda}, c = \frac{9}{8\lambda}$$

Now, condition for line $y = mx + c$ to be normal to ellipse is

$$c = \pm \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2 m^2}}$$

$$\Rightarrow \frac{9}{8\lambda} = \pm \frac{3 \cdot \frac{3}{4\lambda}}{\sqrt{1 + 4 \cdot \frac{9}{16\lambda^2}}}$$

$$\Rightarrow 4\lambda^2 + 9 = 16\lambda^2$$

$$\Rightarrow 4\lambda^2 = 3$$

$$\Rightarrow \lambda = \pm \frac{\sqrt{3}}{2}$$

Thus, normals are $2x \pm \frac{4}{\sqrt{3}}y = -3$.

ILLUSTRATION 6.58

If ω is one of the angles between the normals to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose eccentric angles are θ and

$\frac{\pi}{2} + \theta$, then prove that $\frac{2 \cot \omega}{\sin 2\theta} = \frac{e^2}{\sqrt{1 - e^2}}$.

Sol. The equations of the normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the

points whose eccentric angles are θ and $\frac{\pi}{2} + \theta$ are, respectively,

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

$$\text{and } -ax \operatorname{cosec} \theta - by \sec \theta = a^2 - b^2$$

Since ω is the angle between these two normals, we have

$$\tan \omega = \left| \frac{\frac{a}{b} \tan \theta + \frac{a}{b} \cot \theta}{1 - \frac{a^2}{b^2}} \right|$$

$$= \left| \frac{ab(\tan \theta + \cot \theta)}{b^2 - a^2} \right|$$

$$= \left| \frac{2ab}{(\sin 2\theta)(b^2 - a^2)} \right|$$

$$= \frac{2ab}{(a^2 - b^2) \sin 2\theta} = \frac{2a^2 \sqrt{1 - e^2}}{a^2 e^2 \sin 2\theta}$$

$$\therefore \frac{2 \cot \omega}{\sin 2\theta} = \frac{e^2}{\sqrt{1 - e^2}}$$

ILLUSTRATION 6.59

If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes at G and g , respectively, then find the ratio $PG : Pg$.

Sol. Let $P(a \cos \theta, b \sin \theta)$ be a point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then the equation of the normal at P is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

It meets the axes at

$$G\left(\frac{a^2 - b^2}{a} \cos \theta, 0\right) \text{ and } g\left(0, -\frac{a^2 - b^2}{b} \sin \theta\right)$$

$$PG^2 = \left(a \cos \theta - \frac{a^2 - b^2}{a} \cos \theta\right)^2 + b^2 \sin^2 \theta$$

$$= \frac{b^2}{a^2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta)$$

$$\text{and } Pg^2 = \frac{a^2}{b^2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta)$$

$$PG : Pg = b^2 : a^2$$

ILLUSTRATION 6.60

Find the maximum distance of normal to ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b) \text{ from the centre.}$$

Sol. Normal to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $P(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$$

Its distance from origin is

$$d = \frac{|a^2 - b^2|}{\sqrt{a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta}}$$

$$= \frac{|a^2 - b^2|}{\sqrt{a^2 + b^2 + a^2 \tan^2 \theta + b^2 \cot^2 \theta}}$$

$$= \frac{|a^2 - b^2|}{\sqrt{a^2 + b^2 + 2ab + (a \tan \theta - b \cot \theta)^2}}$$

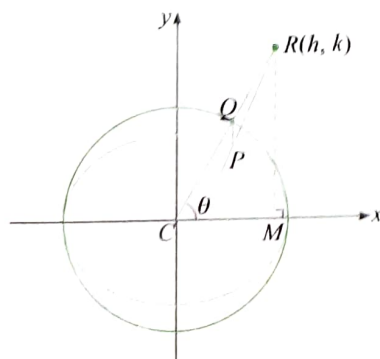
$$\leq \frac{|a^2 - b^2|}{\sqrt{a^2 + b^2 + 2ab}}$$

$$\therefore d_{\max} = a - b$$

ILLUSTRATION 6.61

Point P is on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and Q is the corresponding point on the auxiliary circle of the ellipse. If the line joining centre C to Q meets the normal (drawn at point P on the given ellipse) at R , then find the value of CR .

Sol.



Equation of normal to ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at $P(4 \cos \theta, 3 \sin \theta)$ is

$$\frac{4x}{\cos \theta} - \frac{3y}{\sin \theta} = 7$$

Normal passes through $R(h, k)$.

$$\therefore \frac{4h}{\cos \theta} - \frac{3k}{\sin \theta} = 7$$

$$\Rightarrow 4CR - 3CR = 7$$

$$\Rightarrow CR = 7$$

ILLUSTRATION 6.62

Normal to the ellipse $\frac{x^2}{64} + \frac{y^2}{49} = 1$ intersects the major and minor axes at P and Q , respectively. Find the locus of the point dividing segment PQ in the ratio 2 : 1.

Sol. The equation of normal is $8x \sec \theta - 7y \operatorname{cosec} \theta = 15$.

It meets the axes at

$$P\left(\frac{15}{8} \cos \theta, 0\right) \text{ and } Q\left(0, \frac{-15}{7} \sin \theta\right)$$

Let $M(h, k)$ divides PQ in ratio 2 : 1. Therefore,

$$3h = \frac{15}{8} \cos \theta, \quad 3k = -\frac{30}{7} \sin \theta$$

$$\therefore \cos \theta = \frac{8h}{5}, \quad \sin \theta = \frac{-7k}{10}$$

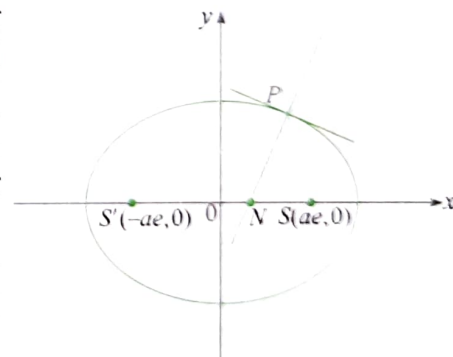
Eliminating θ , we get the locus as

$$\frac{64x^2}{25} + \frac{49y^2}{100} = 1$$



IMPORTANT PROPERTY OF NORMAL

In the properties of tangents, we learned that tangent bisects one of the angles between two focal lengths of point of contact P . Since two angle bisectors of pair of lines are perpendicular, normal at P will be another bisector of angle between these focal lengths.



This can also be proved using angle bisector theorem.

For point $P(x_1, y_1)$ on the ellipse,

$$SP = a - ex_1$$

$$S'P = a + ex_1$$

$$\text{Equation of normal at point } P \text{ is } \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

It meets major axis at point N .

For point N , putting $y = 0$ in the equation of normal, we get

$$x = x_1 \left(1 - \frac{b^2}{a^2}\right) = x_1 e^2$$

$$\Rightarrow SN = ae - x_1 e^2 = e(a - ex_1) = e \cdot SP$$

$$\text{and } S'N = ae + x_1 e^2 = e(a + ex_1) = e \cdot S'P$$

$$\Rightarrow \frac{SP}{S'P} = \frac{SN}{S'N}$$

Therefore, from angle bisector theorem in triangle SPS' , it is proved that PN bisects $\angle SPS'$.

Also, this proves that incident ray from focus S after reflection by ellipse at point P passes through another focus S' .

ILLUSTRATION 6.63

The normal at point $P\left(2, \frac{3\sqrt{3}}{2}\right)$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

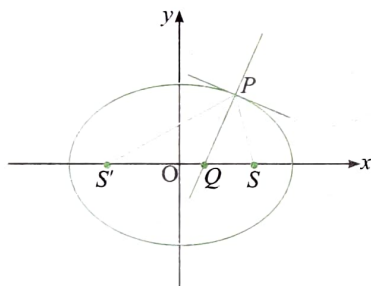
meets the major axis of the ellipse at Q . If S and S' are foci of given ellipse, then find the ratio $SQ : S'Q$.

Sol. Equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

$$\therefore e^2 = 1 - \frac{9}{16} = \frac{7}{16}$$

$$\Rightarrow e = \frac{\sqrt{7}}{4}$$

Hence, foci are $S(\sqrt{7}, 0)$ and $S'(-\sqrt{7}, 0)$.



Normal at P bisects the angle $\angle SPS'$.

Therefore, in triangle SPS' using angle bisector theorem, we get

$$\frac{SQ}{S'Q} = \frac{SP}{S'P} = \frac{a - ex}{a + ex} = \frac{4 - \frac{\sqrt{7}}{4} \times 2}{4 + \frac{\sqrt{7}}{4} \times 2} = \frac{8 - \sqrt{7}}{8 + \sqrt{7}}$$

CONORMAL POINTS

From any point in the plane, maximum four normals can be drawn to an ellipse.

Four feet of normals on the ellipse are called conormal points. If eccentric angles of four conormal points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are α, β, γ and δ , then $\alpha + \beta + \gamma + \delta = (2n + 1)\pi$, $n \in \mathbb{Z}$.

Proof: Equation of normal at $P(a \cos \theta, b \sin \theta)$ on the ellipse is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$$

If it passes through point (h, k) , then

$$\frac{ah}{\cos \theta} - \frac{bk}{\sin \theta} = a^2 - b^2$$

$$\Rightarrow \frac{ah}{1 - \tan^2 \frac{\theta}{2}} - \frac{bk}{2 \tan \frac{\theta}{2}} = a^2 - b^2$$

$$\frac{1 + \tan^2 \frac{\theta}{2}}{2} \cdot \frac{ah}{1 - \tan^2 \frac{\theta}{2}} - \frac{bk}{1 + \tan^2 \frac{\theta}{2}} = a^2 - b^2$$

$$\Rightarrow \frac{(1 + t^2) ah}{(1 - t^2)} - \frac{(1 + t^2) bk}{2t} = a^2 - b^2; \text{ where } t = \tan \frac{\theta}{2}$$

$$\Rightarrow 2t(1 + t^2) ah - (1 - t^2) bk = 2t(1 + t^2)(a^2 - b^2)$$

$$\Rightarrow bk t^4 + [2ah - 2(a^2 - b^2)] t^3 + [2ah - 2(a^2 - b^2)] t - bk = 0$$

This equation has four roots, $\tan \frac{\alpha}{2}, \tan \frac{\beta}{2}, \tan \frac{\gamma}{2}$ and $\tan \frac{\delta}{2}$, where α, β, γ and δ are eccentric angles of feet of normals on the ellipse.

$$\text{Now, } s_1 = \sum \tan \frac{\alpha}{2}$$

$$s_2 = \sum \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = 0$$

$$s_3 = \sum \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$$

$$s_4 = \sum \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \tan \frac{\delta}{2} = -1$$

$$\text{Now, } \tan \left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} \right) = \frac{s_1 + s_3}{1 - s_2 + s_4}$$

$$\text{But } 1 - s_2 + s_4 = 0$$

$$\Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + \frac{\delta}{2} = (2n + 1) \frac{\pi}{2}, n \in \mathbb{Z}$$

$$\Rightarrow \alpha + \beta + \gamma + \delta = (2n + 1)\pi, n \in \mathbb{Z}$$

CONCEPT APPLICATION EXERCISE 6.5

- The line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then prove that $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$.
- Find the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) at the positive end of the latus rectum.
- Find the area of the rectangle formed by the perpendiculars from centre of $\frac{x^2}{9} + \frac{y^2}{4} = 1$, to the tangent and normal at the point whose eccentric angle is $\frac{\pi}{4}$.
- If the straight line $4ax + 3by = 24$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$), then find the coordinates of foci of the ellipse.
- If normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b > 0$) meets the major and minor axes at Q and R , respectively, so that $3PQ = 2PR$, then find the eccentricity of ellipse.

6. If the normal at one end of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one end of the minor axis, then prove that eccentricity is constant.
7. The foci of an ellipse are $S(3, 1)$ and $S'(11, 5)$. If the normal at P is $x + 2y - 15 = 0$, then find the coordinates of point P .
8. If the area of a square circumscribing an ellipse is 10 square units and maximum distance of a normal from the centre of the ellipse is 1 unit, then find the eccentricity of ellipse.

ANSWERS

2. $x - ey - e^3 a = 0$
3. $\frac{30}{13}$ sq. units
4. $(\pm\sqrt{10}, 0)$
5. $\frac{1}{\sqrt{3}}$
7. $(17, -1)$
8. $\sqrt{\frac{3}{4}}$

Solved Examples

EXAMPLE 6.1

Find the range of eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) such that the line segment joining the foci does not subtend a right angle at any point on the ellipse.

Sol. Let any point P on the ellipse be $P(a \cos \theta, b \sin \theta)$. If $\angle S_1 P S_2 = \pi/2$, then P lies on the circle having $S_1 S_2$ as its diameter.

Therefore, the equation of the circle drawn on $S_1 S_2$ as diameter is

$$\begin{aligned} x^2 + y^2 &= a^2 e^2 \\ &= a^2 - b^2 \end{aligned}$$

Since the point P should not lie on the ellipse, there should not be any point on the intersection of

$$x^2 + y^2 = a^2 - b^2 \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore a^2 - b^2 < b^2$$

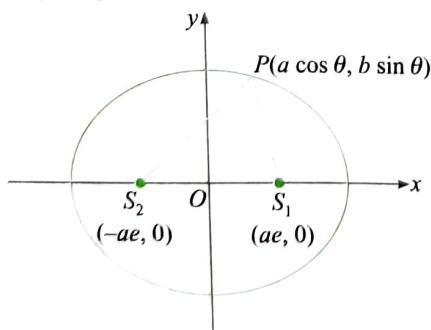
$$\text{or } 2b^2 > a^2$$

$$\text{or } \frac{b^2}{a^2} > \frac{1}{2}$$

$$\text{or } 1 - e^2 > \frac{1}{2}$$

$$\text{or } e^2 < \frac{1}{2}$$

$$\text{or } e \in \left(0, \frac{1}{\sqrt{2}}\right)$$



EXAMPLE 6.2

Find the values of α for which three distinct chords drawn from $(\alpha, 0)$ to the ellipse $x^2 + 2y^2 = 1$ are bisected by the parabola $y^2 = 4x$.

Sol. Let the middle point of the chord be $(t^2, 2t)$.

It must lie inside the ellipse. Therefore,

$$t^4 + 8t^2 - 1 < 0$$

$$\text{or } t^2 \in (0, -4 + \sqrt{17}) \quad (1)$$

Also, the equation of the chord is

$$T = S_1$$

$$\text{or } t^2 x + 4ty = t^4 + 8t^2$$

Let this pass through $(\alpha, 0)$. Then,

$$\alpha t^2 = t^4 + 8t^2$$

$$\text{or } t^4 + (8 - \alpha)t^2 = 0$$

$$\text{i.e., } t^2 = 0 \text{ or } t^2 = \alpha - 8$$

$$\text{i.e., } \alpha = t^2 + 8$$

$$\text{So, } \alpha \in (8, 4 + \sqrt{17})$$

EXAMPLE 6.3

Prove that if any tangent to the ellipse is cut by the tangents at the endpoints of the major axis at T and T' , then the circle whose diameter is TT' will pass through the foci of the ellipse.

Sol. Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

Let any tangent to the ellipse be

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad (1)$$

Tangents at the vertices are $x = a$ and $x = -a$.

Solving with (1), we get the points T and T' , respectively, as

$$T\left(a, \frac{b(1 - \cos \theta)}{\sin \theta}\right), T'\left(-a, \frac{b(1 + \cos \theta)}{\sin \theta}\right)$$

$$\text{i.e., } T\left(a, b \tan \frac{\theta}{2}\right), T'\left(-a, b \cot \frac{\theta}{2}\right)$$

Circle on TT' as diameter is

$$(x - a)(x + a) + \left(y - b \tan \frac{\theta}{2}\right)\left(y - b \cot \frac{\theta}{2}\right) = 0$$

$$\text{or } x^2 - a^2 + y^2 + b^2 - b\left[\tan \frac{\theta}{2} + \cot \frac{\theta}{2}\right]y = 0$$

It will pass through the foci $(\pm ae, 0)$.

If $a^2 e^2 - a^2 + b^2 = 0$, then

$$b^2 = a^2(1 - e^2)$$

which is true.

EXAMPLE 6.4

Let d be the perpendicular distance from the center of the ellipse to any tangent to the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left(1 - \frac{b^2}{a^2}\right)$.

Sol. The equation of the tangent at the point $P(a \cos \theta, b \sin \theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad (1)$$

The perpendicular distance of (1) from the center $(0, 0)$ of the ellipse is given by

$$d = \frac{1}{\sqrt{\frac{1}{a^2} \cos^2 \theta + \frac{1}{b^2} \sin^2 \theta}} = \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}$$

$$\therefore 4a^2\left(1 - \frac{b^2}{a^2}\right) = 4a^2\left\{1 - \frac{b^2 \cos^2 \theta + a^2 \sin^2 \theta}{a^2}\right\}$$

$$= 4(a^2 - b^2) \cos^2 \theta = 4a^2 e^2 \cos^2 \theta \quad (2)$$

The foci are $F_1 \equiv (ae, 0)$ and $F_2 \equiv (-ae, 0)$. Therefore,

$$PF_1 = e(1 - e \cos \theta)$$

and $PF_2 = a(1 + e \cos \theta)$

$$\therefore (PF_1 - PF_2)^2 = 4a^2 e^2 \cos^2 \theta \quad (3)$$

Hence, from (2) and (3), we have

$$(PF_1 - PF_2)^2 = 4a^2 \left(1 - \frac{b^2}{a^2}\right)$$

EXAMPLE 6.5

Let P be a point on an ellipse with eccentricity $1/2$, such that $\angle PS_1 S_2 = \alpha$, $\angle PS_2 S_1 = \beta$, and $\angle S_1 P S_2 = \gamma$, where S_1 and S_2 are the foci of the ellipse. Then prove that $\cot(\alpha/2)$, $\cot(\gamma/2)$, and $\cot(\beta/2)$ are in AP.

Sol. We know that

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{1-e}{1+e}$$

$$= \frac{1-(1/2)}{1+(1/2)} = \frac{1}{3}$$

In triangle ABC , we know that

$$\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{C}{2} \tan \frac{B}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$$

$$\text{or } \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

For $\triangle PS_1 S_2$,

$$\cot \frac{\alpha}{2} + \cot \frac{\beta}{2} + \cot \frac{\gamma}{2} = 3 \cot \frac{\gamma}{2}$$

$$\text{or } 2 \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} + \cot \frac{\beta}{2}$$

Therefore, $\cot \alpha/2$, $\cot \gamma/2$, $\cot \beta/2$ are in AP.

EXAMPLE 6.6

If a triangle is inscribed in an ellipse and two of its sides are parallel to the given straight lines, then prove that the third side touches the fixed ellipse.

Sol. Let the eccentric angles of the vertices P, Q, R of $\triangle PQR$ be $\theta_1, \theta_2, \theta_3$, respectively.

Then the equations of PQ and PR are, respectively,

$$\frac{x}{a} \cos \frac{\theta_1 + \theta_2}{2} + \frac{y}{b} \sin \frac{\theta_1 + \theta_2}{2} = \cos \frac{\theta_1 - \theta_2}{2}$$

$$\text{and } \frac{x}{a} \cos \frac{\theta_2 + \theta_3}{2} + \frac{y}{b} \sin \frac{\theta_2 + \theta_3}{2} = \cos \frac{\theta_2 - \theta_3}{2}$$

If PQ and PR are parallel to the given straight lines, then we have

$$\theta_1 + \theta_2 = \text{Constant} = 2\alpha \text{ (say)}$$

$$\text{and } \theta_1 + \theta_3 = \text{Constant} = 2\beta$$

$$\text{Hence, } \theta_2 - \theta_3 = 2(\alpha - \beta) \quad (1)$$

Now, the equation of QR is

$$\frac{x}{a} \cos \frac{\theta_2 + \theta_3}{2} + \frac{y}{b} \sin \frac{\theta_2 + \theta_3}{2} = \cos \frac{\theta_2 - \theta_3}{2} \quad (2)$$

$$\text{or } \frac{x}{a} \cos \frac{\theta_2 + \theta_3}{2} + \frac{y}{b} \sin \frac{\theta_2 + \theta_3}{2} = \cos(\alpha - \beta) \quad (3)$$

which shows that line (2), for different values of $\theta_2 + \theta_3$, is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(\alpha - \beta)$$

EXAMPLE 6.7

The tangent at a point $P(a \cos \phi, b \sin \phi)$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$ meets its auxiliary circle at two points, the chord joining which subtends a right angle at the center. Find the eccentricity of the ellipse.

Sol. The equation of the auxiliary circle is

$$x^2 + y^2 = a^2 \quad (1)$$

Therefore, the equation of tangent to ellipse at a point $P(a \cos \phi, b \sin \phi)$ is

$$\left(\frac{x}{a}\right) \cos \phi + \left(\frac{y}{b}\right) \sin \phi = 1 \quad (2)$$

which meets the auxiliary circle at points A and B .

Therefore, the equation of the pair of lines OA and OB is obtained by making (1) homogeneous with the help of (2). Then,

$$x^2 + y^2 = a^2 \left(\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi\right)^2$$

But $\angle AOB = 90^\circ$. Therefore,

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\text{or } 1 - \cos^2 \phi + 1 - \frac{a^2}{b^2} \sin^2 \phi = 0$$

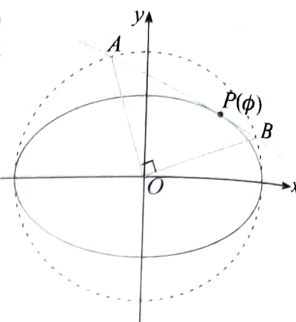
$$\text{or } \sin^2 \phi \left(1 - \frac{a^2}{b^2}\right) + 1 = 0$$

$$\text{or } (a^2 - b^2) \sin^2 \phi = b^2$$

$$\text{or } a^2 e^2 \sin^2 \phi = a^2 (1 - e^2)$$

$$\text{or } (1 + \sin^2 \phi) e^2 = 1$$

$$\text{or } e = \frac{1}{\sqrt{1 + \sin^2 \phi}}$$



EXAMPLE 6.8

Find the locus of point P such that the tangents drawn from it to the given ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the coordinate axes at concyclic points.

Sol. Let $P \equiv (h, k)$. The combined equation of tangents from (h, k) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) - \left(\frac{xh}{a^2} + \frac{yk}{b^2} - 1\right)^2 = 0$$

Any conic that can be drawn through the points of intersection of these tangents with the coordinate axes is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) - \left(\frac{xh}{a^2} + \frac{yk}{b^2} - 1\right)^2 + \lambda xy = 0$$

$$\text{or } x^2 \left(\frac{k^2}{a^2 b^2} - \frac{1}{a^2}\right) + y^2 \left(\frac{h^2}{a^2 b^2} - \frac{1}{b^2}\right) + xy \left(\lambda - \frac{2hk}{a^2 b^2}\right) + \frac{2yk}{b^2} - \frac{h^2}{a^2} - \frac{k^2}{b^2} = 0$$

It should represent a circle. Therefore,

$$\lambda = \frac{2hk}{a^2 b^2}, \frac{k^2}{a^2 b^2} - \frac{1}{a^2} = \frac{h^2}{a^2 b^2} - \frac{1}{b^2}$$

$$\therefore \lambda = \frac{2hk}{a^2 b^2}, h^2 - k^2 = a^2 - b^2$$

Hence, the locus of P is

$$x^2 - y^2 = a^2 - b^2$$

EXAMPLE 6.9

A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q . Prove that the tangents at P and Q to the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Sol. The given ellipses are

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \quad (1)$$

and $\frac{x^2}{6} + \frac{y^2}{3} = 1 \quad (2)$

Then the equation of tangent to (1) at point $T(2 \cos \theta, \sin \theta)$ is given by

$$\frac{x \cos \theta}{2} + \frac{y \sin \theta}{1} = 1 \quad (3)$$

Let this tangent meet the ellipse

(2) at P and Q . Let the tangents drawn to ellipse (2) at P and Q meet each other at $R(h, k)$.

Then PQ is the chord of contact of ellipse (2) with respect to the point $R(h, k)$ and is given by

$$\frac{xh}{6} + \frac{yk}{3} = 1 \quad (4)$$

Clearly, (3) and (4) represent the same line and, hence, should be identical.

Therefore, comparing the ratio of coefficients, we get

$$\frac{(\cos \theta)/2}{h/6} = \frac{\sin \theta}{k/3} = \frac{1}{1}$$

or $h = 3 \cos \theta, k = 3 \sin \theta$

or $h^2 + k^2 = 9$

Therefore, the locus of (h, k) is

$$x^2 + y^2 = 9$$

which is the director circle of the ellipse

$$\frac{x^2}{6} + \frac{y^2}{3} = 1$$

and we know that the director circle is the locus of the point of intersection of the tangents which are at right angle.

Thus, tangents at P and Q are perpendicular.

EXAMPLE 6.10

Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B , and C to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$, meet the ellipse, respectively, at P, Q , and R so that P, Q , and R lie on the same side of the major axis as A, B , and C , respectively. Prove that the normals to the ellipse drawn at the points P, Q , and R are concurrent.

Sol. Let A, B , and C be the points on the circle whose coordinates are

$$A(a \cos \theta, a \sin \theta)$$

$$B\left(a \cos\left(\theta + \frac{2\pi}{3}\right), a \sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

$$C\left(a \cos\left(\theta + \frac{4\pi}{3}\right), a \sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Hence, $P \equiv (a \cos \theta, b \sin \theta)$

$$Q \equiv \left(a \cos\left(\theta + \frac{2\pi}{3}\right), b \sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

[Given]

$$R \equiv \left(a \cos\left(\theta + \frac{4\pi}{3}\right), b \sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

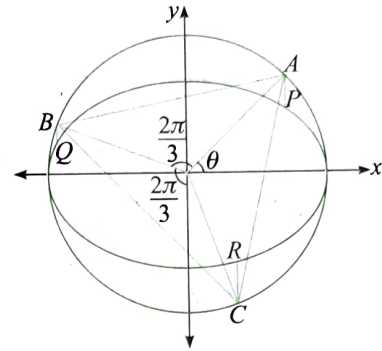
It is given that P, Q , and R are on the same side of the x -axis as A, B , and C .

So, the required normals to the ellipse at P, Q and R are

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2 \quad (1)$$

$$ax \sec\left(\theta + \frac{2\pi}{3}\right) - by \operatorname{cosec}\left(\theta + \frac{2\pi}{3}\right) = a^2 - b^2 \quad (2)$$

$$ax \sec\left(\theta + \frac{4\pi}{3}\right) - by \operatorname{cosec}\left(\theta + \frac{4\pi}{3}\right) = a^2 - b^2 \quad (3)$$



$$\text{Now, } \Delta = \begin{vmatrix} \sec \theta & \operatorname{cosec} \theta & 1 \\ \sec\left(\theta + \frac{2\pi}{3}\right) & \operatorname{cosec}\left(\theta + \frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta + \frac{4\pi}{3}\right) & \operatorname{cosec}\left(\theta + \frac{4\pi}{3}\right) & 1 \end{vmatrix}$$

Multiplying R_1, R_2 and R_3 by $\sin \theta \cos \theta, \sin\left(\theta + \frac{2\pi}{3}\right) \cos\left(\theta + \frac{2\pi}{3}\right)$, and $\sin\left(\theta + \frac{4\pi}{3}\right) \cos\left(\theta + \frac{4\pi}{3}\right)$ respectively, we get

$$\Delta = \frac{1}{k} \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

where

$$k = 2 \sin \theta \cos \theta \sin\left(\theta + \frac{2\pi}{3}\right) \cos\left(\theta + \frac{2\pi}{3}\right) \sin\left(\theta + \frac{4\pi}{3}\right) \cos\left(\theta + \frac{4\pi}{3}\right)$$

Operating $R_2 \rightarrow R_2 + R_3$

$$\Delta = \frac{1}{k} \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ 2 \sin \theta \cdot \cos \frac{2\pi}{3} & 2 \cos \theta \cdot \cos \frac{2\pi}{3} & 2 \sin 2\theta \cdot \cos \frac{4\pi}{3} \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \frac{1}{k} \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ -\sin \theta & -\cos \theta & -\sin 2\theta \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

Hence, $\Delta = 0$.

Exercises

Single Correct Answer Type

- P and Q are the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and B is an end of the minor axis. If PBQ is an equilateral triangle, then the eccentricity of the ellipse is
 (1) $1/\sqrt{2}$ (2) $1/3$ (3) $1/2$ (4) $\sqrt{3}/2$
- S_1 and S_2 are the foci of an ellipse of major axis of length 10 units, and P is any point on the ellipse such that the perimeter of triangle PS_1S_2 is 15. Then the eccentricity of the ellipse is
 (1) 0.5 (2) 0.25 (3) 0.28 (4) 0.75
- If the eccentricity of the ellipse $\frac{x^2}{a^2+1} + \frac{y^2}{a^2+2} = 1$ is $1/\sqrt{6}$, then the latus rectum of the ellipse is
 (1) $5/\sqrt{6}$ (2) $10/\sqrt{6}$
 (3) $8/\sqrt{6}$ (4) none of these
- If the ellipse $\frac{x^2}{4} + y^2 = 1$ meets the ellipse $x^2 + \frac{y^2}{a^2} = 1$ at four distinct points and $a = b^2 - 5b + 7$, then b does not lie in
 (1) $[4, 5]$ (2) $(-\infty, 2) \cup (3, \infty)$
 (3) $(-\infty, 0)$ (4) $[2, 3]$
- A point moves so that its distance from the point $(2, 0)$ is always $\frac{1}{3}$ of its distance from the line $x - 18 = 0$. If the locus of the point is a conic, then length of its latus rectum is
 (1) $\frac{16}{3}$ (2) $\frac{32}{3}$ (3) $\frac{8}{3}$ (4) $\frac{15}{4}$
- If $(-4, 1)$, $(6, 1)$ are the vertices of an ellipse and one of the foci lies on $x - 2y = 2$, then the eccentricity of ellipse is
 (1) $\frac{3}{5}$ (2) $\frac{4}{5}$ (3) $\frac{2}{5}$ (4) $\frac{1}{5}$
- Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) in the first or second quadrants whose foci are S_1 and S_2 . Then the least possible value of the circumradius of ΔPS_1S_2 will be
 (1) ae (2) be (3) $\frac{ae}{b}$ (4) $\frac{ae^2}{b}$
- An equilateral triangle is inscribed in an ellipse whose equation is $x^2 + 4y^2 = 4$. If one vertex of the triangle is $(0, 1)$, then the length of each side is
 (1) $\frac{8\sqrt{3}}{13}$ (2) $\frac{24\sqrt{3}}{13}$
 (3) $\frac{16\sqrt{3}}{13}$ (4) $\frac{48\sqrt{3}}{13}$
- A circle has the same center as an ellipse and passes through the foci F_1 and F_2 of the ellipse, such that the two curves intersect at four points. Let P be any one of their point of intersection. If the major axis of the ellipse is 17 and the area of triangle PF_1F_2 is 30, then the distance between the foci is
 (1) 13 (2) 10
 (3) 11 (4) none of these
- There are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose distances from its center are the same and are equal to $\sqrt{(a^2 + 2b^2)/2}$. Then the eccentricity of the ellipse is
 (1) $1/2$ (2) $1/\sqrt{2}$ (3) $1/3$ (4) $1/3\sqrt{2}$
- The eccentricity of the locus of point $(3h + 2, k)$, where (h, k) lies on the circle $x^2 + y^2 = 1$, is
 (1) $1/3$ (2) $\sqrt{2}/3$ (3) $2\sqrt{2}/3$ (4) $1/\sqrt{3}$
- Let S and S' be two foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If a circle described on SS' as diameter intersects the ellipse at real and distinct points, then the eccentricity e of the ellipse satisfies
 (1) $e = 1/\sqrt{2}$ (2) $e \in (1/\sqrt{2}, 1)$
 (3) $e \in (0, 1/\sqrt{2})$ (4) none of these
- Point P represents the complex number $z = x + iy$ and point Q represents the complex number $z + 1/z$. If P moves on the circle $|z| = 2$, then the eccentricity of locus of point Q is
 (1) $3/5$ (2) $4/5$
 (3) $3/4$ (4) $1/2$
- The locus of the point which divides the double ordinates of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the ratio $1 : 2$ internally is
 (1) $\frac{x^2}{a^2} + \frac{9y^2}{b^2} = 1$ (2) $\frac{x^2}{a^2} + \frac{9y^2}{b^2} = \frac{1}{9}$
 (3) $\frac{9x^2}{a^2} + \frac{9y^2}{b^2} = 1$ (4) none of these
- The equation of the line passing through the center and bisecting the chord $7x + y - 1 = 0$ of the ellipse $\frac{x^2}{1} + \frac{y^2}{7} = 1$ is
 (1) $x = y$ (2) $2x = y$ (3) $x = 2y$ (4) $x + y = 0$
- A parabola is drawn with focus at one of the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, and directrix passing through the other focus and perpendicular to the major axis of the ellipse. If the latus rectum of the ellipse and that of the parabola are same, then the eccentricity of the ellipse is
 (1) $1 - \frac{1}{\sqrt{2}}$ (2) $2\sqrt{2} - 2$
 (3) $\sqrt{2} - 1$ (4) none of these

17. P is any point lying on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) whose foci are S and S' . If $\angle PSS' = \alpha$ and $\angle PS'S = \beta$, then the value of $\tan \frac{\alpha}{2} \tan \frac{\beta}{2}$ is
- (1) $\frac{1+e}{1-e}$ (2) $\frac{1-e^2}{1+e^2}$
 (3) $\frac{1-e}{1+e}$ (4) 1
18. With a given point and line as focus and directrix, a series of ellipses are described. The locus of the extremities of their minor axis is
- (1) an ellipse (2) a parabola
 (3) a hyperbola (4) none of these
19. The angle subtended by common tangents of two ellipses $4(x-4)^2 + 25y^2 = 100$ and $4(x+1)^2 + y^2 = 4$ at the origin is
- (1) $\pi/3$ (2) $\pi/4$ (3) $\pi/6$ (4) $\pi/2$
20. The length of the major axis of the ellipse $(5x-10)^2 + (5y+15)^2 = \frac{(3x-4y+7)^2}{4}$ is
- (1) 10 (2) 20/3 (3) 20/7 (4) 4
21. The foci of an ellipse are $S(-1, -1)$ and $S'(0, -2)$. If the eccentricity is $\frac{1}{2}$, then the equation of the directrix corresponding to the focus S is
- (1) $x-y+3=0$ (2) $x-y+7=0$
 (3) $x-y+5=0$ (4) $x-y+4=0$
22. If the curves $\frac{x^2}{4} + y^2 = 1$ and $\frac{x^2}{a^2} + y^2 = 1$ for a suitable value of a cut on four concyclic points, the equation of the circle passing through these four points is
- (1) $x^2 + y^2 = 2$ (2) $x^2 + y^2 = 1$
 (3) $x^2 + y^2 = 4$ (4) none of these
23. If the maximum distance of any point on the ellipse $x^2 + 2y^2 + 2xy = 1$ from its center is r , then r is equal to
- (1) $3 + \sqrt{3}$ (2) $2 + \sqrt{2}$
 (3) $\sqrt{2}/\sqrt{3} - \sqrt{5}$ (4) $\sqrt{2} - \sqrt{2}$
24. If S and S' are the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, and P is any point on it, then the range of values of $SP \cdot S'P$ is
- (1) $9 \leq f(\theta) \leq 16$ (2) $9 \leq f(\theta) \leq 25$
 (3) $16 \leq f(\theta) \leq 25$ (4) $1 \leq f(\theta) \leq 16$
25. A man running around a race course notes that the sum of the distances of two flagposts from him is always 10 m and the distance between the flag posts is 8 m. Then the area of the path he encloses in square meters is
- (1) 15π (2) 20π (3) 27π (4) 30π
26. The eccentric angle of a point on the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at a distance of $5/4$ units from the focus on the positive x -axis is
- (1) $\cos^{-1}(3/4)$ (2) $\cos^{-1}(4/5)$
 (3) $\cos^{-1}(3/5)$ (4) none of these
27. If PQR is an equilateral triangle inscribed in the auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($a > b$), and $P'Q'R'$ is the corresponding triangle inscribed within the ellipse, then the centroid of triangle $P'Q'R'$ lies at
- (1) center of ellipse
 (2) focus of ellipse
 (3) between focus and center on major axis
 (4) none of these
28. Let d_1 and d_2 be the lengths of the perpendiculars drawn from the foci S and S' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the tangent at any point P on the ellipse. Then, $SP : S'P =$
- (1) $d_1 : d_2$ (2) $d_2 : d_1$ (3) $d_1^2 : d_2^2$ (4) $\sqrt{d_1} : \sqrt{d_2}$
29. The slopes of the common tangents of the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ and the circle $x^2 + y^2 = 3$ are
- (1) ± 1 (2) $\pm\sqrt{2}$
 (3) $\pm\sqrt{3}$ (4) none of these
30. If the line $y = mx + c$ is a tangent to the ellipse $x^2 + 2y^2 = 4$, then c cannot take the value
- (1) $-\sqrt{2}$ (2) $\sqrt{2}$ (3) 2 (4) 1
31. If $(\sqrt{3})bx + ay = 2ab$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the eccentric angle of the point of contact is
- (1) $\pi/6$ (2) $\pi/4$ (3) $\pi/3$ (4) $\pi/2$
32. Length of the perpendicular from the centre of the ellipse $27x^2 + 9y^2 = 243$ on a tangent drawn to it which makes equal intercepts on the coordinate axes is
- (1) 3/2 (2) $3\sqrt{2}$
 (3) $3/\sqrt{2}$ (4) 6
33. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and its corresponding point Q on the auxiliary circle meet on the line
- (1) $x = ae$ (2) $x = 0$
 (3) $y = 0$ (4) none of these
34. The area of the quadrilateral formed by the tangents at the endpoint of the latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is
- (1) 27/4 sq. units (2) 9 sq. units
 (3) 27/2 sq. units (4) 27 sq. units
35. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the midpoint of the intercept made by the tangents between the coordinate axes is
- (1) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (2) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
 (3) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (4) $\frac{x^2}{4} + \frac{y^2}{2} = 1$

36. The minimum area of the triangle formed by the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the coordinate axes is

(1) ab sq. units (2) $\frac{a^2 + b^2}{2}$ sq. units

(3) $\frac{(a+b)^2}{2}$ sq. units (4) $\frac{a^2 + ab + b^2}{3}$ sq. units

37. Tangents are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($a > b$), and the circle $x^2 + y^2 = a^2$ at the points where a common ordinate cuts them (on the same side of the x -axis). Then the greatest acute angle between these tangents is given by

(1) $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$ (2) $\tan^{-1}\left(\frac{a+b}{2\sqrt{ab}}\right)$

(3) $\tan^{-1}\left(\frac{2ab}{\sqrt{a-b}}\right)$ (4) $\tan^{-1}\left(\frac{2ab}{\sqrt{a+b}}\right)$

38. Let P be any point on a directrix of an ellipse of eccentricity e , S be the corresponding focus, and C the center of the ellipse. The line PC meets the ellipse at A . The angle between PS and tangent at A is α . Then α is equal to

(1) $\tan^{-1}e$ (2) $\pi/2$
(3) $\tan^{-1}(1-e^2)$ (4) none of these

39. If a tangent of slope 2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is normal to the circle $x^2 + y^2 + 4x + 1 = 0$, then the maximum value of ab is

(1) 4 (2) 2
(3) 1 (4) none of these

40. If the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ make angles α and β with the major axis such that $\tan \alpha + \tan \beta = \lambda$, then the locus of their point of intersection is

(1) $x^2 + y^2 = a^2$ (2) $x^2 + y^2 = b^2$
(3) $x^2 - a^2 = 2\lambda xy$ (4) $\lambda(x^2 - a^2) = 2xy$

41. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($b > a$) and the parabola $y^2 = 4ax$ cut at right angle, then eccentricity of the ellipse is

(1) $\frac{3}{5}$ (2) $\frac{2}{3}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{2}$

42. If $a - \beta = \text{constant}$, then the locus of the point of intersection of tangents at $P(a \cos \alpha, b \sin \alpha)$ and $Q(a \cos \beta, b \sin \beta)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(1) a circle (2) a straight line
(3) an ellipse (4) a parabola

43. The locus of the point of intersection of tangents to an ellipse at two points, sum of whose eccentric angles is constant, is $a/\tan$

(1) parabola (2) circle
(3) ellipse (4) straight line

44. A tangent of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is intercepted by auxiliary circle such that the intercepted portion of the tangent subtends equal angles at the foci. Length of the tangent is

(1) 6 (2) 3 (3) 4 (4) 8

45. A tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ at any point P meets the line $x = 0$ at a point Q . Let R be the image of Q in the line $y = x$. Then the circle whose extremities of a diameter are Q and R passes through a fixed point. The fixed point is

(1) (3, 0) (2) (5, 0) (3) (0, 0) (4) (4, 0)

46. For the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with vertices A and A' , tangent drawn at the point P in the first quadrant meets the y -axis at Q and the chord $A'P$ meets the y -axis at M . If O is the origin, then $OQ^2 - MQ^2$ is equal to

(1) 9 (2) 13 (3) 4 (4) 5

47. If the ellipse $\frac{x^2}{a^2 - 7} + \frac{y^2}{13 - 5a} = 1$ is inscribed in a square of side length $a\sqrt{2}$, then a is equal to

(1) $6/5$ (2) $11/10$
(3) $13/10$ (4) no such a exists

48. The locus of the point of intersection of the tangent at the endpoints of the focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($b < a$) is $a/\tan$

(1) circle (2) ellipse
(3) hyperbola (4) pair of straight lines

49. From point $P(8, 27)$, tangents PQ and PR are drawn to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Then the angle subtended by QR at the origin is

(1) $\tan^{-1}\frac{\sqrt{6}}{65}$ (2) $\tan^{-1}\frac{4\sqrt{6}}{65}$
(3) $\tan^{-1}\frac{8\sqrt{2}}{65}$ (4) $\tan^{-1}\frac{48\sqrt{6}}{455}$

50. If tangents PQ and PR are drawn from a point on the circle $x^2 + y^2 = 25$ to the ellipse $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$, ($b < 4$), so that the fourth vertex S of parallelogram $PQSR$ lies on the circumcircle of triangle PQR , then the eccentricity of the ellipse is

(1) $\sqrt{5}/4$ (2) $\sqrt{7}/3$ (3) $\sqrt{7}/4$ (4) $\sqrt{5}/3$

51. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is inscribed in a rectangle whose length to breadth ratio is 2 : 1, then the area of the rectangle is

(1) $4 \cdot \frac{a^2 + b^2}{7}$ (2) $4 \cdot \frac{a^2 + b^2}{3}$
(3) $12 \cdot \frac{a^2 + b^2}{5}$ (4) $8 \cdot \frac{a^2 + b^2}{5}$

52. An ellipse has semi-major axis of length 2 and semi-minor axis of length 1. It slides between the coordinate axes in the first quadrant while maintaining contact with both x -axis and y -axis. The locus of the centre of the ellipse is
- $x^2 + y^2 = 8$
 - $\frac{x^2}{9} + \frac{y^2}{4} = 1$
 - $y^2 = 4x$
 - $x^2 + y^2 = 5$
53. The number of points on the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ from which a pair of perpendicular tangents is drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is
- 0
 - 2
 - 1
 - 4
54. Let P_i and P'_i be the feet of the perpendiculars drawn from the foci S and S' on a tangent T_i to an ellipse whose length of semi-major axis is 20. If $\sum_{i=1}^{10} (SP_i)(S'P'_i) = 2560$, then the value of eccentricity is
- 1/5
 - 2/5
 - 3/5
 - 4/5
55. The normal at a variable point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of eccentricity e meets the axes of the ellipse at Q and R . Then the locus of the midpoint of QR is a conic with eccentricity e' such that
- e' is independent of e
 - $e' = 1$
 - $e' = e$
 - $e' = 1/e$
56. Any ordinate MP of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ meets the auxiliary circle at Q . Then locus of the point of intersection of normals at P and Q to the respective curves is
- $x^2 + y^2 = 8$
 - $x^2 + y^2 = 34$
 - $x^2 + y^2 = 64$
 - $x^2 + y^2 = 15$
57. The number of distinct normal lines that can be drawn to the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$ from the point $P(0, 6)$ is
- one
 - two
 - three
 - four
58. The line $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$ is normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m belonging to
- $(0, 1)$
 - $(0, \infty)$
 - R
 - none of these
2. The equation $3x^2 + 4y^2 - 18x + 16y + 43 = k$
- represents an empty set, if $k < 0$
 - represents an ellipse, if $k > 0$
 - represents a point, if $k = 0$
 - cannot represent a real pair of straight lines for any value of k
3. If the equation of the ellipse is $3x^2 + 2y^2 + 6x - 8y + 5 = 0$, then which of the following is/are true?
- $e = 1/\sqrt{3}$.
 - Center is $(-1, 2)$.
 - Foci are $(-1, 1)$ and $(-1, 3)$.
 - Directrices are $y = 2 \pm \sqrt{3}$.
4. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the endpoints of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are
- $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
 - $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
 - $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
 - $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$
5. $ABCD$ is a rhombus with $AC = 2BD$. Diagonals AC and BD intersect at P . E_1, E_2, E_3 and E_4 are four ellipses passing through P and their foci are A and B , B and C , C and D and D and A , respectively. If for $i = 1, 2, 3, 4$, e_i are the eccentricities of E_i , then
- $e_1 = e_3$
 - $e_2 = e_4$
 - $e_1 = 2e_2$
 - $e_1 = e_2$
6. The major axis and minor axis of an ellipse are, respectively, $x - 2y - 5 = 0$ and $2x + y + 10 = 0$. If one end of latus rectum is $(3, 4)$, then the foci are
- $(5, 0)$
 - $(-7, -6)$
 - $(-11, -8)$
 - $(11, 3)$
7. If the chord through the points whose eccentric angles are θ and ϕ on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ passes through a focus, then the value of $\tan(\theta/2) \tan(\phi/2)$ is
- 1/9
 - 9
 - 1/9
 - 9
8. Which of the following is/are true about the ellipse $x^2 + 4y^2 - 2x - 16y + 13 = 0$?
- The latus rectum of the ellipse is 1.
 - The distance between the foci of the ellipse is $4\sqrt{3}$.
 - The sum of the focal distances of a point $P(x, y)$ on the ellipse is 4.
 - Line $y = 3$ meets the tangents drawn at the vertices of the ellipse at points P and Q . Then PQ subtends a right angle at any of its foci.
9. Which of the following is/are true?
- There are infinite positive integral values of a for which $(13x - 1)^2 + (13y - 2)^2 = \left(\frac{5x + 12y - 1}{a}\right)^2$ represents an ellipse.
 - The minimum distance of a point $(1, 2)$ from the ellipse $4x^2 + 9y^2 + 8x - 36y + 4 = 0$ is 1.

Multiple Correct Answers Type

1. $\frac{x^2}{r^2 - r - 6} + \frac{y^2}{r^2 - 6r + 5} = 1$ will represent an ellipse if r lies in the interval
- $(-\infty, -2)$
 - $(3, \infty)$
 - $(5, \infty)$
 - $(1, \infty)$

- (3) If from a point $P(0, \alpha)$ two normals other than the axes are drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, then $|\alpha| < 9/4$.
- (4) If the length of the latus rectum of an ellipse is one-third of its major axis, then its eccentricity is equal to $1/\sqrt{3}$.
10. Let E_1 and E_2 , respectively, be two ellipses $\frac{x^2}{a^2} + y^2 = 1$, and $x^2 + \frac{y^2}{a^2} = 1$ (where a is a parameter). Then the locus of the points of intersection of the ellipses E_1 and E_2 is a set of curves comprising
- (1) two straight lines (2) one straight line
(3) one circle (4) one parabola
11. Consider the ellipse $\frac{x^2}{f(k^2 + 2k + 5)} + \frac{y^2}{f(k + 11)} = 1$. If $f(x)$ is a positive decreasing function, then
- (1) the set of values of k for which the major axis is the x -axis is $(-3, 2)$
(2) the set of values of k for which the major axis is the y -axis is $(-\infty, 2)$
(3) the set of values of k for which the major axis is the y -axis is $(-\infty, -3) \cup (2, \infty)$
(4) the set of values of k for which the major axis is the y -axis is $(-3, \infty)$
12. Two concentric ellipses are such that the foci of one are on the other and their major axes are equal. Let e and e' be their eccentricities. Then,
- (1) the quadrilateral formed by joining the foci of the two ellipses is a parallelogram
(2) the angle θ between their axes is given by $\theta = \cos^{-1} \sqrt{\frac{1}{e^2} + \frac{1}{e'^2} - \frac{1}{e^2 e'^2}}$
(3) if $e^2 + e'^2 = 1$, then the angle between the axes of the two ellipses is 90°
(4) none of these
13. A point moves such that the sum of the square of its distances from two fixed straight lines intersecting at angle 2α is a constant. The locus of point is an ellipse of eccentricity
- (1) $\frac{\sqrt{\cos 2\alpha}}{\sin \alpha}$ if $\alpha < \frac{\pi}{4}$ (2) $\frac{\sqrt{-\cos 2\alpha}}{\cos \alpha}$ if $\alpha > \frac{\pi}{4}$
(3) $\frac{\sqrt{\cos 2\alpha}}{\cos \alpha}$ if $\alpha < \frac{\pi}{4}$ (4) $\frac{\sqrt{-\cos 2\alpha}}{\sin \alpha}$ if $\alpha > \frac{\pi}{4}$
14. The coordinates $(2, 3)$ and $(1, 5)$ are the foci of an ellipse which passes through the origin. Then the equation of the
- (1) tangent at the origin is $(3\sqrt{2} - 5)x + (1 - 2\sqrt{2})y = 0$
(2) tangent at the origin is $(3\sqrt{2} + 5)x + (1 + 2\sqrt{2})y = 0$
(3) normal at the origin is $(3\sqrt{2} + 5)x - (2\sqrt{2} + 1)y = 0$
(4) normal at the origin is $x(3\sqrt{2} - 5) - y(1 - 2\sqrt{2}) = 0$
15. If a pair of variable straight lines $x^2 + 4y^2 + \alpha xy = 0$ (where α is a real parameter) cuts the ellipse $x^2 + 4y^2 = 4$ at two points A and B , then the locus of the point of intersection of tangents at A and B is
- (1) $x - 2y = 0$ (2) $2x - y = 0$
(3) $x + 2y = 0$ (4) $2x + y = 0$
16. A point on the ellipse $x^2 + 3y^2 = 37$ where the normal is parallel to the line $6x - 5y = 2$ is
- (1) $(5, -2)$ (2) $(5, 2)$ (3) $(-5, 2)$ (4) $(-5, -2)$
17. The locus of the image of the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, ($a > b$), with respect to any of the tangents to the ellipse is
- (1) $(x + 4)^2 + y^2 = 100$ (2) $(x + 2)^2 + y^2 = 50$
(3) $(x - 4)^2 + y^2 = 100$ (4) $(x - 2)^2 + y^2 = 50$
18. If the tangent drawn at point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is the same as the normal drawn at point $(\sqrt{5} \cos \theta, 2 \sin \theta)$ on the ellipse $4x^2 + 5y^2 = 20$, then
- (1) $\theta = \cos^{-1} \left(-\frac{1}{\sqrt{5}} \right)$ (2) $\theta = \cos^{-1} \left(\frac{1}{\sqrt{5}} \right)$
(3) $t = -\frac{2}{\sqrt{5}}$ (4) $t = -\frac{1}{\sqrt{5}}$
19. If a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $4 - 3y = 7$ and its eccentricity is $\frac{\sqrt{7}}{4}$, then the value of $L.R.$ can be
- (1) $\frac{9}{\sqrt{2}}$ (2) $\frac{9}{2}$
(3) $\frac{8\sqrt{337}}{19}$ (4) $\frac{3\sqrt{337}}{8}$

Linked Comprehension Type

For Problems 1–3

An ellipse E , $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, centered at point O has AB and CD as its major and minor axes, respectively. Let S_1 be one of the foci of the ellipse, the radius of the incircle of triangle OCS_1 be 1 unit and $OS_1 = 6$ units.

1. The perimeter of $\triangle OCS_1$ is

- (1) 20 units (2) 10 units (3) 15 units (4) 25 units

2. The equation of the director circle of E is

- (1) $x^2 + y^2 = 48.5$ (2) $x^2 + y^2 = \sqrt{97}$
(3) $x^2 + y^2 = 97$ (4) $x^2 + y^2 = \sqrt{48.5}$

3. The area of ellipse (E) is

- (1) $65\pi/4$ (2) $64\pi/5$ (3) 64π (4) 65π

For Problems 4–6

Consider the ellipse whose major and minor axes are x -axis and y -axis, respectively. ϕ is the angle between CP and the normal at point P on the ellipse, and the greatest value of $\tan \phi$ is $3/2$ (where C is the center of the ellipse). Also, the length of the semi-major axis is 10 units.

4. The eccentricity of the ellipse is
 (1) $1/2$ (2) $1/3$
 (3) $\sqrt{3}/2$ (4) none of these
5. A rectangle is inscribed in the ellipse whose sides are parallel to the coordinate axes. Then the maximum area of the rectangle is
 (1) 50 units (2) 100 units
 (3) 25 units (4) none of these
6. The locus of the point of intersection of perpendicular tangents to the ellipse is
 (1) $x^2 + y^2 = 125$ (2) $x^2 + y^2 = 150$
 (3) $x^2 + y^2 = 200$ (4) none of these

For Problems 7–9

A curve is represented by $C = 21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$.

7. The eccentricity of the curve is
 (1) $1/3$ (2) $1/\sqrt{3}$ (3) $2/3$ (4) $2/\sqrt{5}$
8. The lengths of axes are
 (1) $6, 2\sqrt{6}$ (2) $5, 2\sqrt{5}$
 (3) $4, 4\sqrt{5}$ (4) none of these
9. The center of the conic C is
 (1) $(1, 0)$ (2) $(0, 0)$
 (3) $(0, 1)$ (4) none of these

For Problems 10–12

For all real p , the line $2px + y\sqrt{1-p^2} = 1$ touches a fixed ellipse whose axes are the coordinate axes.

10. The eccentricity of the ellipse is
 (1) $2/3$ (2) $\sqrt{3}/2$ (3) $1/\sqrt{3}$ (4) $1/2$
11. The foci of the ellipse are
 (1) $(0, \pm\sqrt{3})$ (2) $(0, \pm 2/3)$
 (3) $(\pm\sqrt{3}/2, 0)$ (4) none of these
12. The locus of the point of intersection of perpendicular tangents is
 (1) $x^2 + y^2 = 5/4$ (2) $x^2 + y^2 = 3/2$
 (3) $x^2 + y^2 = 2$ (4) none of these

For Problems 13–15

Let S and S' be the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentricity is e . P is a variable point on the ellipse. Consider the locus of the incentre of $\triangle PSS'$.

13. The locus of the incentre is a/an
 (1) ellipse (2) hyperbola
 (3) parabola (4) circle
14. The eccentricity of the locus of P is
 (1) $\sqrt{\frac{2e}{1-e}}$ (2) $\sqrt{\frac{2e}{1+e}}$
 (3) 1 (4) none of these

15. The maximum area of rectangle inscribed in the locus is

- (1) $\frac{2abe^2}{1+e}$ (2) $\frac{2abe}{1-e}$
 (3) $\frac{abe}{1+e}$ (4) none of these

For Problems 16–18

$C_1: x^2 + y^2 = r^2$ and $C_2: \frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at four distinct points A, B, C , and D . Their common tangents form a parallelogram $A'B'C'D'$.

16. If $ABCD$ is a square, then r is equal to

- (1) $\frac{12}{5}\sqrt{2}$ (2) $\frac{12}{5}$
 (3) $\frac{12}{5\sqrt{5}}$ (4) none of these

17. If $A'B'C'D'$ is a square, then r is equal to

- (1) $\sqrt{20}$ (2) $\sqrt{12}$
 (3) $\sqrt{15}$ (4) none of these

18. If $A'B'C'D'$ is a square, then the ratio of the area of circle C_1 to the area of circumcircle of $\triangle A'B'C'$ is

- (1) $9/16$ (2) $3/4$
 (3) $1/2$ (4) none of these

For Problems 19–21

A coplanar beam of light emerging from a point source has the equation $\lambda x - y + 2(1 + \lambda) = 0$, $\lambda \in R$. The rays of the beam strike an elliptical surface and get reflected. The reflected rays form another convergent beam having equation $\mu x - y + 2(1 - \mu) = 0$, $\mu \in R$. Further, it is found that the foot of the perpendicular from the point $(2, 2)$ upon any tangent to the ellipse lies on the circle $x^2 + y^2 - 4y - 5 = 0$.

19. The eccentricity of the ellipse is equal to

- (1) $1/3$ (2) $1/\sqrt{3}$ (3) $2/3$ (4) $1/2$

20. The area of the largest triangle that an incident ray and the corresponding reflected ray can enclose with the axis of the ellipse is equal to

- (1) $4\sqrt{5}$ (2) $2\sqrt{5}$
 (3) $\sqrt{5}$ (4) none of these

21. The total distance travelled by an incident ray starting at one focus and the corresponding reflected ray terminating at another focus is

- (1) 9 (2) 8 (3) 7 (4) 6

For Problems 22–24

The tangent at any point P of the circle $x^2 + y^2 = 16$ meets the tangent at a fixed point A at T . Point T is joined to B , the other end of the diameter, through A .

22. The locus of the intersection of AP and BT is a conic whose eccentricity is

- (1) $1/2$ (2) $1/\sqrt{2}$ (3) $1/3$ (4) $1/\sqrt{3}$

23. The sum of focal distances of any point on the curve is

- (1) 12 (2) 16 (3) 20 (4) 8

24. Which of the following does not change by changing the radius of the circle?

- (1) Coordinates of foci (2) Length of the major axis
 (3) Eccentricity (4) Length of the minor axis

For Problems 25–27

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is such that it has the least area but contains the circle $(x-1)^2 + y^2 = 1$.

25. The eccentricity of the ellipse is

- (1) $\sqrt{2/3}$ (2) $1/\sqrt{3}$
(3) $1/2$ (4) none of these

26. The equation of the auxiliary circle of ellipse is

- (1) $x^2 + y^4 = 6.5$ (2) $x^2 + y^4 = 5$
(3) $x^2 + y^4 = 4.5$ (4) none of these

27. The length of the latus rectum of the ellipse is

- (1) 2 units (2) $\sqrt{2}$ units
(3) 3 units (4) 2.5 units

Matrix Match Type

1. Match the following lists:

List I	List II
a. The distance between the points on the curve $4x^2 + 9y^2 = 1$, where the tangent is parallel to the line $8x = 9y$, is less than	p. 1
b. The sum of the distance between the foci of the curve $25(x+1)^2 + 9(y+2)^2 = 225$ from $(-1, 0)$ is more than	q. 4
c. The sum of distances from the x-axis of the points on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, where the normal is parallel to the line $2x + y = 1$, is less than	r. 7
d. Tangents are drawn from the points on the line $x - y + 2 = 0$ to the ellipse $x^2 + 2y^2 = 2$. Then all the chords of contact pass through the point whose distance from $(2, 1/2)$ is more than	s. 5

2. The tangents drawn from a point P to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ make angles α and β with the major axis.

Now, match the following lists:

List I	List II
a. If $\alpha + \beta = \frac{c\pi}{2}$, ($c \in \mathbb{N}$), then the locus of P can be	p. Circle
b. If $\tan \alpha \tan \beta = c$, ($c \in \mathbb{R}$), then the locus of P can be	q. Ellipse
c. If $\tan \alpha + \tan \beta = c$, (where $c \in \mathbb{R}$), then the locus of P can be	r. Hyperbola
d. If $\cot \alpha + \cot \beta = c$, (where $c \in \mathbb{R}$), then the locus of P can be	s. Pair of straight lines

3. Match the following lists:

List I	List II
a. If the tangent to the ellipse $x^2 + 4y^2 = 16$ at the point $P(\phi)$ is a normal to the circle $x^2 + y^2 - 8x - 4y = 0$, then $\phi/2$ may be	p. 0
b. The eccentric angle(s) of a point on the ellipse $x^2 + 3y^2 = 6$ at a distance of 2 units from the center of the ellipse is/are	q. $\cos^{-1}(-\frac{2}{3})$
c. The angle of intersection of the ellipse $x^2 + 4y^2 = 4$ and the parabola $x^2 + 1 = y$ is	r. $\frac{\pi}{4}$
d. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersects it again at the point $Q(2\theta)$, then θ is	s. $\frac{5\pi}{4}$

4. Match the following lists:

List I	List II
a. An ellipse passing through the origin has its foci $(3, 4)$ and $(6, 8)$. Then the length of its minor axis is	p. 8
b. If PQ is a focal chord of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which passes through $S \equiv (3, 0)$ and $PS = 2$, then the length of chord PQ is	q. $10\sqrt{2}$
c. If the line $y = x + K$ touches the ellipse $9x^2 + 16y^2 = 144$, then the difference of values of K is	r. 10
d. The sum of distances of a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ from the foci is	s. 12

5. Match the following lists and then choose the correct code.

List I	List II
a. A stick of length 10 m slides on the coordinate axes. Then locus of the point dividing this stick from the x-axis in the ratio 6 : 4 is a curve whose eccentricity is e . Then $3e$ is equal to	p. $\sqrt{6}$
b. AA' is the major axis of the ellipse $3x^2 + 2y^2 + 6x - 4y - 1 = 0$ and P is a variable point on it. Then the greatest area of triangle $AP A'$ is	q. $2\sqrt{7}$
c. The distance between the foci of the curve represented by the equation $x = 1 + 4 \cos \theta$, $y = 2 + 3 \sin \theta$ is	r. $\frac{128}{3}$
d. Tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ at the endpoints of the latus rectum. The area of the quadrilateral so formed is	s. $\sqrt{5}$

Codes:

- a b c d
(1) p r q q
(2) q r p s
(3) s p q r
(4) r p s q

6. Match the following lists and then choose the correct code.

List I	List II
a. If the vertices of a rectangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are extremities of latus rectum, then the eccentricity of the ellipse is	p. $\frac{2}{\sqrt{5}}$
b. If the extremities of the diameter of the circle $x^2 + y^2 = 16$ are the foci of the ellipse, then the eccentricity of the ellipse, if its size is just sufficient to contain the circle, is	q. $\frac{1}{\sqrt{2}}$
c. If the normal at point (6, 2) to the ellipse passes through its nearest focus (5, 2), having center at (4, 2), then its eccentricity is	r. $\frac{1}{3}$
d. If the extremities of the latus rectum of the parabola $y^2 = 24x$ are the foci of ellipse, and if the ellipse passes through the vertex of the parabola, then its eccentricity is	s. $\frac{1}{2}$

Codes:

- | | | | | |
|-----|---|---|---|---|
| | a | b | c | d |
| (1) | q | p | s | r |
| (2) | q | r | s | q |
| (3) | s | p | q | r |
| (4) | q | q | s | p |

Numerical Value Type

- If $x, y \in R$, satisfies the equation $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$, then the difference between the largest and the smallest value of the expression $\frac{x^2}{4} + \frac{y^2}{9}$ is _____.
- The value of a for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($a > b$), if the extremities of the latus rectum of the ellipse having

positive ordinates lie on the parabola $x^2 = -2(y-2)$ is _____.

- If the variable line $y = kx + 2h$ is tangent to an ellipse $2x^2 + 3y^2 = 6$, then the locus of $P(h, k)$ is a conic C whose eccentricity is e . Then the value of $3e^2$ is _____.
- Tangents drawn from the point $P(2, 3)$ to the circle $x^2 + y^2 - 8x + 6y + 1 = 0$ touch circle at points A and B . The circumcircle of $\triangle PAB$ cuts the director circle of ellipse $\frac{(x+5)^2}{9} + \frac{(y-3)^2}{b^2} = 1$ orthogonally. Then the value of b^2 is _____.
- Points P and D are taken on the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$. If a, b, c and d are the lengths of the sides of quadrilateral $PADB$, where A and B are foci of the ellipse, then maximum value of $(abcd)$ is _____.
- If $a(x^2 + y^2 + 2y + 1) = (x - 2y + 3)^2$ is an ellipse and $a \in (b, \infty)$, then the value of b is _____.
- If the midpoint of a chord of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is $(0, 3)$ then the length of the chord is _____.
- Let the distance between a focus and the corresponding directrix of an ellipse be 8 and the eccentricity be $1/2$. If the length of the minor axis is k , then $\sqrt{3}k/2$ is _____.
- Square of the radius of director circle of auxiliary circle of ellipse $(3x+4y-1)^2 + 5(4x-3y+2)^2 = 250$ is _____.
- Suppose x and y are real numbers and that $x^2 + 9y^2 - 4x + 6y + 4 = 0$. Then the maximum value of $(4x - 9y)/2$ is _____.
- Rectangle $ABCD$ has area 200. An ellipse with area 200π passes through A and C and has foci at B and D . Then the perimeter of the rectangle is _____.

Archives

JEE MAIN

Single Correct Answer Type

- The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which is in turn inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is
 - $x^2 + 16y^2 = 16$
 - $x^2 + 12y^2 = 16$
 - $4x^2 + 48y^2 = 48$
 - $4x^2 + 64y^2 = 48$

(AIEEE 2009)
- Equation of the ellipse whose axes are the axes of coordinates and which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$ is
 - $5x^2 + 3y^2 - 32 = 0$
 - $3x^2 + 5y^2 - 32 = 0$
 - $5x^2 + 3y^2 - 48 = 0$
 - $3x^2 + 5y^2 - 15 = 0$

(AIEEE 2011)

- Statement 1:** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement 2: If the line $y = mx + \frac{4\sqrt{3}}{m}$, ($m \neq 0$) is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.

- Statement 1 is false, statement 2 is true.
- Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.
- Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.
- Statement 1 is true, statement 2 is false.

(AIEEE 2012)

4. An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is

- (1) $4x^2 + y^2 = 4$ (2) $x^2 + 4y^2 = 8$
(3) $4x^2 + y^2 = 8$ (4) $x^2 + 4y^2 = 16$

(AIEEE 2012)

5. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 9$, and having centre at $(0, 3)$ is:

- (1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$
(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

(JEE Main 2013)

6. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is

- (1) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (2) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(3) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (4) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

(JEE Main 2014)

7. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the lateral recta to the ellipse

$$\frac{x^2}{9} + \frac{y^2}{5} = 1, \text{ is}$$

- (1) 27/4 (2) 18 (3) 27/2 (4) 27

(JEE Main 2015)

8. The eccentricity of an ellipse whose center is at the origin is $1/2$. If one of its directrices is $x = -4$, then the equation of the normal to it at $(1, 3/2)$ is

- (1) $x + 2y = 4$ (2) $2y - x = 2$
(3) $4x - 2y = 1$ (4) $4x + 2y = 7$

(JEE Main 2017)

9. Two sets A and B are as under:

$$A = \{(a, b) \in R \times R : |a - 5| < 1 \text{ and } |b - 5| < 1\};$$

$$B = \{(a, b) \in R \times R : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}. \text{ Then}$$

- (1) neither $A \subset B$ nor $B \subset A$
(2) $B \subset A$
(3) $A \subset B$
(4) $A \cap B = \phi$ (an empty set)

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M . Then the area of the triangle with vertices at A , M , and O (the origin) is

- (1) 31/10 (2) 29/10
(3) 21/10 (4) 27/10 (IIT-JEE 2009)

2. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x -axis at Q . If M is the midpoint of the line segment PQ , then the locus of M intersects the latus rectums of the given ellipse at points

- (1) $(\pm(3\sqrt{5})/2, \pm 2/7)$ (2) $(\pm(3\sqrt{5})/2, \pm\sqrt{19}/7)$
(3) $(\pm 2\sqrt{3}, \pm 1/7)$ (4) $(\pm 2\sqrt{3}, \pm 4\sqrt{3}/7)$

(IIT-JEE 2009)

3. The ellipse $E_1: \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle

R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R . The eccentricity of the ellipse E_2 is

- (1) $\sqrt{2}/2$ (2) $\sqrt{3}/2$
(3) $1/2$ (4) $3/4$ (IIT-JEE 2012)

Multiple Correct Answers Type

1. In a triangle ABC with fixed base BC , the vertex A moves such that $\cos B + \cos C = 4 \sin^2 \frac{A}{2}$. If a , b , and c denote the lengths of the sides of the triangle opposite to the angles A , B , and C , respectively, then

- (1) $b + c = 4a$
(2) $b + c = 2a$
(3) the locus of point A is an ellipse
(4) the locus of point A is a pair of straight lines

(IIT-JEE 2009)

2. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x -axis and the y -axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves S , E_1 and E_2 at

P , Q and R , respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If

e_1 and e_2 are the eccentricities of E_1 and E_2 , respectively, then the correct expression(s) is (are)

- (1) $e_1^2 + e_2^2 = \frac{43}{40}$ (2) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
(3) $|e_1^2 - e_2^2| = \frac{5}{8}$ (4) $e_1 e_2 = \frac{\sqrt{3}}{4}$

(JEE Advanced 2015)

3. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q . Consider the ellipse whose centre is at the origin $O(0, 0)$ and whose semi-major axis is OQ . If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE?

- (1) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1
(2) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$
(3) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$
(4) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

(JEE Advanced 2018)

Linked Comprehension Type

For Problems 1–3

Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B .

(IIT-JEE 2010)

1. The coordinates of A and B are, respectively,

- (1) $(3, 0)$ and $(0, 2)$
- (2) $(-8/5, 2\sqrt{161}/15)$ and $(-9/5, 8/5)$
- (3) $(-8/5, 2\sqrt{161}/15)$ and $(0, 2)$
- (4) $(3, 0)$ and $(-9/5, 8/5)$

2. The orthocenter of triangle PAB is

- (1) $(5, 8/7)$
- (2) $(7/5, 25/8)$
- (3) $(11/5, 8/5)$
- (4) $(8/25, 7/5)$

3. The equation of the locus of the point whose distances from the point P and the line AB are equal is

- (1) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
- (2) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$
- (3) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
- (4) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

For Problems 4 and 5

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$, for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

(JEE Advanced 2016)

4. The orthocentre of the triangle F_1MN is

- (1) $\left(-\frac{9}{10}, 0\right)$
- (2) $\left(\frac{2}{3}, 0\right)$
- (3) $\left(\frac{9}{10}, 0\right)$
- (4) $\left(\frac{2}{3}, \sqrt{6}\right)$

5. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x -axis at Q , then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is

- (1) $3 : 4$
- (2) $4 : 5$
- (3) $5 : 8$
- (4) $2 : 3$

Matrix Match Type

1. Match the conics in List I with the statements/expressions in List II.

List I	List II
a. Circle	p. The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
b. Parabola	q. Points z in the complex plane satisfying $ z + 2 - z - 2 = \pm 3$

c. Ellipse	r. Points of the conic have parametric representation $x = \sqrt{3} \left(\frac{1-t^2}{1+t^2} \right), y = \frac{2t}{1+t^2}$
d. Hyperbola	s. The eccentricity of the conic lies in the interval $1 \leq e < \infty$
	t. Points z in the complex plane satisfying $\operatorname{Re}(z + 1)^2 = z ^2 + 1$

(IIT-JEE 2009)

2. Match the following lists:

List I	List II
a. Let $y(x) = \cos(3 \cos^{-1} x)$, $x \in [-1, 1], x \neq \pm \frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)} \left\{ (x^2 - 1) \frac{d^2 y(x)}{dx^2} + x \frac{dy(x)}{dx} \right\}$ equals	p. 1
b. Let $A_1, A_2, \dots, A_n (n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point $A_k, k = 1, 2, \dots, n$. If $\left \sum_{k=1}^{n-1} (\vec{a}_k \times \vec{a}_{k+1}) \right = \left \sum_{k=1}^{n-1} (\vec{a}_k \cdot \vec{a}_{k+1}) \right $, then the minimum value of n is	q. 2
c. If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	r. 8
d. Number of positive solutions satisfying the equation $\tan^{-1} \left(\frac{1}{2x+1} \right) + \tan^{-1} \left(\frac{1}{4x+1} \right) = \tan^{-1} \left(\frac{2}{x^2} \right)$ is	s. 9

Codes:

- | | | | | |
|-----|---|---|---|---|
| | a | b | c | d |
| (1) | s | r | q | p |
| (2) | q | s | r | p |
| (3) | s | r | p | q |
| (4) | q | s | p | r |

(JEE Advanced 2014)

Numerical Value Type

1. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at points P and Q . Let the tangents to the ellipse at P and Q meet at point R . If $\Delta(h) = \text{area of triangle } PQR$, $\Delta_1 = \max_{1/2 \leq h \leq 1} \Delta(h)$, and $\Delta_2 = \min_{1/2 \leq h \leq 1} \Delta(h)$, then $\frac{8}{\sqrt{5}} \Delta_1 - 8 \Delta_2 =$ _____.

(JEE Advanced 2013)

2. Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at $(0, 0)$ and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2 , then the value of $\left(\frac{1}{m_1^2} + m_2^2 \right)$ is _____.

(JEE Advanced 2015)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (3) | 2. (1) | 3. (2) | 4. (4) | 5. (2) |
| 6. (1) | 7. (1) | 8. (3) | 9. (1) | 10. (2) |
| 11. (3) | 12. (2) | 13. (2) | 14. (1) | 15. (1) |
| 16. (3) | 17. (3) | 18. (2) | 19. (4) | 20. (2) |
| 21. (1) | 22. (2) | 23. (3) | 24. (3) | 25. (1) |
| 26. (1) | 27. (1) | 28. (1) | 29. (2) | 30. (4) |
| 31. (1) | 32. (2) | 33. (3) | 34. (4) | 35. (1) |
| 36. (1) | 37. (1) | 38. (2) | 39. (1) | 40. (4) |
| 41. (3) | 42. (3) | 43. (4) | 44. (1) | 45. (3) |
| 46. (3) | 47. (4) | 48. (4) | 49. (4) | 50. (3) |
| 51. (4) | 52. (4) | 53. (4) | 54. (3) | 55. (3) |
| 56. (3) | 57. (3) | 58. (3) | | |

Multiple Correct Answers Type

- | | |
|------------------|-----------------------|
| 1. (1), (3) | 2. (1), (2), (3), (4) |
| 3. (1), (2), (3) | 4. (2), (3) |
| 5. (1), (2), (4) | 6. (1), (3) |
| 7. (3), (4) | 8. (1), (3), (4) |
| 9. (1), (2), (3) | 10. (1), (3) |
| 11. (1), (3) | 12. (1), (2), (3) |
| 13. (3), (4) | 14. (1), (3) |
| 15. (1), (3) | 16. (2), (4) |
| 17. (1), (3) | 18. (1), (4) |
| 19. (1), (4) | |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (3) | 2. (1) | 3. (1) | 4. (3) | 5. (2) |
| 6. (1) | 7. (2) | 8. (1) | 9. (3) | 10. (2) |
| 11. (4) | 12. (1) | 13. (1) | 14. (2) | 15. (1) |
| 16. (1) | 17. (4) | 18. (3) | 19. (3) | 20. (2) |
| 21. (4) | 22. (2) | 23. (4) | 24. (3) | 25. (1) |
| 26. (3) | 27. (2) | | | |

Matrix Match Type

- $a \rightarrow p, q, r, s; b \rightarrow p, q, r, s; c \rightarrow q, r, s; d \rightarrow p$
- $a \rightarrow r, s; b \rightarrow p, q, r, s; c \rightarrow r, s; d \rightarrow r, s$
- $a \rightarrow p; b \rightarrow r, s; c \rightarrow p; d \rightarrow q$
- $a \rightarrow q; b \rightarrow r; c \rightarrow r; d \rightarrow p$
- (3)
- (4)

Numerical Value Type

- | | | | | |
|----------|----------|--------|---------|---------|
| 1. (8) | 2. (2) | 3. (7) | 4. (54) | 5. (16) |
| 6. (5) | 7. (6.4) | 8. (8) | 9. (20) | 10. (8) |
| 11. (80) | | | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (2) | 2. (2) | 3. (2) | 4. (4) | 5. (1) |
| 6. (3) | 7. (4) | 8. (3) | 9. (3) | |

JEE ADVANCED

Single Correct Answer Type

- | | | |
|--------|--------|--------|
| 1. (4) | 2. (3) | 3. (3) |
|--------|--------|--------|

Multiple Correct Answers Type

- | | | |
|-------------|-------------|-------------|
| 1. (2), (3) | 2. (1), (2) | 3. (1), (3) |
|-------------|-------------|-------------|

Linked Comprehension Type

- | | | | | |
|--------|--------|--------|--------|--------|
| 1. (4) | 2. (3) | 3. (1) | 4. (1) | 5. (3) |
|--------|--------|--------|--------|--------|

Matrix Match Type

- | | |
|----------------------|--------|
| 1. $c \rightarrow r$ | 2. (1) |
|----------------------|--------|

Numerical Value Type

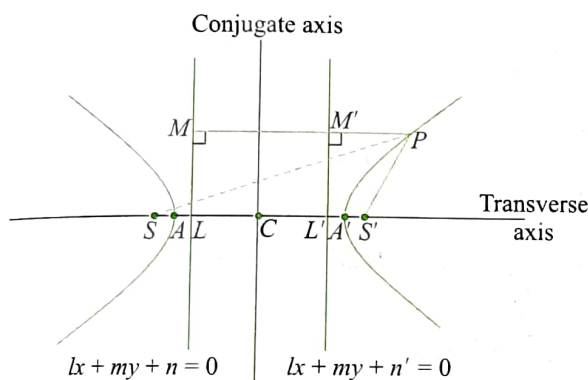
- | | |
|--------|--------|
| 1. (9) | 2. (4) |
|--------|--------|

DEFINITION

We know that hyperbola is one of the conic sections. Hyperbola is locus of a point such that ratio of its distances from a fixed point (focus) and a fixed line (directrix) is constant, the value of which is more than 1. The constant ratio is called eccentricity ' e '. Thus, for hyperbola, $e \in (1, \infty)$.

For any hyperbola, let the focus be $S(\alpha, \beta)$ and the directrix be $lx + my + n = 0$.

Also, let PM be the perpendicular from point P upon directrix.



The locus of variable point $P(x, y)$ is hyperbola if $\frac{SP}{PM} = e$ (constant), where $e \in (1, \infty)$.

$$\therefore SP = e \cdot PM$$

Therefore, equation of hyperbola is

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}, e > 1$$

The above equation takes the form $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, which represents hyperbola if $h^2 - ab > 0$ and $\Delta = abc + 2hgf - af^2 - bg^2 - ch^2 \neq 0$.

We observe that graph of the hyperbola is open curve and has two branches.

The line passing through focus S and perpendicular to the directrix is called the transverse axis or major axis (focal axis) of the hyperbola.

In the figure, transverse axis intersects the hyperbola at points A and A' which are two vertices of the hyperbola. Here, AA' is the length of the transverse axis or length of major axis or simply transverse axis. There is another axis of the hyperbola which is called the conjugate axis (or minor axis) and it is perpendicular bisector of AA' . Graph of the hyperbola is symmetrical about the conjugate axis also.

Both the axes intersect at C , which is called the centre of the hyperbola. Centre bisects every chord of the hyperbola passing through it.

Because of symmetry of the hyperbola about its axes, it is possible to define hyperbola using another focus $S'(\alpha', \beta')$ and directrix $lx + my + n' = 0$, which are symmetrical about the conjugate axis. In this case, $\frac{S'P}{PM'} = e$.

In the figure, $CS = CS'$ and $CL = CL'$.

SP and $S'P$ are called focal distances or focal radii of point P .

For point P on hyperbola, we have

$$\begin{aligned} SP - S'P &= e \cdot PM - e \cdot PM' \\ &= e \cdot (PM - PM') \\ &= e \cdot MM' \\ &= \frac{e}{2} \times 2LL' \\ &= \frac{e}{2} \times (L'A - LA + LA' - L'A') \\ &= \frac{e}{2} \times \left(\frac{AS'}{e} - \frac{AS}{e} + \frac{A'S}{e} - \frac{A'S'}{e} \right) \\ &= \frac{1}{2} \times (AS' - A'S' + A'S - AS) \\ &= \frac{1}{2} (AA' + AA') \\ &= AA' \\ &= \text{Transverse axis} \end{aligned}$$

Note that point P is lying on the right branch of the hyperbola.

For point P lying on the left branch, we have $S'P - SP = AA'$.

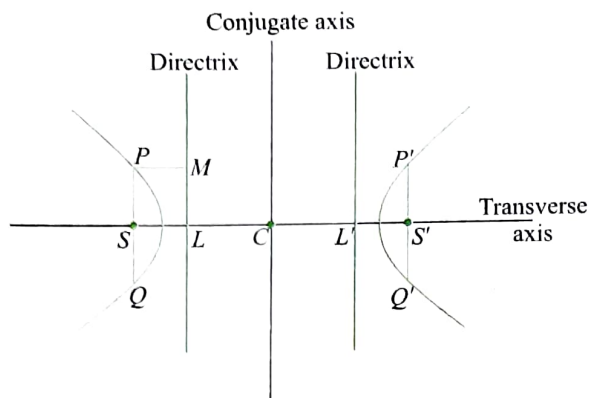
Thus, difference of two focal radii of variable point on the hyperbola is constant which is equal to the length of transverse axis.

Further in triangle PSS' , SS' is fixed and difference of two variable sides SP and $S'P$ is constant. This leads to another definition of hyperbola as follows:

If in a triangle, one side (SS') is fixed and variable vertex (P) moves in such a way that the difference of two variable sides of triangle remains the same (constant), then locus of variable vertex is hyperbola.

Latus Rectum of Hyperbola

We know that double ordinate through focus terminated by conic is its latus rectum.



In the above figure, double ordinate through foci S and S' are PQ and $P'Q'$, respectively.

$$\begin{aligned}\text{Latus rectum} &= PQ (= P'Q') \\ &= 2 \times SP \\ &= 2 \times e \times PM \quad (\text{from definition of hyperbola}) \\ &= 2 \times e \times SL \\ &= 2 \times e \times (\text{distance of focus from directrix})\end{aligned}$$

ILLUSTRATION 7.1

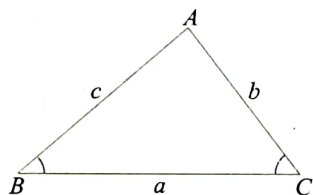
If base of triangle and ratio of tangents of half of the base angles are given, then prove that the locus of opposite vertex is hyperbola.

Sol. In triangle ABC , base BC is given.

i.e., $BC = a$ (constant)

Also, given that

$$\frac{\tan \frac{B}{2}}{\tan \frac{C}{2}} = k \text{ (constant)}$$



$$\therefore \frac{\sin \frac{B}{2} \cos \frac{C}{2}}{\sin \frac{C}{2} \cos \frac{B}{2}} = k$$

$$\Rightarrow \frac{\sin \frac{B}{2} \cos \frac{C}{2} - \sin \frac{C}{2} \cos \frac{B}{2}}{\sin \frac{B}{2} \cos \frac{C}{2} + \sin \frac{C}{2} \cos \frac{B}{2}} = \frac{k-1}{k+1}$$

$$\Rightarrow \frac{\sin \frac{B-C}{2}}{\sin \frac{B+C}{2}} = \frac{k-1}{k+1}$$

$$\Rightarrow \frac{2 \sin \frac{B-C}{2} \cos \frac{B+C}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B+C}{2}} = \frac{k-1}{k+1}$$

$$\Rightarrow \frac{\sin B - \sin C}{\sin(B+C)} = \frac{k-1}{k+1}$$

$$\Rightarrow \frac{\sin B - \sin C}{\sin A} = \frac{k-1}{k+1} \quad [\text{As in triangle } \sin(B+C) = \sin A]$$

$$\Rightarrow \frac{b-c}{a} = \frac{k-1}{k+1} \quad (\text{using sine rule})$$

$$\Rightarrow b-c = a \left(\frac{k-1}{k+1} \right)$$

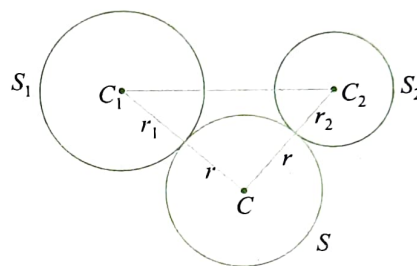
$$\Rightarrow AC - AB = a \left(\frac{k-1}{k+1} \right) \quad (= \text{constant})$$

Thus, difference of two variable sides AC and AB is constant. Hence, locus of vertex A is hyperbola with B and C as foci.

ILLUSTRATION 7.2

Prove that the locus of centre of the circle which touches two given disjoint circles externally is hyperbola.

Sol. As shown in the figure, variable circle S with centre C and radius r touches two given disjoint circles S_1 and S_2 having centres C_1 and C_2 and radii r_1 and r_2 , respectively.



Clearly, $CC_1 = r + r_1$ and $CC_2 = r + r_2$

$$\therefore CC_1 - CC_2 = r_1 - r_2 \quad (= \text{constant})$$

Thus, locus of centre C is hyperbola having foci C_1 and C_2 .

ILLUSTRATION 7.3

The equation of one directrix of a hyperbola is $2x + y = 1$, the corresponding focus is $(1, 2)$ and eccentricity is $\sqrt{3}$. Find the equation of the hyperbola and coordinates of the centre and second focus.

Sol. Given focus of hyperbola is $F_1(1, 2)$ and corresponding directrix is $2x + y - 1 = 0$.

Let $P(x, y)$ be the point on the hyperbola.

Foot of perpendicular from P on the directrix is M .

So, from the definition of hyperbola, we have

$$F_1P = e \cdot PM$$

$$\therefore \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{3} \left(\frac{|2x+y-1|}{\sqrt{2^2+1^2}} \right)$$

Squaring and simplifying, we get equation of hyperbola as follows:

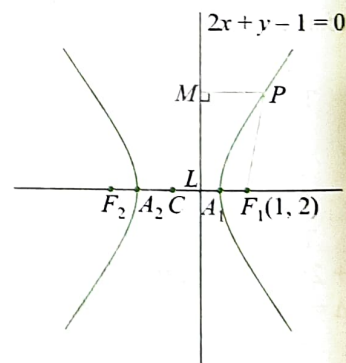
$$7x^2 + 12xy - 2y^2 - 2x + 14y - 22 = 0 \quad (1)$$

Transverse axis passes through focus F_1 and perpendicular to directrix.

So, equation of transverse axis is $x - 2y + 3 = 0$.

Solving transverse axis and directrix, we get $L \equiv (-1/5, 7/5)$.

Since A_1 and A_2 divide F_1L in the ratio $\sqrt{3} : 1$ internally and externally, respectively, we have



$$A_1 \equiv \left(\frac{5-\sqrt{3}}{5(\sqrt{3}+1)}, \frac{10+7\sqrt{3}}{5(\sqrt{3}+1)} \right)$$

$$\text{and } A_2 \equiv \left(-\frac{5+\sqrt{3}}{5(\sqrt{3}-1)}, \frac{7\sqrt{3}-10}{5(\sqrt{3}-1)} \right)$$

Centre C of hyperbola is midpoint of A_1A_2 , which is $(-4/5, 11/10)$.
 F_2 is second focus and C is midpoint of F_1F_2 .

$$\therefore F_2 \equiv \left(-\frac{13}{5}, \frac{1}{5} \right)$$

ILLUSTRATION 7.4

Prove that the equation $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$ represents a hyperbola.

Sol. We have $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$

Comparing this with general second-degree equation i.e.,
 $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, we get

$$a = 2, b = 2, c = -4, h = 5/2, g = -11/2, f = -7/2$$

$$\text{Now, } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & 5/2 & -11/2 \\ 5/2 & 2 & -7/2 \\ -11/2 & -7/2 & -4 \end{vmatrix} = \frac{81}{4} \neq 0$$

$$\text{Also, } h^2 - ab = (5/2)^2 - (2)(2) > 0$$

So, given equation represents a hyperbola.

ILLUSTRATION 7.5

If equation $\left| \sqrt{(x - \tan \theta)^2 + (y - \sqrt{3} \tan \theta)^2} - \sqrt{(x - 2 \tan \theta)^2 + y^2} \right| = 2, \theta \in [0, \pi] - \left\{ \frac{\pi}{2} \right\}$ represents hyperbola, then find the values of θ .

Sol. We have

$$\left| \frac{\sqrt{(x - \tan \theta)^2 + (y - \sqrt{3} \tan \theta)^2}}{\text{Distance between } F_1(\tan \theta, \sqrt{3} \tan \theta) \text{ and } P(x, y)} - \frac{\sqrt{(x - 2 \tan \theta)^2 + y^2}}{\text{Distance between } F_2(2 \tan \theta, 0) \text{ and } P(x, y)} \right| = 2$$

$$\text{So, } |F_1P - F_2P| = 2,$$

where $P \equiv (x, y)$, $F_1 \equiv (\tan \theta, \sqrt{3} \tan \theta)$ and $F_2 \equiv (2 \tan \theta, 0)$.

Therefore, given equation represents a hyperbola if

$$F_1F_2 > 2$$

$$\Rightarrow \sqrt{(\tan \theta - 2 \tan \theta)^2 + (\sqrt{3} \tan \theta - 0)^2} > 2$$

$$\Rightarrow |2 \tan \theta| > 2$$

$$\Rightarrow |\tan \theta| > 1$$

$$\text{Hence, } \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4} \right).$$

CONCEPT APPLICATION EXERCISE 7.1

1. Find the eccentricity of the hyperbola

$$\sqrt{(x+4)^2 + (y+2)^2} - \sqrt{(x-4)^2 + (y-2)^2} = 8.$$

2. OA and OB are fixed straight lines, P is any point and PM and PN are the perpendiculars from P on OA and OB , respectively. Prove that the locus of P is hyperbola if the quadrilateral $OPMN$ is of constant area.

3. Find the equation of the transverse axis of the hyperbola $(x-3)^2 + (y+1)^2 = (4x+3y)^2$.

ANSWERS

$$1. e = \frac{\sqrt{5}}{2}$$

$$3. 3x - 4y = 13$$

EQUATION OF HYPERBOLA AND ITS PROPERTIES

STANDARD EQUATION OF HYPERBOLA

For standard equation of hyperbola, we consider coordinate axes as axes of the hyperbola. So, we have following two cases:

Transverse Axis is x-Axis and Conjugate Axis is y-Axis

Here, centre of hyperbola is origin.

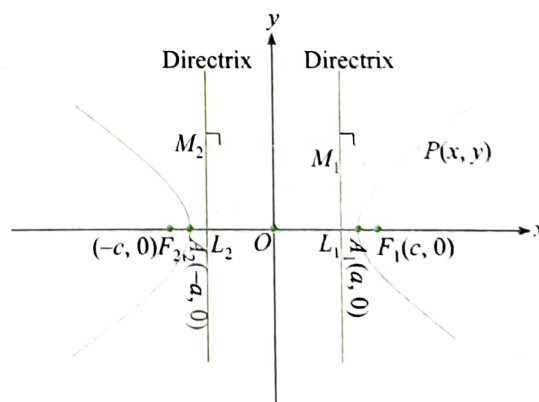
Let foci be $F_1(c, 0)$ and $F_2(-c, 0)$.

Also, let hyperbola intersect x-axis at $A_1(a, 0)$ and $A_2(-a, 0)$.

According to the definition of the hyperbola, for any point $P(x, y)$ lying on the hyperbola, we have

$$PF_2 - PF_1 = 2a$$

Here, we have taken point P on the right branch of the hyperbola such that $PF_2 > PF_1$.



$$\therefore \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a \quad (1)$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a + \sqrt{(x-c)^2 + y^2}$$

Squaring both sides, we get

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$$\Rightarrow \frac{cx}{a} - a = \sqrt{(x-c)^2 + y^2}$$

Squaring again and further simplifying, we get

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Here, $c^2 - a^2 = b^2$)

This is the equation of hyperbola in standard form.

For the points lying on the left branch in the figure, we get the same equation.

Here, the length of transverse axis AA' is $2a$. The length of conjugate axis is $2b$, however, this length is not intercepted on conjugate axis.

Also, a is called semi-transverse axis and b is called semi-conjugate axis.

Like ellipse, eccentricity of hyperbola is also defined as

$$e = \frac{c}{a} = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}}$$

$$\therefore c = ae$$

So, foci are $F_1(ae, 0)$ and $F_2(-ae, 0)$.

$$\text{Thus, } e = \frac{2ae}{2a} = \frac{F_1F_2}{2a} = \frac{\text{Distance between foci}}{\text{Transverse axis}}$$

Also, from $c^2 - a^2 = b^2$, we have

$$b^2 = a^2e^2 - a^2$$

$$\text{or } b^2 = a^2(e^2 - 1)$$

For vertex A_1 , on hyperbola we have

$$F_1A_1 = e \times A_1L_1$$

$$\therefore ae - a = e \times (a - OL_1)$$

$$\therefore OL_1 = \frac{a}{e}$$

So, equation of directrix corresponding to focus $F_1(ae, 0)$ is $x = \frac{a}{e}$.

From symmetry, directrix corresponding to focus $F_2(-ae, 0)$ is $x = -\frac{a}{e}$.

For point P on hyperbola,

$$PF_1 = e \cdot PM_1 = e \left(x - \frac{a}{e} \right) = ex - a$$

$$\text{and } PF_2 = e \cdot PM_2 = e \left(x + \frac{a}{e} \right) = ex + a$$

$$\therefore PF_2 - PF_1 = 2a$$

For point P lying on the left branch, we find that $PF_2 = ex - a$ and $PF_1 = ex + a$.

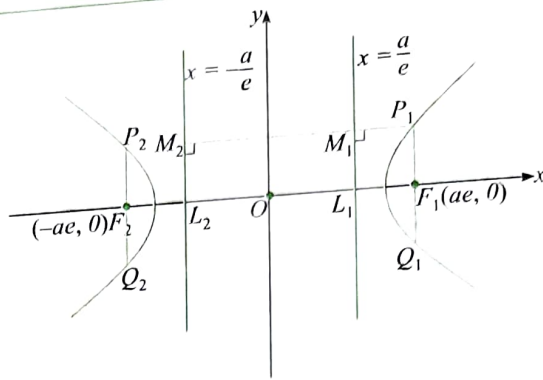
Clearly, $PF_1 - PF_2 = 2a$.

For both the branches of hyperbola, we have $|PF_1 - PF_2| = 2a$.

Latus rectum:

In the following figure, latus rectum = $P_1Q_1 (= P_2Q_2)$.

Equations of line passing through foci are $x = \pm ae$.



Solving with hyperbola, we get

$$\frac{a^2e^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore y^2 = b^2(e^2 - 1) = b^2 \frac{b^2}{a^2} = \frac{b^4}{a^2}$$

$$\therefore y = \pm \frac{b^2}{a}$$

So, endpoints of latus rectum are

$$P_1 \left(ae, \frac{b^2}{a} \right), Q_1 \left(ae, -\frac{b^2}{a} \right)$$

$$\text{and } P_2 \left(-ae, \frac{b^2}{a} \right), Q_2 \left(-ae, -\frac{b^2}{a} \right)$$

$$\text{Length of latus rectum} = P_1Q_1 = P_2Q_2 = \frac{2b^2}{a}$$

Also,

Length of latus rectum

$$= 2 \times e \times \text{Distance of focus from directrix}$$

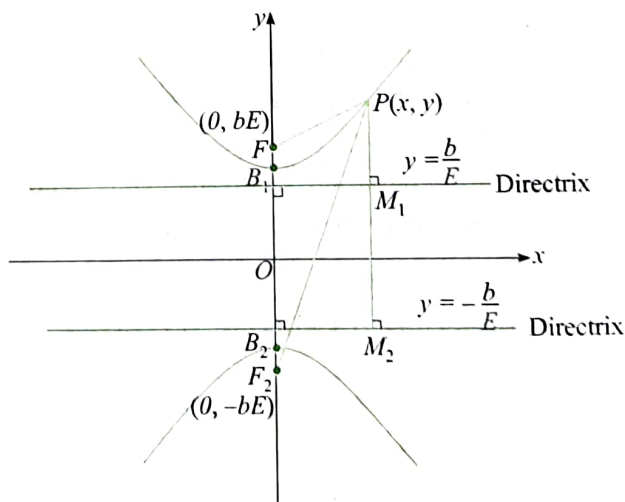
$$= 2 \times e \times F_1L_1$$

$$= 2 \times e \times \left(ae - \frac{a}{e} \right) = 2a(e^2 - 1) = 2a \frac{b^2}{a^2} = \frac{2b^2}{a}$$

Transverse Axis is y-Axis and Conjugate Axis is x-Axis

In this case, equation of hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = -1$.

Clearly, this hyperbola does not meet x -axis, but it meets y -axis.



The points where hyperbola meets y -axis are $B_1(0, b)$ and $B_2(0, -b)$, which are vertices of hyperbola.

The foci are $F_1(0, bE)$ and $F_2(0, -bE)$ and two directrices are $y = \pm \frac{b}{E}$, where E is eccentricity of hyperbola.

Here, the length of transverse axis (B_1B_2) is $2b$ and that of conjugate axis is $2a$.

For any point $P(x, y)$ on the hyperbola, we have

$$PF_2 - PF_1 = 2b \quad (PF_2 > PF_1)$$

$$\therefore \sqrt{x^2 + (y + bE)^2} - \sqrt{x^2 + (y - bE)^2} = 2b \quad (1)$$

Simplifying, we get equation of hyperbola as

$$\frac{x^2}{b^2(E^2 - 1)} - \frac{y^2}{b^2} = -1$$

$$\text{or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad [\text{Here, } a^2 = b^2(E^2 - 1)]$$

This hyperbola is called conjugate hyperbola of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

For any point $P(x, y)$ on the hyperbola, we have

$$PF_1 = E \times PM_1 = E \left(y - \frac{b}{E} \right) = Ey - b$$

$$\text{and } PF_2 = E \times PM_2 = E \left(\frac{b}{E} + y \right) = b + Ey$$

$$\therefore PF_2 - PF_1 = 2b$$

$$\text{The end points of latus rectum are } \left(\pm \frac{a^2}{b}, \pm bE \right).$$

$$\text{Also, length of latus rectum is } \frac{2a^2}{b}.$$

It can be observed that in all the aspects of conjugate hyperbola, except its equation, a and b are interchanged.

$$\text{Now, } \frac{1}{E^2} = \frac{b^2}{a^2 + b^2}$$

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\frac{1}{e^2} = \frac{a^2}{a^2 + b^2}$, where ' e ' is eccentricity.

$$\text{Therefore, } \frac{1}{e^2} + \frac{1}{E^2} = 1.$$

This is the relation between eccentricities of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and its conjugate hyperbola.}$$

Rectangular Hyperbola (Equilateral Hyperbola)

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $b^2 = a^2(e^2 - 1)$, where e is eccentricity.

$$\text{If } a = b, \text{ then } e^2 - 1 = 1$$

$$\text{or } e^2 = 2$$

$$\therefore e = \sqrt{2}$$

No, equation of hyperbola becomes $x^2 - y^2 = a^2$.

This hyperbola is called rectangular hyperbola or equilateral hyperbola.

Foci of this hyperbola are $(\pm\sqrt{2}a, 0)$ and two directrix are $x = \pm \frac{a}{\sqrt{2}}$.

Also, for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, we have $a^2 = b^2(E^2 - 1)$, where E is eccentricity.

If $a = b$, then $E = \sqrt{2}$ and equation of hyperbola becomes $x^2 - y^2 = -a^2$, which is rectangular hyperbola.

ILLUSTRATION 7.6

Find the standard equation of hyperbola in each of the following cases:

- Distance between the foci of hyperbola is 16 and its eccentricity is $\sqrt{2}$.
- Vertices of hyperbola are $(\pm 4, 0)$ and foci of hyperbola are $(\pm 6, 0)$.
- Foci of hyperbola are $(0, \pm\sqrt{10})$ and it passes through the point $(2, 3)$.
- Distances of one of the vertices of hyperbola from the foci are 3 and 1.

Sol.

- Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
Foci are $(\pm ae, 0)$.

$$\text{Distance between foci} = 2ae = 16 \text{ (given)}$$

$$\text{Also, } e = \sqrt{2} \text{ (given)}$$

$$\therefore a = 4\sqrt{2}$$

We know that,

$$b^2 = a^2(e^2 - 1)$$

$$\therefore b^2 = (4\sqrt{2})^2 [(\sqrt{2})^2 - 1] = 32$$

So, the equation of hyperbola is

$$\frac{x^2}{32} - \frac{y^2}{32} = 1$$

$$\text{or } x^2 - y^2 = 32$$

If the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, then its equation will be $x^2 - y^2 = -32$.

- Vertices are $(\pm 4, 0)$ and foci are $(\pm 6, 0)$.

$$\therefore a = 4 \text{ and } ae = 6$$

$$\Rightarrow e = \frac{6}{4} = \frac{3}{2}$$

$$\text{Now, } b^2 = a^2(e^2 - 1)$$

$$\therefore b^2 = 16 \left(\frac{9}{4} - 1 \right) = 20$$

$$\text{So, the equation of hyperbola is } \frac{x^2}{16} - \frac{y^2}{20} = 1.$$

- Since foci $(0, \pm\sqrt{10})$ are on y -axis, consider the equation of hyperbola as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

$$\therefore be = \sqrt{10}$$

$$\text{Also, } a^2 = b^2(e^2 - 1)$$

$$\therefore a^2 = b^2e^2 - b^2 = 10 - b^2$$

$\therefore$ Equation of the hyperbola becomes

$$\frac{x^2}{10 - b^2} - \frac{y^2}{b^2} = -1$$

Since, hyperbola passes through the point (2, 3), we have

$$\therefore \frac{4}{10 - b^2} - \frac{9}{b^2} = -1$$

$$\Rightarrow 4b^2 - 9(10 - b^2) = -b^2(10 - b^2)$$

$$\Rightarrow b^4 - 23b^2 + 90 = 0$$

$$\Rightarrow (b^2 - 18)(b^2 - 5) = 0$$

$$\Rightarrow b^2 = 5 \quad (b^2 = 18 \text{ not possible as } a^2 + b^2 = 10)$$

$$\Rightarrow a^2 = 10 - 5 = 5$$

So, the equation of hyperbola is

$$\frac{x^2}{5} - \frac{y^2}{5} = -1$$

$$\text{or } y^2 - x^2 = 5$$

(iv) According to the equation

$$ae - a = 1 \text{ and } ae + a = 3$$

$$\therefore \frac{e + 1}{e - 1} = 3 \text{ or } e = 2$$

$$\therefore a = 1$$

$$\text{So, } b^2 = a^2(e^2 - 1) = 3$$

Therefore, equation is $x^2 - \frac{y^2}{3} = 1$ or it can be $\frac{x^2}{3} - y^2 = -1$.

ILLUSTRATION 7.7

If ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ are confocal, then find the value of b^2 .

Sol. Given hyperbola is

$$\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$$

$$\text{or } \frac{x^2}{144/25} - \frac{y^2}{81/25} = 1$$

$$\text{So, foci of hyperbola are } \left(\pm \sqrt{\frac{144}{25} + \frac{81}{25}}, 0 \right) \equiv (\pm 3, 0).$$

These are foci of ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ also.

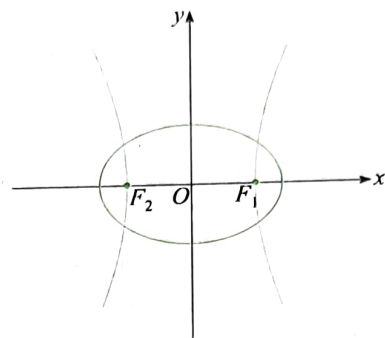
$$\therefore 3 = \sqrt{16 - b^2}$$

$$\therefore b^2 = 7$$

ILLUSTRATION 7.8

If hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the foci of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then find the eccentricities of ellipse and hyperbola.

Sol. Let e_1 and e_2 be the eccentricities of ellipse and hyperbola, respectively.



Since hyperbola passes through the foci of ellipse, we have

$$b = ae_1 \quad (1)$$

$$\text{Also for ellipse, } b^2 = a^2(1 - e_1^2) \quad (2)$$

$$\text{and for hyperbola, } a^2 = b^2(e_2^2 - 1) \quad (3)$$

From (1) and (2), we get

$$2e_1^2 = 1 \text{ or } e_1 = \frac{1}{\sqrt{2}}$$

So, from (1) and (3), we get

$$2 = e_2^2 - 1 \text{ or } e_2 = \sqrt{3}$$

ILLUSTRATION 7.9

Find the eccentricity of the hyperbola given by equations

$$x = \frac{e^t + e^{-t}}{2} \text{ and } y = \frac{e^t - e^{-t}}{3}, t \in R.$$

Sol. Given equations

$$x = \frac{e^t + e^{-t}}{2} \text{ and } y = \frac{e^t - e^{-t}}{3}$$

$$\text{or } 2x = e^t + e^{-t} \text{ and } 3y = e^t - e^{-t}$$

Squaring and subtracting, we get

$$4x^2 - 9y^2 = 4$$

$$\text{or } \frac{x^2}{1} - \frac{y^2}{4/9} = 1$$

$$\text{Now, } b^2 = a^2(e^2 - 1)$$

$$\therefore e^2 = \frac{4}{9} + 1 = \frac{13}{9}$$

$$\text{or } e = \frac{\sqrt{13}}{3}$$

ILLUSTRATION 7.10

An ellipse and a hyperbola have their principal axes along the coordinate axes and have common foci separated by a distance $2\sqrt{3}$. The difference of their focal semi-axes is equal to 4. If the ratio of their eccentricities is $3/7$, find the equation of these curves.

Sol. Let the semi-major axis, semi-minor axis, and eccentricity of the ellipse be a , b , and e , respectively, and those of hyperbola be a' , b' , and e' , respectively.

Given that $a - a' = 4$

$$2ae = 2\sqrt{13} \text{ or } ae = \sqrt{13}$$

$$2a'e' = 2\sqrt{13} \text{ or } a'e' = \sqrt{13}$$

and $\frac{e}{e'} = \frac{3}{7}$

From (2), $a = \frac{\sqrt{13}}{e}$

From (3), $a' = \frac{\sqrt{13}}{e'}$

Putting in (1), we get

$$\frac{\sqrt{13}}{e} - \frac{\sqrt{13}}{e'} = 4$$

or $\sqrt{13}(e' - e) = 4ee'$

Putting $e = 3e'/7$, we get

$$\sqrt{13}\left(e' - \frac{3}{7}e'\right) = 4 \times \frac{3}{7}(e')^2$$

$$\therefore e' = \frac{\sqrt{13}}{3} \text{ and } e = \frac{3}{7}e' = \frac{\sqrt{13}}{7}$$

$$a = \frac{\sqrt{13}}{e} = \frac{\sqrt{13}}{\frac{\sqrt{13}}{7}} \times 7 = 7$$

$$a' = \frac{\sqrt{13}}{e'} = \frac{\sqrt{13}}{\frac{\sqrt{13}}{3}} \times 3 = 3$$

$$b^2 = a^2(1 - e^2) = 49\left(1 - \frac{13}{49}\right) = 36$$

$$(b')^2 = (a')^2\{(e')^2 - 1\} = 9\left(\frac{13}{9} - 1\right) = 4$$

The equation of ellipse is

$$\frac{x^2}{49} + \frac{y^2}{36} = 1$$

and the equation of hyperbola is

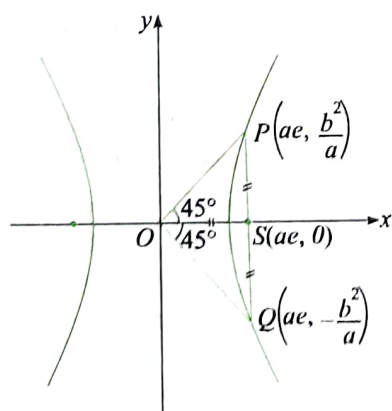
$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

ILLUSTRATION 7.11

If the latus rectum subtends a right angle at the centre of the

hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then find its eccentricity.

Sol.



- (1)
- (2)
- (3)
- (4)
- (5)
- (6)

In the figure, latus rectum PQ subtends right angle at the centre of the hyperbola.

$$\therefore OS = SP$$

$$\Rightarrow ae = \frac{b^2}{a}$$

$$\Rightarrow e = \frac{b^2}{a^2} = e^2 - 1$$

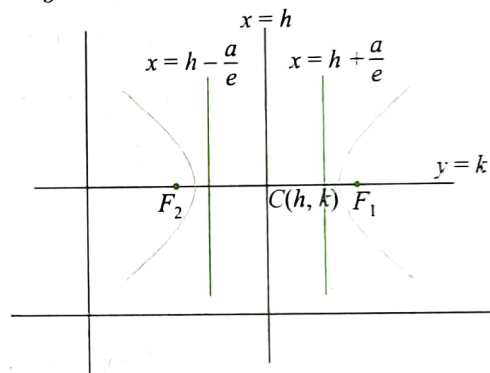
$$\Rightarrow e^2 - e - 1 = 0$$

$$\Rightarrow e = \frac{1 + \sqrt{5}}{2}$$

HYPERBOLA HAVING AXES PARALLEL TO COORDINATE AXES

If hyperbola having equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is shifted (translated) such that its centre is at (h, k) , then equation of hyperbola becomes

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$



Here, transverse axis is $y = k$ and conjugate axis is $x = h$.

So, centre of hyperbola is $C(h, k)$.

Eccentricity is given by relation $b^2 = a^2(e^2 - 1)$.

Foci are $F_1(h + ae, k)$ and $F_2(h - ae, k)$.

Equation of two directrix are given by $x = h \pm \frac{a}{e}$.

If the equation of hyperbola is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$, then transverse axis is $x = h$ and conjugate axis $y = k$.

Foci are $F_1(h, k + be)$ and $F_2(h, k - be)$.

Equation of two directrix are given by $y = k \pm \frac{b}{e}$.

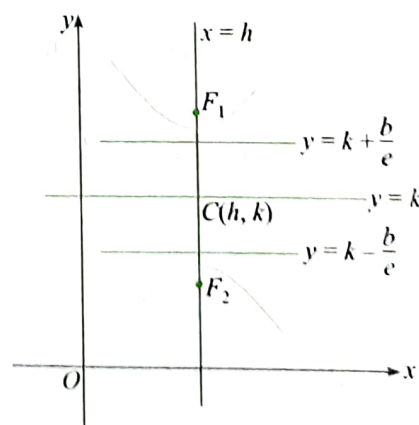


ILLUSTRATION 7.12

Find the equation of hyperbola in each of the following cases:

- Centre is (1, 0), one focus is (6, 0) and transverse axis 6
- Centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2)
- Centre is (-3, 2), one vertex is (-3, 4) and eccentricity is $5/2$
- Foci are (4, 2), (8, 2) and eccentricity is 2

Sol.

- (i) Centre is (1, 0), one focus is (6, 0).
So, transverse axis is along x-axis.

Therefore, equation of hyperbola is $\frac{(x-1)^2}{a^2} - \frac{y^2}{b^2} = 1$.

Given that $2a = 6$.

$$\therefore a = 3$$

Also, distance between centre and focus is 'ae'.

$$\therefore ae = 6 - 1 = 5$$

$$\text{So, } b^2 = a^2 e^2 - a^2 = 25 - 9 = 16$$

Therefore, equation of hyperbola is $\frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$.

- (ii) Centre is (3, 2) and one focus is (5, 2).
Thus, transverse axis is horizontal, along line $y = 2$.

Therefore, equation of hyperbola is $\frac{(x-3)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1$.

Distance between centre and focus is 'ae'.

$$\therefore ae = 5 - 3 = 2$$

Also, one vertex is (4, 2).

$$\therefore a = 4 - 3 = 1$$

$$\text{So, } b^2 = a^2 e^2 - a^2 = 4 - 1 = 3$$

Therefore, equation of hyperbola is $(x-3)^2 - \frac{(y-2)^2}{3} = 1$.

- (iii) Centre is (-3, 2) and one vertex is (-3, 4).
So, transverse axis is vertical, along line $x = -3$.

Thus, equation of hyperbola is $\frac{(x+3)^2}{a^2} - \frac{(y-2)^2}{b^2} = -1$.

Distance between centre and vertex is 'a'.

$$\therefore a = 4 - 2 = 2$$

Given that, $e = 5/2$

$$\therefore b^2 = a^2 e^2 - a^2 = 25 - 4 = 21$$

Therefore, equation of hyperbola is $\frac{(x+3)^2}{4} - \frac{(y-2)^2}{21} = -1$.

- (iv) Foci are (4, 2) and (8, 2).

So, transverse axis is horizontal, along line $y = 2$.

Also, centre is mid-point of foci, which is (6, 2).

Thus, equation of hyperbola is $\frac{(x-6)^2}{a^2} - \frac{(y-2)^2}{b^2} = 1$.

Distance between foci, $2ae = 4$.

Also, it is given that $e = 2$.

$$\therefore a = 1$$

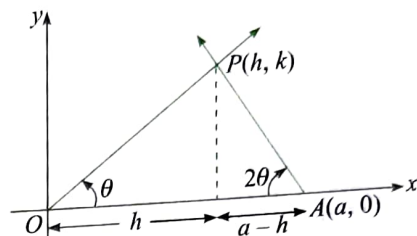
$$\text{So, } b^2 = a^2 e^2 - 1 = 4 - 1 = 3$$

Therefore, equation of hyperbola is $(x-6)^2 - \frac{(y-2)^2}{3} = 1$.

ILLUSTRATION 7.13

Two rods are rotating about two fixed points in opposite directions. If they start from their position of coincidence and one rotates at the rate double that of the other, then prove that the locus of point of the intersection of the two rods is hyperbola.

Sol. Suppose two rods are coincident on the x-axis. One rotates about point O and the other about point A(a, 0).



If they rotate according to the question, then at some time t , they are in the position as shown in the figure. From the figure,

$$\tan \theta = \frac{k}{h} \text{ and } \tan 2\theta = \frac{k}{a-h}$$

$$\text{or } \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{k}{a-h}$$

$$\text{or } \frac{2k/h}{1 - (k^2/h^2)} = \frac{k}{a-h}$$

$$\text{or } \frac{2hk}{h^2 - k^2} = \frac{k}{a-h}$$

$$\text{or } 2h(a-h) = h^2 - k^2$$

$$\text{or } 2ah - 2h^2 = h^2 - k^2$$

$$\text{or } 3x^2 - y^2 - 2ax = 0$$

Therefore, the locus is a hyperbola.

ILLUSTRATION 7.14

Find the coordinates of vertices, foci, eccentricity, latus-rectum and the equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Sol. We have hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

$$\text{or } 9(x^2 - 8x + 16) - 16(y^2 - 6y + 9) = 144$$

$$\text{or } \frac{(x-4)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1$$

Clearly, centre of the hyperbola is C(4, 3).

Transverse axis is $y - 3 = 0$ or $y = 3$ and conjugate axis is $x - 4 = 0$ or $x = 4$.

Also, semi transverse axis is $a = 4$ and semi conjugate axis is $b = 3$.

From $b^2 = a^2(e^2 - 1)$, we get eccentricity $e = 5/4$.

Vertices are at distance 'a' units from the centre.

So, vertices are $(4 - 4, 3) \equiv (0, 3)$ and $(4 + 4, 3) \equiv (8, 3)$.

Foci are at distance 'ae' from the centre.

Now, $ae = 5$.

So, foci are $(4 - 5, 3) \equiv (-1, 3)$ and $(4 + 5, 3) \equiv (9, 3)$.

Two directrices are at distance ' a/e ' from the conjugate axis.

Thus, directrices are given by

$$x - 4 = \pm a/e$$

$$x - 4 = \pm 16/5$$

$$5x - 36 = 0 \text{ and } 5x - 4 = 0$$

$$\text{Length of latus rectum} = 2b^2/a = 2 \times 9/4 = 9/2.$$

EQUATION OF HYPERBOLA REFERRED TO TWO PERPENDICULAR LINES AS AXES

Like ellipse, equation of hyperbola whose transverse axis is $L_2 \equiv b_1x - a_1y + c_2 = 0$ and conjugate axis is $L_1 \equiv a_1x + b_1y + c_1 = 0$ is given by

$$\frac{\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} - \frac{\left(\frac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1,$$

where $b^2 = a^2(e^2 - 1)$ in which e is eccentricity of hyperbola.

The centre of the hyperbola is the point of intersection of the lines L_1 and L_2 .

ILLUSTRATION 7.15

Find the centre, eccentricity, foci and equations of directrices

of the hyperbola $\frac{(3x - 4y - 12)^2}{100} - \frac{(4x + 3y - 12)^2}{225} = 1$.

Sol. We have

$$\frac{(3x - 4y - 12)^2}{100} - \frac{(4x + 3y - 12)^2}{225} = 1$$

$$\text{or } \frac{\left(\frac{3x - 4y - 12}{\sqrt{3^2 + (-4)^2}}\right)^2}{4} - \frac{\left(\frac{4x + 3y - 12}{\sqrt{4^2 + 3^2}}\right)^2}{9} = 1$$

Transverse axis is $4x + 3y - 12 = 0$ and conjugate axis is $3x - 4y - 12 = 0$.

Centre is intersection of axes of hyperbola, which is $\left(\frac{84}{25}, -\frac{12}{25}\right)$.
Here, $a = 2$ and $b = 3$.

$$\therefore e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{9}{4} = \frac{13}{4}$$

$$\therefore e = \frac{\sqrt{13}}{2}$$

$$\text{So, } ae = \sqrt{13}.$$

Now, slope of transverse axis is $-\frac{4}{3} = \tan \theta$, where θ is inclination of line with x -axis.

Foci lie on transverse axis at distance ' ae ' from the centre.

$$\text{Therefore, foci are } \left(\frac{84}{25} \pm \sqrt{13} \cos \theta, -\frac{12}{25} \pm \sqrt{13} \sin \theta\right)$$

$$\equiv \left(\frac{84}{25} \mp \sqrt{13} \times \frac{3}{5}, -\frac{12}{25} \pm \sqrt{13} \times \frac{4}{5}\right)$$

$$\equiv \left(\frac{84 \mp 15\sqrt{13}}{25}, \frac{-12 \pm 20\sqrt{13}}{25}\right)$$

Directrix lie at distance $\frac{a}{e}$ from the conjugate axis.

$$\text{Now, } \frac{a}{e} = \frac{2}{\sqrt{13}/2} = \frac{4}{\sqrt{13}}.$$

Thus, distance between conjugate axis and directrix is $\frac{4}{\sqrt{13}}$.

So, equations of directrices are given by $3x - 4y + c = 0$, where

$$\frac{|c + 12|}{\sqrt{3^2 + (-4)^2}} = \frac{4}{\sqrt{13}}.$$

$$\therefore c = -12 \pm \frac{20}{\sqrt{13}}$$

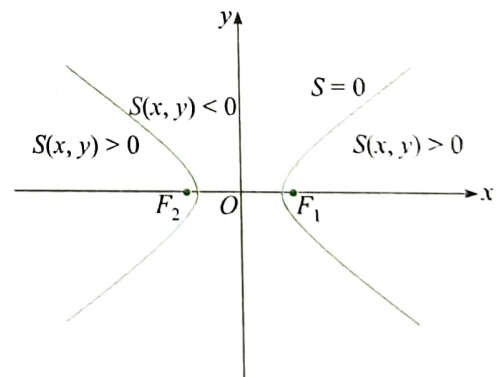
So, equations of directrices are $3x - 4y - 12 \pm \frac{20}{\sqrt{13}} = 0$.

DIVISION OF PLANE BY HYPERBOLA

Consider standard equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$.

$$\text{Let } S(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1.$$

Now, graph of hyperbola divides plane in two regions, one which contains centre (origin) and other which contains foci.



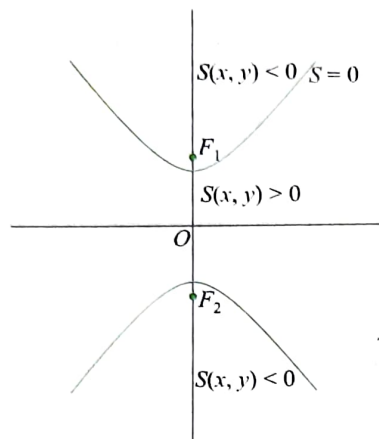
For origin, $S(0, 0) = 0 - 0 - 1 < 0$.

So for all the points lying in the region where origin lies, $S(x, y) < 0$.

Therefore, for all the points lying in the region where foci lie, $S(x, y) > 0$.

For hyperbola having equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$, $S(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1$.

For origin, $S(0, 0) = 0 - 0 + 1 > 0$.



So for all the points lying in the region where origin lies, $S(x, y) > 0$.

Therefore, for all the points lying in the region where foci lie, $S(x, y) < 0$.

ILLUSTRATION 7.16

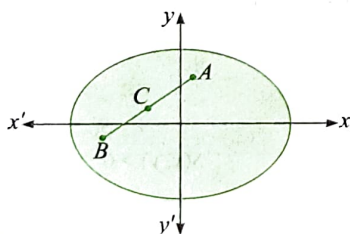
Each of the four inequalities given below defines a region in the xy plane. One of these four regions does not have the following property. For any two points (x_1, y_1) and (x_2, y_2) in the region, the point $((x_1 + x_2)/2, (y_1 + y_2)/2)$ is also in the region. The inequality defining this region is

- (1) $x^2 + 2y^2 \leq 1$ (2) $\max\{|x|, |y|\} \leq 1$
 (3) $x^2 - y^2 \leq 1$ (4) $y^2 - x \leq 0$

Sol. We have given points $A(x_1, y_1)$ and $B(x_2, y_2)$ and its midpoint is $C\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

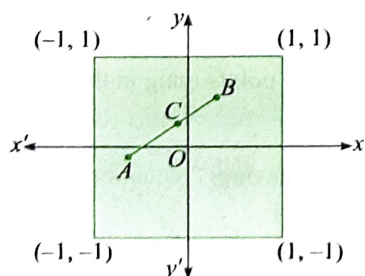
- (1) $x^2 + 2y^2 \leq 1$ represents interior region of ellipse $x^2 + 2y^2 = 1$ as shown in the given figure.

Clearly, for any two points A and B in the shaded region, mid point of AB also lies in the shaded region.



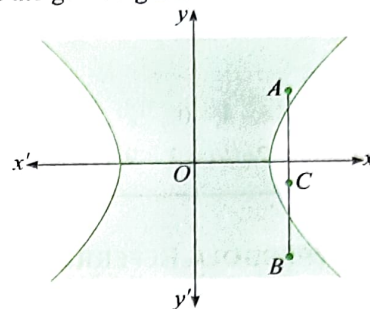
- (2) $\max\{|x|, |y|\} \leq 1$
 $\Rightarrow |x| \leq 1$ and $|y| \leq 1$
 $\Rightarrow -1 \leq x \leq 1$ and $-1 \leq y \leq 1$

This represents the interior region of a square with its sides $x = \pm 1$ and $y = \pm 1$ as shown in the given figure.



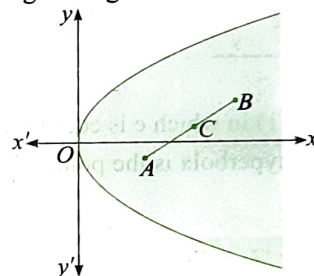
Clearly, for any two points A and B in the shaded region mid point of AB also lies in the shaded region

- (3) $x^2 - y^2 \leq 1$ represents the exterior region of hyperbola as shown in the given figure.



As shown in the given figure for points A and B selected in the region, the midpoint does not necessarily lie in the same region.

- (4) $y^2 \leq x$ represents interior region of the parabola $y^2 = x$ as shown in the given figure.



Clearly, for any two points A and B in the shaded region, mid point of AB also lies in the shaded region.

IMPORTANT PROPERTY OF FOCAL CHORD

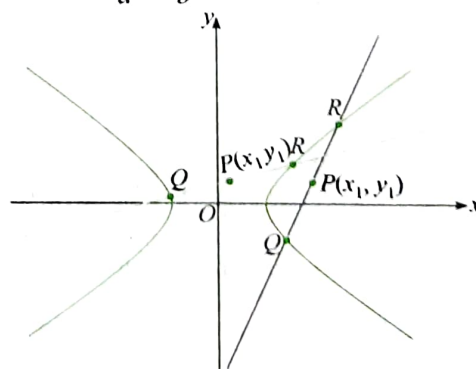
We know that chord of a hyperbola passing through its focus is called focal chord. Consider focal chord PQ of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ through one of its foci S . This chord has two segments SP and SQ . The harmonic mean of SP and SQ is equal to semi latus rectum of the hyperbola.

$$\therefore \frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$$

This can be proved in the same way as we did in the previous Chapter **Ellipse**.

EQUATION OF CHORD OF HYPERBOLA BISECTED AT GIVEN POINT

Consider hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



Equation of chord of hyperbola which is bisected at point $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

or $T = S_1$, where $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$ and $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$.

In the figure, chord QR is bisected at point P , i.e., P is midpoint of QR .

This equation can be determined in the same way as we did in previous Chapter **Ellipse**.

Equation of chord of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ which is bisected at point $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} + 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} + 1$$

In general, to get expression T , we replace x^2 by xx_1 , y^2 by yy_1 , $2x$ by $(x + x_1)$ and $2y$ by $(y + y_1)$ in the equation of conic.

ILLUSTRATION 7.17

Find the locus of the midpoints of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$.

Sol. Using $T = S_1$, the equation of chord whose midpoint is (h, k) is

$$3xh - 2yk + 2(x + h) - 3(y + k) = 3h^2 - 2k^2 + 4h - 6k$$

or $x(3h+2) - y(2k+3) + \dots = 0$

Its slope is $\frac{3h+2}{2k+3} = 2$ as it is parallel to $y = 2x$.

$$\Rightarrow 3h - 4k = 4$$

$$\Rightarrow 3x - 4y = 4, \text{ which is required locus.}$$

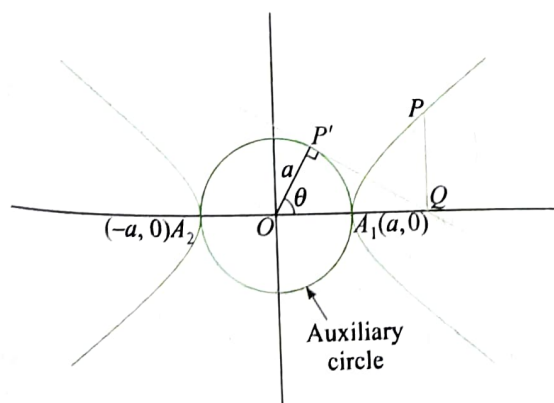
PARAMETRIC FORM OF HYPERBOLA

To get the parametric form of hyperbola, we first define auxiliary circle of the hyperbola.

Auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the circle

described on transverse axis as a diameter, i.e., circle for which vertices of hyperbola are the end points of diameter.

Clearly, equation of this circle is $x^2 + y^2 = a^2$.



From point $P(x, y)$ on the hyperbola, we draw a line segment PQ perpendicular to transverse axis. From point Q , we draw tangent to auxiliary circle touching it at P' . If the inclination of radius OP' with x -axis is θ , coordinates of point P' are $(a \cos \theta, a \sin \theta)$.

Now, in triangle $OP'Q$, $OQ = a \sec \theta$, which is abscissa of point P .

Putting $x = a \sec \theta$ in the equation of hyperbola, we get $y = b \tan \theta$.

Thus, parametric form of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$x = a \sec \theta, y = b \tan \theta,$$

where $\theta \in [0, 2\pi) - \{\pi/2, 3\pi/2\}$.

So, variable point on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be taken as

$P(a \sec \theta, b \tan \theta)$. This point is also represented as $P(\theta)$. Here, θ is called eccentric angle of point P .

Since hyperbola is unbounded curve, we get unbounded functions like $\sec \theta$ and $\tan \theta$ in its parametric form.

Points P and P' are called corresponding points on the hyperbola and auxiliary circle, respectively.

Here, we have to be careful about quadrants of points P and P' .

We have following table for quadrants in which point P' of auxiliary circle and its corresponding point P on the hyperbola lie.

Quadrant of angle θ	Quadrant of point $P'(a \cos \theta, a \sin \theta)$	Quadrant of point $P(a \sec \theta, b \tan \theta)$
I	I	I
II	II	III
III	III	II
IV	IV	IV

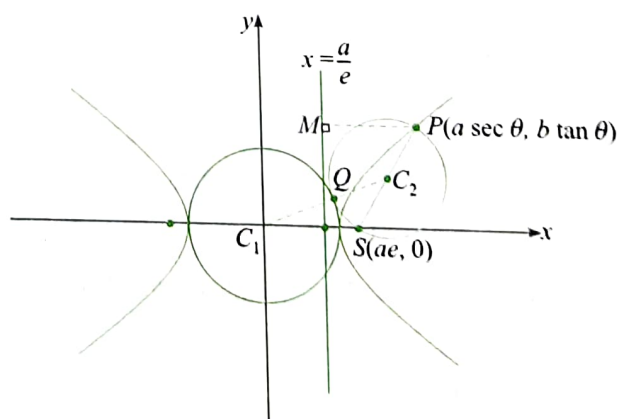
For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, standard parametric form is taken as $x = a \tan \theta$ and $y = b \sec \theta$.

IMPORTANT PROPERTY OF AUXILIARY CIRCLE

Circle described on focal length as diameter always touches the auxiliary circle.

Proof:

Consider hyperbola having equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



Let the coordinates of point P on the hyperbola be $(a \sec \theta, b \tan \theta)$.

From the definition of the hyperbola, we have

$$SP = e \cdot PM = e \left(a \sec \theta - \frac{a}{e} \right) \\ = a(e \sec \theta - 1)$$

Equation of auxiliary circle is $x^2 + y^2 = a^2$, having centre $C_1(0, 0)$ and radius $C_1Q = r_1 = a$.

Circle described on SP as a diameter has centre C_2

$$\left(\frac{ae + a \sec \theta}{2}, \frac{b \tan \theta}{2} \right) \text{ and radius, } C_2Q = r_2 = \frac{a(e \sec \theta - 1)}{2}.$$

$$\text{Now, } r_1 + r_2 = \frac{a(1 + e \sec \theta)}{2} = C_1Q + C_2Q$$

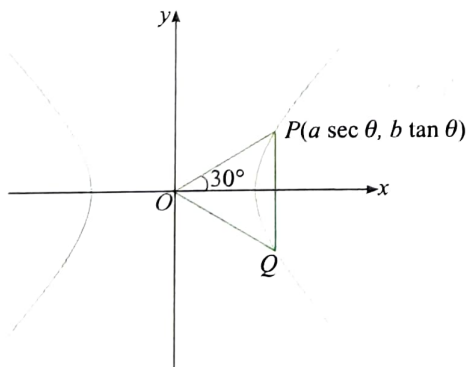
$$\begin{aligned} \text{and } C_1C_2 &= \sqrt{\frac{a^2(e + \sec \theta)^2}{4} + \frac{b^2 \tan^2 \theta}{4}} \\ &= \frac{a}{2} \sqrt{e^2 + 2e \sec \theta + \sec^2 \theta + \frac{b^2}{a^2} \tan^2 \theta} \\ &= \frac{a}{2} \sqrt{e^2 + 2e \sec \theta + \sec^2 \theta + (e^2 - 1) \tan^2 \theta} \\ &= \frac{a}{2} \sqrt{e^2 \sec^2 \theta + 2e \sec \theta + 1} \\ &= \frac{a}{2} (e \sec \theta + 1) \\ &= r_1 + r_2 \\ &= C_1Q + C_2Q \end{aligned}$$

Hence, circle on SP as a diameter touches the auxiliary circle externally.

ILLUSTRATION 7.18

If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then find range of the eccentricity (e) of the hyperbola.

Sol.



Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and any double ordinate PQ be such that $P \equiv (a \sec \theta, b \tan \theta)$.

$$\therefore Q \equiv (a \sec \theta, -b \tan \theta)$$

According to the question, triangle OPQ is equilateral.

$$\therefore \tan 30^\circ = \frac{b \tan \theta}{a \sec \theta}$$

$$\Rightarrow 3 \frac{b^2}{a^2} = \operatorname{cosec}^2 \theta$$

$$\Rightarrow 3(e^2 - 1) = \operatorname{cosec}^2 \theta$$

Now, $\operatorname{cosec}^2 \theta \geq 1$

$$\Rightarrow 3(e^2 - 1) \geq 1$$

$$\Rightarrow e^2 \geq \frac{4}{3}$$

$$\Rightarrow e \geq \frac{2}{\sqrt{3}}$$

EQUATION OF CHORD OF HYPERBOLA

Consider hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation of the chord passing through the points $P(a \sec \alpha, b \tan \alpha)$ and $Q(a \sec \beta, b \tan \beta)$ is

$$\begin{aligned} y - b \tan \alpha &= \frac{b \tan \beta - b \tan \alpha}{a \sec \beta - a \sec \alpha} (x - a \sec \alpha) \\ \Rightarrow b(\tan \beta - \tan \alpha)x - ay(\sec \beta - \sec \alpha) &= ab(\tan \beta \sec \alpha - \tan \alpha \sec \beta) \\ \Rightarrow \frac{x}{a} \sin(\beta - \alpha) - \frac{y}{b} (\cos \alpha - \cos \beta) &= \sin \beta - \sin \alpha \\ \Rightarrow 2 \frac{x}{a} \sin \frac{\beta - \alpha}{2} \cos \frac{\beta + \alpha}{2} - 2 \frac{y}{b} \sin \frac{\beta - \alpha}{2} \sin \frac{\beta + \alpha}{2} &= 2 \sin \frac{\beta - \alpha}{2} \cos \frac{\beta + \alpha}{2} \\ \Rightarrow \frac{x}{a} \cos \frac{\alpha - \beta}{2} - \frac{y}{b} \sin \frac{\alpha + \beta}{2} &= \cos \frac{\alpha + \beta}{2} \end{aligned}$$

ILLUSTRATION 7.19

If $(a \sec \theta, b \tan \theta)$ and $(a \sec \phi, b \tan \phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that $\tan \frac{\theta}{2} \tan \frac{\phi}{2} = \frac{1 - e}{1 + e}$.

Sol. The equation of the chord joining $(a \sec \theta, b \tan \theta)$ and $(a \sec \phi, b \tan \phi)$ is

$$\frac{x}{a} \cos \left(\frac{\theta - \phi}{2} \right) - \frac{y}{b} \sin \left(\frac{\theta + \phi}{2} \right) = \cos \left(\frac{\theta + \phi}{2} \right)$$

This passes through $(ae, 0)$. Therefore,

$$e \cos \left(\frac{\theta - \phi}{2} \right) = \cos \left(\frac{\theta + \phi}{2} \right)$$

$$\text{or } e = \frac{\cos \left(\frac{\theta + \phi}{2} \right)}{\cos \left(\frac{\theta - \phi}{2} \right)}$$

$$\text{or } \frac{e - 1}{e + 1} = \frac{\cos \left(\frac{\theta + \phi}{2} \right) - \cos \left(\frac{\theta - \phi}{2} \right)}{\cos \left(\frac{\theta + \phi}{2} \right) + \cos \left(\frac{\theta - \phi}{2} \right)}$$

$$\text{or } \frac{e - 1}{e + 1} = -\tan \frac{\theta}{2} \tan \frac{\phi}{2}$$

$$\text{or } \tan \frac{\theta}{2} \tan \frac{\phi}{2} = \frac{1 - e}{1 + e}$$

CONCEPT APPLICATION EXERCISE 7.2

- Find all the aspects of hyperbola $16x^2 - 9y^2 = -144$.
- If the latus rectum of a hyperbola forms an equilateral triangle with the vertex at the center of the hyperbola, then find the eccentricity of the hyperbola.
- The distance between two directrices of a rectangular hyperbola is 10 units. Find the distance between its foci.
- An ellipse and a hyperbola are confocal (have the same focus) and the conjugate axis of the hyperbola is equal to the minor axis of the ellipse. If e_1 and e_2 are the eccentricities of the ellipse and the hyperbola, respectively, then prove that $\frac{1}{e_1} + \frac{1}{e_2} = 2$.
- If S and S' are the foci, C is the center, and P is a point on a rectangular hyperbola, show that $SP \times S'P = (CP)^2$.
- Find the equation of the hyperbola whose foci are $(8, 3)$ and $(0, 3)$ and eccentricity is $4/3$.
- Find all the aspects of hyperbola $16x^2 - 3y^2 - 32x + 12y - 44 = 0$.
- Show that the locus represented by $x = \frac{1}{2}a(t + \frac{1}{t})$, $y = \frac{1}{2}a(t - \frac{1}{t})$ is a rectangular hyperbola.
- Two straight lines pass through the fixed points $(\pm a, 0)$ and have slopes whose product is $p > 0$. Show that the locus of the points of intersection of the lines is a hyperbola.
- If AOB and COD are two straight lines which bisect one another at right angles, show that the locus of a point P which moves so that $PA \times PB = PC \times PD$ is a hyperbola. Find its eccentricity.
- Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at the point $(5, 3)$.
- Foot of perpendicular from point P on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its transverse axis is N . If A and A' are vertices of hyperbola and Q divides AP in the ratio $a^2 : b^2$, then prove that NQ is perpendicular to $A'P$. Vertex A is near to point P .

ANSWERS

- $e = 5/4$, Vertices $\equiv (0, \pm 4)$, Foci $\equiv (0, \pm 5)$, L.R. = $9/2$, Directrix: $y = \pm 16/5$
- $\frac{1+\sqrt{13}}{2\sqrt{3}}$ 3. 20 6. $\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1$
- $e = \sqrt{19/3}$, Vertices $\equiv (1 \pm \sqrt{3}, 2)$, Foci $\equiv (1 \pm \sqrt{19}, 2)$, Directrix: $x - 1 = \pm 3/\sqrt{19}$
- $e = \sqrt{2}$ 11. $125x - 48y = 481$

TANGENT TO HYPERBOLA

A LINE AND A HYPERBOLA

A line may meet the hyperbola in one point or two distinct points or it may not meet the hyperbola at all.

Consider the standard equation of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (1)$$

and the line having equation

$$y = mx + c \quad (2)$$

Solving line and hyperbola, we get

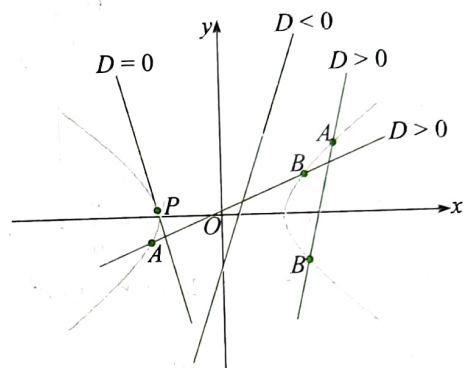
$$b^2x^2 - a^2(mx + c)^2 = a^2b^2$$

$$\text{or } (b^2 - a^2m^2)x^2 - 2a^2cmx - (a^2c^2 + a^2b^2) = 0 \quad (3)$$

Discriminant of this equation is

$$\begin{aligned} D &= 4a^4c^2m^2 + 4(a^2c^2 + a^2b^2)(b^2 - a^2m^2) \\ &= 4a^2b^2(c^2 + b^2 - a^2m^2) \end{aligned}$$

Now, this equation has either two distinct real roots or two equal roots or has non-real roots.



If line meets the hyperbola in two distinct points (A and B), then equation (3) has two distinct real roots.

$$\therefore D > 0 \text{ or } c^2 > a^2m^2 - b^2$$

If line meets the hyperbola in one point (P), i.e., if it touches the hyperbola, then equation (3) has two equal roots.

$$\therefore D = 0 \text{ or } c^2 = a^2m^2 - b^2$$

If line doesn't meet the hyperbola, then equation (3) has imaginary roots.

$$\therefore D < 0 \text{ or } c^2 < a^2m^2 - b^2$$

ILLUSTRATION 7.20

Find the point on the hyperbola $x^2 - 9y^2 = 9$ where the line $5x + 12y = 9$ touches it.

Sol. Solving $5x + 12y = 9$ or $y = \frac{9-5x}{12}$ and $x^2 - 9y^2 = 9$, we have

$$x^2 - 9\left(\frac{9-5x}{12}\right)^2 = 9$$

$$\text{or } x^2 - \frac{1}{16}(9-5x)^2 = 9$$

$$\text{or } 16x^2 - (25x^2 - 90x + 81) = 144$$

$$\text{or } 9x^2 - 90x + 225 = 0$$

$$\text{or } x^2 - 10x + 25 = 0$$

$$\text{or } x = 5$$

$$\Rightarrow y = \frac{9-25}{12} = -\frac{4}{3}$$

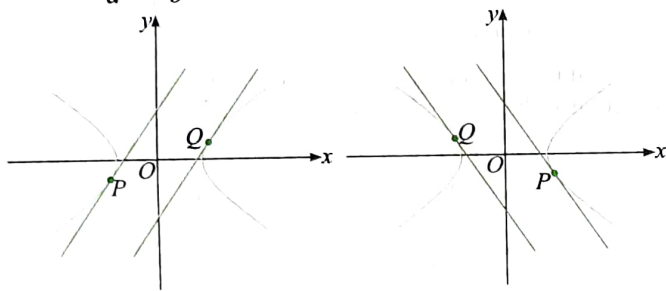
So, point of contact is $\left(5, -\frac{4}{3}\right)$.

EQUATION OF TANGENT TO HYPERBOLA HAVING GIVEN SLOPE

We have already learned that if line $y = mx + c$ meets the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in one point, then $c^2 = a^2m^2 - b^2$. In that case, equation of line is

$$y = mx \pm \sqrt{a^2m^2 - b^2}, m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right) \quad (4)$$

These equations are equations of two parallel tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slope m .



Solving $y = mx + c$ and hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

For equation $(b^2 - a^2m^2)x^2 - 2a^2cmx - (a^2c^2 + a^2b^2) = 0$, if $D = 0$, then we have

$$x = \frac{a^2cm}{b^2 - a^2m^2} = \frac{a^2cm}{-c^2} = \frac{-a^2m}{c}$$

$$\therefore y = m\left(\frac{-a^2m}{c}\right) + c = \frac{-a^2m^2 + c^2}{c} = \frac{-b^2}{c}$$

Therefore, line $y = mx + c$ touches hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$.

For tangent $y = mx + \sqrt{a^2m^2 - b^2}$, point of contact is $P\left(\frac{-a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{-b^2}{\sqrt{a^2m^2 - b^2}}\right)$.

For tangent $y = mx - \sqrt{a^2m^2 - b^2}$, point of contact is $Q\left(\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$.

ILLUSTRATION 7.21

Find the value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$.

Sol. If $y = mx + c$ touches $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 - b^2$.

Here, $c = 6$, $a^2 = 100$, and $b^2 = 49$. Therefore,

$$36 = 100m^2 - 49$$

$$\text{or } 100m^2 = 85$$

$$\text{or } m = \sqrt{\frac{17}{20}}$$

ILLUSTRATION 7.22

Find the equations of tangents to the curve $4x^2 - 9y^2 = 1$ which are parallel to $4y = 5x + 7$.

Sol. Let m be the slope of the tangent to $4x^2 - 9y^2 = 1$. Then $m = (\text{Slope of the line } 4y = 5x + 7) = \frac{5}{4}$

$$\text{We have } \frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$$

$$\text{where } a^2 = \frac{1}{4}, b^2 = \frac{1}{9}$$

The equations of the tangents are

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\text{or } y = \frac{5}{4}x \pm \sqrt{\frac{25}{64} - \frac{1}{9}}$$

$$\text{or } 30x - 24y \pm \sqrt{161} = 0$$

$$\text{or } 24y - 30x = \pm\sqrt{161}$$

ILLUSTRATION 7.23

If it is possible to draw the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slope 2, then find its range of eccentricity.

Sol. For the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the tangent having slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

The tangent having slope 2 is $y = 2x \pm \sqrt{4a^2 - b^2}$, which is real if

$$4a^2 - b^2 \geq 0$$

$$\text{or } \frac{b^2}{a^2} \leq 4$$

$$\text{or } e^2 - 1 \leq 4$$

$$\text{or } e^2 \leq 5$$

$$\text{or } 1 < e < \sqrt{5}$$

ILLUSTRATION 7.24

Find the equation of tangents to hyperbola $x^2 - y^2 - 4x - 2y = 0$ having slope 2.

Sol. We have hyperbola

$$x^2 - y^2 - 4x - 2y = 0$$

$$\text{or } (x-2)^2 - (y+1)^2 = 3$$

$$\text{or } \frac{(x-2)^2}{3} - \frac{(y+1)^2}{3} = 1 \quad (1)$$

Equation of tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slope m is given by

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

So, equation of tangents to hyperbola (1) having slope 2 is given by

$$y+1 = 2(x-2) \pm \sqrt{3 \times 4 - 3}$$

$$\text{or } y+1 = 2x-4 \pm 3$$

Therefore, the equations of tangents to given hyperbola are $2x - y - 2 = 0$ and $2x - y - 8 = 0$.

ILLUSTRATION 7.25

Find the minimum value of $(2 - a - 4 \sec \theta)^2 + (a - 3 \tan \theta)^2$,
 $a \in \mathbb{R}$.

Sol. We have $f(\theta) = (2 - a - 4 \sec \theta)^2 + (a - 3 \tan \theta)^2$
 This is square of the distance between the points $P(2 - a, a)$ and
 $Q(4 \sec \theta, 3 \tan \theta)$.

Point P lies on the line

$$x + y - 2 = 0 \quad (1)$$

and Q lies on hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

So, minimum value of $f(\theta)$ is square of the distance between the
 line (1) and the tangent to hyperbola at point P which is parallel
 to the line (1).

Equations of tangents to $\frac{x^2}{16} - \frac{y^2}{9} = 1$ having slope -1 are given by

$$y = -x \pm \sqrt{7}$$

$$\text{or } x + y + \sqrt{7} = 0 \quad (2)$$

$$\text{and } x + y - \sqrt{7} = 0 \quad (3)$$

Distance between lines (1) and (2) is $\frac{\sqrt{7} + 2}{\sqrt{2}}$.

Distance between lines (1) and (3) is $\frac{\sqrt{7} - 2}{\sqrt{2}}$.

So, required minimum distance is $\frac{\sqrt{7} - 2}{\sqrt{2}}$.

$$\therefore f(\theta) = \left(\frac{\sqrt{7} - 2}{\sqrt{2}} \right)^2$$

ILLUSTRATION 7.26

Find the locus of the midpoints of the chords of the circle
 $x^2 + y^2 = 16$, which are tangent to the hyperbola $9x^2 - 16y^2 = 144$.

Sol. Equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Let (h, k) be the midpoint of the chord of the circle $x^2 + y^2 = 16$.

So, the equation of the chord will be

$$hx + ky = h^2 + k^2$$

$$\Rightarrow y = \underbrace{\frac{-h}{k}}_m x + \underbrace{\frac{h^2 + k^2}{k}}_c$$

This will touch the hyperbola if

$$c^2 = a^2 m^2 - b^2$$

$$\Rightarrow \left(\frac{h^2 + k^2}{k} \right)^2 = 16 \left(\frac{-h}{k} \right)^2 - 9$$

$$\Rightarrow (h^2 + k^2)^2 = 16h^2 - 9k^2$$

Therefore, required locus is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.

EQUATION OF TANGENT TO HYPERBOLA AT A POINT

ON IT

We know that line $y = mx + c$ touches hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at

$$\left(\frac{-a^2 m}{c}, \frac{-b^2}{c} \right).$$

Let $\frac{-a^2 m}{c} = x_1$ and $\frac{-b^2}{c} = y_1$.

So, equation of tangent $y = mx + c$ becomes

$$y = \frac{b^2 x_1}{a^2 y_1} x - \frac{b^2}{y_1}$$

$$\text{or } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

$$\text{or } T = 0, \text{ where } T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$$

This is the equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at
 point (x_1, y_1) on it.

The slope of the tangent can also be obtained using
 differentiation.

Differentiating equation of hyperbola w.r.t. x , we get

$$\frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$\therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$ = Slope of tangent to hyperbola at any point
 on it

$$\therefore \left(\frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{b^2 x_1}{a^2 y_1}$$

So, equation of tangent at point (x_1, y_1) on the hyperbola
 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y - y_1 = \frac{b^2 x_1}{a^2 y_1} (x - x_1)$$

Simplifying, we get equation as

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

If $(x_1, y_1) \equiv (a \sec \theta, b \tan \theta)$, then equation of tangent becomes

$$\frac{x(a \sec \theta)}{a^2} - \frac{y(b \tan \theta)}{b^2} - 1 = 0$$

$$\text{or } \frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta - 1 = 0$$

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ equation of tangent at point

$$P(x_1, y_1) \text{ on is given by } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} + 1 = 0.$$

Equation of tangent at point $P(a \tan \theta, b \sec \theta)$ on it is

$$\frac{x}{a} \tan \theta - \frac{y}{b} \sec \theta + 1 = 0.$$

ILLUSTRATION 7.27

Find the equation of the tangent to the conic $x^2 - y^2 - 8x + 2y + 11 = 0$ at $(2, 1)$.

Sol. The equation of the tangent to $x^2 - y^2 - 8x + 2y + 11 = 0$ at $(2, 1)$ is

$$2x - y - 4(x + 2) + (y + 1) + 11 = 0$$

or $x = 2$

ILLUSTRATION 7.2B

A tangent to the hyperbola $x^2 - 2y^2 = 4$ meets x -axis at P and y -axis at Q . Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where O is origin). Find the locus of R .

Sol. Equation of tangent at point $(a \sec \theta, b \tan \theta)$ is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

It meets axis at $P(a \cos \theta, 0)$ and $Q(0, -b \cot \theta)$.

Now, rectangle $OPRQ$ is completed.

Let the coordinates of point R be (h, k) .

$$\therefore h = a \cos \theta \text{ and } k = -b \cot \theta$$

$$\therefore \sec \theta = \frac{a}{h} \text{ and } \tan \theta = \frac{-b}{k}$$

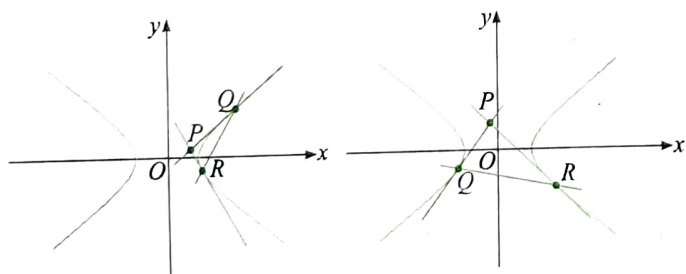
Squaring and subtracting, we get $\frac{a^2}{h^2} - \frac{b^2}{k^2} = 1$.

So, required locus is $\frac{4}{x^2} - \frac{2}{y^2} = 1$.

TANGENTS TO HYPERBOLA FROM A POINT NOT LYING ON IT AND CHORD OF CONTACT

Consider hyperbola having equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Two tangents can be drawn to this hyperbola from a point $P(x_1, y_1)$ if it lies in the region where points satisfy $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 < 0$.



To get the slope of the tangents, we consider one of the equations of tangents to hyperbola having slope m .

Thus, tangent having equation $y = mx + \sqrt{a^2 m^2 - b^2}$ passes through $P(x_1, y_1)$, then

$$y_1 = mx_1 + \sqrt{a^2 m^2 - b^2}$$

$$\text{or } (y_1 - mx_1)^2 = a^2 m^2 - b^2$$

Solving this quadratic equation, we get two values of slopes of tangents PQ and PR which pass through point P .

The line joining points of contact Q and R is called chord of contact.

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, equation of chord of contact w.r.t. point $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$$

$$\text{or } T = 0$$

Equation of Pair of Tangents from External Point

Equation of pair of tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ from a point $P(x_1, y_1)$ not lying on it is given by

$$\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 \right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \right) \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \right)$$

$$\text{or } T^2 = SS_1$$

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, two tangents can be drawn from a point $P(x_1, y_1)$ if it lies in the region where points satisfy

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 > 0.$$

ILLUSTRATION 7.29

Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from $(3, 2)$.

Sol. The equation of the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{1} = 1$$

The equation of the tangent having slope m is

$$y = mx \pm \sqrt{9m^2 - 1} \quad (1)$$

It passes through $(3, 2)$. Therefore,

$$2 = 3m \pm \sqrt{9m^2 - 1}$$

$$\text{or } 4 + 9m^2 - 12m = 9m^2 - 1$$

$$\text{i.e., } m = \frac{5}{12} \text{ or } m = \infty$$

Hence, the equations of the tangents are $y - 3 = \frac{5}{12}(x - 2)$ and $x = 3$.

ILLUSTRATION 7.30

Find the equation of pair of tangents drawn from point $(4, 3)$ to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Also, find the angle between the tangents.

Sol. Equation of pair of tangents is $T^2 = SS_1$.

$$\therefore \left(\frac{4x}{16} - \frac{3y}{9} - 1 \right)^2 = \left(\frac{x^2}{16} - \frac{y^2}{9} - 1 \right) (-1)$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} + 1 - \frac{xy}{6} - \frac{x}{2} + \frac{2y}{3} = -\frac{x^2}{16} + \frac{y^2}{9} + 1$$

$$\Rightarrow \frac{x^2}{8} - \frac{xy}{6} - \frac{x}{2} + \frac{2y}{3} = 0$$

$\Rightarrow 3x^2 - 4xy - 12x + 16y = 0$, which is required equation of pair of tangents.

Comparing it with standard second-degree equation, we have

$$a = 3, b = 0 \text{ and } h = -2$$

$\therefore$ Angle between tangents,

$$\begin{aligned}\theta &= \tan^{-1} \frac{2\sqrt{h^2 - ab}}{|a + b|} \\ &= \tan^{-1} \frac{2\sqrt{(-2)^2 - (3)(0)}}{|3 + 0|} \\ &= \tan^{-1} \frac{4}{3}\end{aligned}$$

ILLUSTRATION 7.31

Tangents drawn from the point (c, d) to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ make angles } \alpha \text{ and } \beta \text{ with the } x\text{-axis.}$$

If $\tan \alpha \tan \beta = 1$, then find the value of $c^2 - d^2$.

Sol. One of the equations of tangents to the hyperbola having slope m is $y = mx + \sqrt{a^2 m^2 - b^2}$. It passes through (c, d) . So,

$$d = mc + \sqrt{a^2 m^2 - b^2}$$

$$\text{or } (d - mc)^2 = a^2 m^2 - b^2$$

$$\text{or } (c^2 - a^2)m^2 - 2cdm + d^2 + b^2 = 0$$

$$\text{or Product of roots} = m_1 m_2 = \frac{d^2 + b^2}{c^2 - a^2}$$

$$\text{or } \tan \alpha \tan \beta = \frac{d^2 + b^2}{c^2 - a^2} = 1$$

$$\text{or } d^2 + b^2 = c^2 - a^2$$

$$\text{or } c^2 - d^2 = a^2 + b^2$$

DIRECTOR CIRCLE

Like ellipse in hyperbola also, locus of point of intersection of perpendicular tangents is circle which is called director circle.

Consider hyperbola having equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$.

Let perpendicular tangents to hyperbola intersect at point $P(h, k)$.

So, equation of pair of tangents through point P is

$$\left(\frac{hx}{a^2} - \frac{ky}{b^2} + 1 \right)^2 = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 \right) \left(\frac{h^2}{a^2} - \frac{k^2}{b^2} + 1 \right)$$

Since tangents are perpendicular, in above equation, sum of coefficients of x^2 and y^2 is 0.

$$\therefore \frac{1}{a^2} \left(\frac{h^2}{a^2} - \frac{k^2}{b^2} + 1 \right) - \frac{1}{b^2} \left(\frac{h^2}{a^2} - \frac{k^2}{b^2} + 1 \right) - \frac{h^2}{a^4} - \frac{k^2}{b^4} = 0$$

$$\Rightarrow -\frac{k^2}{a^2 b^2} - \frac{1}{a^2} - \frac{h^2}{a^2 b^2} + \frac{1}{b^2} = 0$$

$$\Rightarrow h^2 + k^2 = a^2 - b^2$$

Therefore, locus of point P is $x^2 + y^2 = a^2 - b^2$, which is director circle of hyperbola.

Alternative method:

One of the equations of tangents to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ having slope m is

$$y = mx + \sqrt{a^2 m^2 - b^2}$$

If it passes through the point $P(h, k)$, then

$$k = mh + \sqrt{a^2 m^2 - b^2}$$

$$\text{or } (a^2 m^2 - b^2) = (k - mh)^2$$

$$\text{or } (h^2 - a^2)m^2 - 2hkm + k^2 + b^2 = 0$$

This equation has two roots m_1 and m_2 , which are slopes of two tangents to hyperbola drawn from point P .

Since tangents are perpendicular,

$$m_1 m_2 = -1$$

$$\Rightarrow \frac{k^2 + b^2}{h^2 - a^2} = -1$$

$$\Rightarrow h^2 + k^2 = a^2 - b^2$$

So, locus of point P is $x^2 + y^2 = a^2 - b^2$.

Clearly, this circle exists if $a > b$.

$$\text{For } a > b, e^2 = 1 + \frac{b^2}{a^2} < 2 \text{ or } e < \sqrt{2}.$$

For $a < b$, we have $e > \sqrt{2}$. In this case, director circle does not exist.

For $a = b$, we have director circle as $x^2 + y^2 = 0$, which represents point $(0, 0)$. This is the only point on the plane of hyperbola from which perpendicular tangents can be drawn.

Director circle for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is $x^2 + y^2 = b^2 - a^2$, where $b > a$

ILLUSTRATION 7.32

On which curve does the perpendicular tangents drawn to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ intersect?

Sol. The locus of the point of intersection of perpendicular tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the director circle given by

$$x^2 + y^2 = a^2 - b^2$$

Hence, the perpendicular tangents drawn to

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

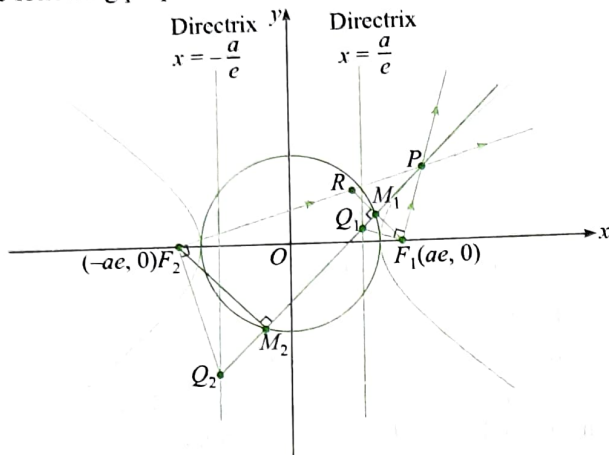
intersect on the curve

$$x^2 + y^2 = 25 - 16 = 9$$

PROPERTIES OF TANGENT TO HYPERBOLA

We have many properties of tangent to hyperbola many of which are similar to the properties of tangent to ellipse.

Consider standard equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. We have following properties of tangent.



- Feet of perpendiculars from foci upon any tangent lie on the auxiliary circle.
In the figure, tangent is drawn to hyperbola at point P on it. Feet of perpendiculars from foci F_1 and F_2 on the tangent are M_1 and M_2 , which lie on auxiliary circle.
- Product of perpendiculars from foci upon any tangent of hyperbola is equal to square of the semi-conjugate axis.
Here, $F_1M_1 \times F_2M_2 = b^2$.
- Length of tangent between the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.
In the figure, tangent meets directrix $x = \frac{a}{e}$ at Q_1 .

So, PQ_1 subtends right angle at corresponding focus F_1 .

Also, tangent meets directrix $x = -\frac{a}{e}$ at Q_2 .

So, PQ_2 subtends right angle at corresponding focus F_2 .

- Tangent at the extremities of latus rectum passes through the corresponding foot of directrix on major axis.
- Chord of contact w.r.t. any point on the directrix passes through the corresponding focus.
- Tangent at point P on the hyperbola bisects the angle between two focal radii of point P.

In the figure, tangent at P bisects the angle F_1PF_2 .

- The above property spells out the reflection property of hyperbola.

If incident ray emanating from one of the foci gets reflected by the hyperbola, then reflected ray passes through the other focus.

In the figure, incident ray from F_1 strikes the hyperbola at point P. Then the reflected ray passes through the focus F_2 . Similarly, if incident ray is F_2P , then reflected ray passes through F_1 .

- Image of one of the foci in the tangent lies on the line joining other focus and point of contact.

In the figure, image R of focus F_1 in the tangent lies on the line joining focus F_2 and point of contact P.

ILLUSTRATION 7.33

Find the equation of hyperbola having foci $S(2, 1)$ and $S'(10, 1)$ and a straight line $x + y - 9 = 0$ as its tangent. Also, find the equation of its director circle.

Sol. Foci are $S(2, 1)$, $S'(10, 1)$.

So transverse axis is horizontal and has the equation $y = 1$.

Centre is midpoint of SS' , which is $(6, 1)$.

So, equation of hyperbola is

$$\frac{(x-6)^2}{a^2} - \frac{(y-1)^2}{b^2} = 1$$

Now, tangent is $x + y - 9 = 0$.

Product of length of perpendicular from foci on the tangent is b^2 .

$$\therefore b^2 = \frac{|2+1-9|}{\sqrt{2}} \times \frac{|10+1-9|}{\sqrt{2}} = \left(\frac{6}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right)$$

$$\therefore b^2 = 6$$

Distance between foci, $2ae = 8$

$$\therefore ae = 4$$

$$\text{Now, } a^2 + b^2 = a^2e^2$$

$$\Rightarrow a^2 + 6 = 16$$

$$\Rightarrow a^2 = 10$$

Therefore, equation of hyperbola is

$$\frac{(x-6)^2}{10} - \frac{(y-1)^2}{6} = 1$$

Equation of director circle is

$$(x-6)^2 + (y-1)^2 = a^2 - b^2$$

$$\text{or } (x-6)^2 + (y-1)^2 = 4$$

CONCEPT APPLICATION EXERCISE 7.3

- Find the equation of tangent to the hyperbola $16x^2 - 25y^2 = 400$ perpendicular to the line $x - 3y = 4$.
- Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point $(0, 5/2)$. Find their equations.
- Find the common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$.
- Find the locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- A tangent to the hyperbola $4x^2 - 9y^2 = 36$ meets the axes at points A and B. Find the locus of midpoint of AB.
- A point P moves such that the chord of contact of the pair of tangents from P on the parabola $y^2 = 4ax$ touches the rectangular hyperbola $x^2 - y^2 = c^2$. Show that the locus of P is the ellipse $\frac{x^2}{c^2} + \frac{y^2}{(2a)^2} = 1$.
- If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at A and B, then find the locus of point of intersection of tangents at A and B.

8. If the chords of contact of tangents from two points $(-4, 2)$ and $(2, 1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are at right angle, then find the eccentricity of the hyperbola.
9. If from any point $P(x_1, y_1)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that corresponding chord of contact touches the another branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.
10. Let p be the perpendicular distance from the centre C of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to the tangent drawn at a point R on the hyperbola. If S and S' are the two foci of the hyperbola, then prove that $(RS + RS')^2 = 4a^2 \left(1 + \frac{b^2}{p^2}\right)$.

ANSWERS

1. $y = -3x \pm \sqrt{209}$
2. $y = \frac{5}{2} \pm \frac{3}{2}x$
3. $y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$
4. $\frac{x^2}{b^2/a^2} - \frac{y^2}{b^2} = 1$
5. $\frac{9}{4x^2} - \frac{1}{y^2} = 1$
7. $y^2 = -\frac{b^4}{a^3}x$
8. $\sqrt{\frac{3}{2}}$

ASYMPTOTES OF HYPERBOLA (TANGENT AT INFINITY)

We know that an asymptote is a line that a curve approaches, as it heads towards infinity.

In other words, asymptote of a curve is a line such that the distance between the curve and the line approaches to zero when one or both of x and y coordinates approach to infinity. Thus, asymptotes are tangents to curve at infinity.

Let $y = mx + c$ be the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = k$. Solving this line with hyperbola, we get

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(kb^2 + c^2) = 0 \quad (1)$$

$$\text{or } (b^2 - a^2m^2)x - 2a^2mc - \frac{a^2(kb^2 + c^2)}{x} = 0 \quad (2)$$

If both roots of eq. (1) tends to infinity, then from eq. (2), we must have

$$b^2 - a^2m^2 = 0 \text{ and } 2a^2mc = 0$$

$$\therefore m^2 = \frac{b^2}{a^2} \text{ or } m = \pm \frac{b}{a} \text{ and } c = 0$$

Therefore, equation of line becomes $y = \pm \frac{b}{a}x$ or $bx \pm ay = 0$.

These are equations of asymptotes of hyperbola, which pass through the centre of the hyperbola.

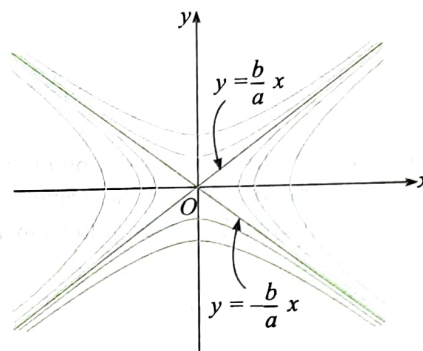
The combined equation of asymptotes is $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Thus, equation of hyperbola and that of combined equation of pair of asymptotes differ by constant (k) only.

For given equation of hyperbola, we have unique pair of asymptotes.

But for given pair of asymptotes, there exist innumerable hyperbolas having same eccentricity and centre.

The curves of hyperbola for different values of k lie between these two asymptotes as shown in the figure.



We observe that when x and y tend to infinity, the curves of hyperbola flatten and approach to the asymptotes.

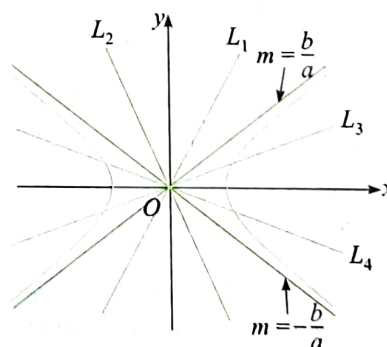
Technically, to trace the hyperbola we should first draw the asymptotes and then draw the arcs of the hyperbola which approach to asymptotes at infinity.

Also, we observe that axes of hyperbola are angle bisectors of the pair of asymptotes.

We know that for hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = k$, values of slopes of tangents lie in $\left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right)$.

So, slopes of asymptotes $\left(\pm \frac{b}{a}\right)$ are limiting values of slopes of tangents.

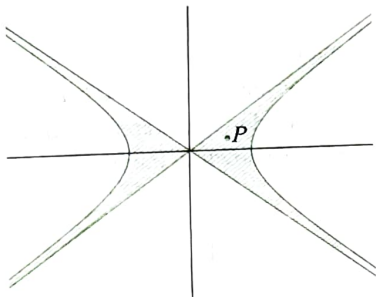
This fact can be seen in the following figure.



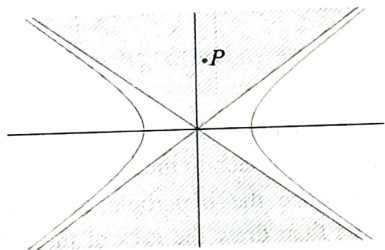
Line L_1 has slope greater than $\frac{b}{a}$ and L_2 has slope less than $-\frac{b}{a}$. Clearly, we find the tangents to hyperbola parallel to these lines (translate the lines and check).

Line L_3 has slope less than $\frac{b}{a}$ and L_4 has slope more than $-\frac{b}{a}$. Translating these lines, we find that they intersect the hyperbola in two points but, never touch the hyperbola.

Further, two tangents drawn to hyperbola from point P lying in the region between the asymptotes and the branches of the hyperbola are to the same branch of the hyperbola.



Two tangents drawn to hyperbola from point P lying in the region between the asymptotes where the branch of hyperbola does not lie are to the different branches of the hyperbola.



PROPERTIES OF ASYMPTOTES

- The acute angle θ between the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = k \text{ is given by}$$

$$\tan \theta = \left| \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \frac{b}{a} \left(-\frac{b}{a}\right)} \right| = \left| \frac{2ab}{a^2 - b^2} \right|$$

$$\text{Also, } \left| \frac{\frac{2b}{a}}{1 - \frac{b^2}{a^2}} \right| = \tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

$$\therefore \tan \frac{\theta}{2} = \frac{b}{a} = \sqrt{e^2 - 1}$$

$$\therefore e = \sec \frac{\theta}{2}$$

For rectangular hyperbola $x^2 - y^2 = a^2$, equations of asymptotes are $y = \pm x$ which are perpendicular.

- The length of any tangent intercepted between the asymptotes is bisected at the point of contact.
- Area of triangle formed by any tangent and asymptotes

is constant. For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, this area is ab sq. units.

ILLUSTRATION 7.34

Find the eccentricity of the hyperbola with asymptotes $3x + 4y = 2$ and $4x - 3y = 2$.

Sol. Since the asymptotes are perpendicular, the hyperbola is rectangular and, hence, eccentricity is $\sqrt{2}$.

ILLUSTRATION 7.35

Find the equation of the hyperbola which has $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ as its asymptotes and which passes through the origin.

Sol. The combined equation of the asymptotes is

$$(3x - 4y + 7)(4x + 3y + 1) = 0$$

$$\text{or } 12x^2 - 7xy - 12y^2 + 31x + 17y + 7 = 0 \quad (1)$$

Since the equation of hyperbola and the combined equation of its asymptotes differ by a constant, the equation of the hyperbola may be taken as

$$12x^2 - 7xy - 12y^2 + 31x + 17y + k = 0 \quad (2)$$

As (2) passes through the origin $(0, 0)$, we have $k = 0$.

Hence, the equation to the required hyperbola is

$$12x^2 - 7xy - 12y^2 + 31x + 17y = 0$$

ILLUSTRATION 7.36

Find the equations of the asymptotes of the hyperbola $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$.

Sol. Since the equation of hyperbola and the combined equation of its asymptotes differ by a constant, the equations of the asymptotes should be

$$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0 \quad (1)$$

Now, λ is to be chosen so that (1) represents a pair of straight lines.

Comparing (1) with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (2)$$

we have

$$a = 3, b = 8, h = 5, g = 7, f = 11, c = \lambda$$

We know that (2) represents a pair of straight lines if

$$abc + 2hgf - af^2 - bg^2 - ch^2 = 0$$

$$\text{or } 3 \times 8 \times \lambda + 2 \times 7 \times 11 \times 5 - 3 \times 121 - 8 \times 49 - \lambda \times 25 = 0$$

$$\text{or } \lambda = 15$$

Hence, the combined equation of the asymptotes is $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$.

ILLUSTRATION 7.37

If a hyperbola passing through the origin has $3x - 4y - 1 = 0$ and $4x - 3y - 6 = 0$ as its asymptotes, then find the equations of its transverse and conjugate axes.

Sol. The axes of a hyperbola are the bisectors of the pair of asymptotes.

The transverse axis is the bisector which contains the origin and is given by

$$\frac{3x - 4y - 1}{5} = + \frac{4x - 3y - 6}{5}$$

$$\text{or } x + y - 5 = 0$$

The conjugate axis is

$$\frac{3x - 4y - 1}{5} = -\frac{4x - 3y - 6}{5}$$

or $x - y - 1 = 0$

ILLUSTRATION 7.38

The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C , meets the asymptotes in Q and R . Prove that area of triangle CQR is constant. Also, prove that P is midpoint of QR .

Sol. Let point P be $(a \sec \theta, b \tan \theta)$.

Equation of tangent at point P is $\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$.

Equations of asymptotes are $y = \pm \frac{b}{a} x$.

Solving asymptotes with tangent, we get

$$Q \equiv \left(\frac{a}{\sec \theta - \tan \theta}, \frac{b}{\sec \theta - \tan \theta} \right)$$

$$\text{and } R \equiv \left(\frac{a}{\sec \theta + \tan \theta}, \frac{-b}{\sec \theta + \tan \theta} \right)$$

$$\therefore \text{Area of triangle } CQR = \frac{1}{2} \begin{vmatrix} 0 & 0 \\ a & b \\ \frac{a}{\sec \theta - \tan \theta} & \frac{b}{\sec \theta - \tan \theta} \\ \frac{a}{\sec \theta + \tan \theta} & \frac{-b}{\sec \theta + \tan \theta} \\ 0 & 0 \end{vmatrix}$$

$$= \frac{1}{2} \left| -\frac{a}{\sec \theta - \tan \theta} \cdot \frac{b}{\sec \theta + \tan \theta} - \frac{a}{\sec \theta - \tan \theta} \cdot \frac{b}{\sec \theta + \tan \theta} \right|$$

$= ab$, which is constant.

Also, midpoint of QR is

$$\left(\frac{\frac{a}{\sec \theta - \tan \theta} + \frac{a}{\sec \theta + \tan \theta}}{2}, \frac{\frac{b}{\sec \theta - \tan \theta} - \frac{b}{\sec \theta + \tan \theta}}{2} \right)$$

$$\equiv (a \sec \theta, b \tan \theta), \text{ which is point } P.$$

ILLUSTRATION 7.39

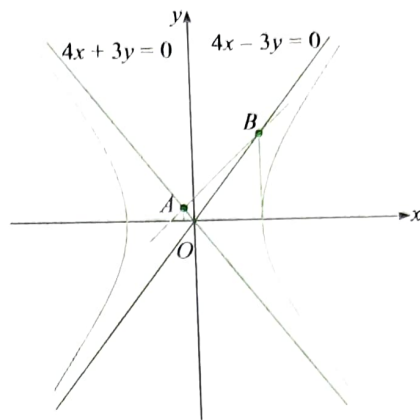
If two distinct tangents can be drawn from the point $(\alpha, \alpha + 1)$ on different branches of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then find the values of α .

Sol. Point $P(\alpha, \alpha + 1)$ lies on the line

$$y = x + 1 \quad (1)$$

It is given that from point P two tangents can be drawn to different branches of hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

So, point P lies in one of the four regions formed by asymptotes where branch of hyperbola does not lie.



Equations of asymptotes are

$$4x - 3y = 0 \quad (2)$$

$$\text{and } 4x + 3y = 0 \quad (3)$$

If point P lies on the line (2), then $4\alpha - 3(\alpha + 1) = 0$ or $\alpha = 3$.

If point P lies on the line (3), then $4\alpha + 3(\alpha + 1) = 0$ or $\alpha = -3/7$.

Thus, $\alpha \in (-3/7, 3)$

ILLUSTRATION 7.40

From a point $P(1, 2)$, pair of tangents are drawn to a hyperbola, one tangent to each arm of hyperbola. Equations of asymptotes of hyperbola are $\sqrt{3}x - y + 5 = 0$ and $\sqrt{3}x + y - 1 = 0$. Find the eccentricity of hyperbola.

Sol. We know that if angle between asymptotes is θ , then eccentricity of hyperbola is $\sec \frac{\theta}{2}$.

Acute angle between asymptotes $\sqrt{3}x - y + 5 = 0$ and $\sqrt{3}x + y - 1 = 0$ is $\frac{\pi}{3}$.

Let $L_1(x, y) = \sqrt{3}x - y + 5$ and $L_2(x, y) = \sqrt{3}x + y - 1$.

$$\therefore L_1(0, 0) = 5 \text{ and } L_2(0, 0) = -1$$

$$\text{Also, } a_1 a_2 + b_1 b_2 = (\sqrt{3})(\sqrt{3}) + (-1)(1) = 2 > 0.$$

Thus, origin lies in acute angle.

$$\text{Now, } L_1(1, 2) > 0 \text{ and } L_2(1, 2) > 0$$

So, $(1, 2)$ lies in obtuse angle formed by asymptotes.

Since tangents drawn from point $(1, 2)$ are to different branches of hyperbola, branches of hyperbola lie in acute angles formed by asymptotes.

Therefore, eccentricity of hyperbola is $e = \sec \frac{\theta}{2} = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$.

CONCEPT APPLICATION EXERCISE 7.4

- Find the angle between the asymptotes of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
- Find the asymptotes of the curve $xy - 3y - 2x = 0$.

- If asymptotes of hyperbola bisect the angles between the transverse axis and conjugate axis of hyperbola, then what is the eccentricity of hyperbola?
- The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2 \tan \lambda$ in magnitude then find its eccentricity.
- If the foci of a hyperbola lie on $y = x$ and one of the asymptotes is $y = 2x$, then find the equation of the hyperbola, given that it passes through (3, 4).

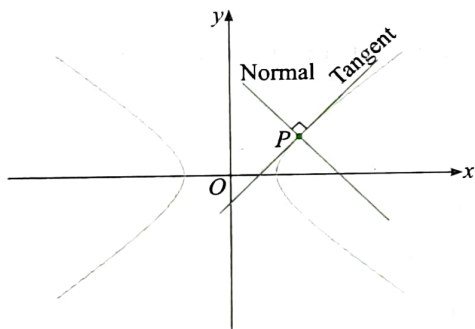
ANSWERS

- $\tan^{-1} \frac{24}{7}$
- $x - 3 = 0$ and $y - 2 = 0$
- $\sqrt{2}$
- $e = \sec \lambda$
- $2x^2 + 2y^2 - 5xy + 10 = 0$

NORMAL TO HYPERBOLA**EQUATION OF NORMAL TO HYPERBOLA AT A POINT ON IT**

Equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point $P(x_1, y_1)$ on it is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$.

Slope of tangent is $\frac{b^2 x_1}{a^2 y_1}$.



Thus, slope of normal to hyperbola at point P is $-\frac{a^2 y_1}{b^2 x_1}$.
So, equation of normal at point P is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1} (x - x_1)$$

$$\Rightarrow \frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

If $(x_1, y_1) \equiv (a \sec \theta, b \tan \theta)$, then equation of normal is

$$\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2$$

Clearly, transverse axis is normal to hyperbola at its vertices.

Normal other than transverse axis, never passes through focus, which can be verified by putting x and y values of the foci in the equation of normal.

Equation of normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at point $P(a \tan \theta, b \sec \theta)$ is

$$\frac{ax}{\tan \theta} + \frac{by}{\sec \theta} = a^2 + b^2$$

EQUATION OF NORMAL TO HYPERBOLA HAVING GIVEN SLOPE

The equation of normal to the given hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point $P(a \sec \theta, b \tan \theta)$ is

$$ax \cos \theta + by \cot \theta = a^2 + b^2$$

$$\Rightarrow y = -\frac{a \sin \theta}{b} x + \frac{a^2 + b^2}{b \cot \theta}$$

$$\text{Let } -\frac{a}{b} \sin \theta = m.$$

$$\therefore \sin \theta = -\frac{bm}{a}$$

$$\therefore \cot \theta = \pm \frac{\sqrt{a^2 - b^2 m^2}}{bm}$$

Hence, the equation of the normal becomes

$$y = mx \pm \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2 m^2}}, \text{ where } m \in \left[-\frac{a}{b}, \frac{a}{b}\right]$$

ILLUSTRATION 7.41

Find the equation of normal to the hyperbola $x^2 - 9y^2 = 7$ at point (4, 1).

Sol. Differentiating the equation of hyperbola $x^2 - 9y^2 = 7$ w.r.t. x , we get

$$2x - 18y \frac{dy}{dx} = 0$$

$$\text{or } \frac{dy}{dx} = \frac{x}{9y}$$

Now, slope of normal at any point on the curve is $-\frac{dx}{dy} = -\frac{9y}{x}$.

Therefore, the slope of normal at point (4, 1) is

$$\left(-\frac{dx}{dy}\right)_{(4,1)} = -\frac{9}{4}$$

Therefore, the equation of normal at point (4, 1) is $y - 1 = -\frac{9}{4}(x - 4)$

$$\text{or } 9x + 4y = 40$$

Alternative method:

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, equation of normal at point $P(x_1, y_1)$ is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

So, for given hyperbola $\frac{x^2}{7} - \frac{y^2}{7/9} = 1$, equation of normal at point $P(4, 1)$ is

$$\frac{7x}{4} + \frac{(7/9)y}{1} = 7 + \frac{7}{9}$$

$$\Rightarrow 9x + 4y = 40$$

ILLUSTRATION 7.42

Find the equation of normal to the hyperbola $3x^2 - y^2 = 1$ having slope $1/3$.

Sol. Differentiating the equation of hyperbola $3x^2 - y^2 = 1$ w.r.t. x , we get

$$6x - 2y \frac{dy}{dx} = 0$$

$$\text{or } \frac{dy}{dx} = \frac{3x}{y}$$

Let the point on the curve where the normal has slope $1/3$ be (x_1, y_1) . Therefore,

$$-\frac{dx}{dy} = -\frac{y_1}{3x_1} = \frac{1}{3} \text{ or } y_1 = -x_1 \quad (1)$$

Also, P lies on the curve. Therefore,

$$3x_1^2 - y_1^2 = 1 \quad (2)$$

Solving (1) and (2), we get

$$x_1^2 = \frac{1}{2} \text{ or } x_1 = \pm \frac{1}{\sqrt{2}}$$

$$\therefore y = \mp \frac{1}{\sqrt{2}}$$

Therefore, the points on the curve are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$.

Hence, the equations of normal are

$$y - \frac{1}{\sqrt{2}} = \frac{1}{3} \left(x + \frac{1}{\sqrt{2}} \right)$$

$$\text{and } y + \frac{1}{\sqrt{2}} = \frac{1}{3} \left(x - \frac{1}{\sqrt{2}} \right)$$

$$\text{or } \sqrt{2}(x - 3y) = 4 \text{ and } \sqrt{2}(x - 3y) = -4$$

Alternative method:

Equation of normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ having slope m is

$$y = mx \pm \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2 m^2}}$$

So, for given hyperbola equation of normal having slope $\frac{1}{3}$ is

$$y = \frac{1}{3}x \pm \frac{\left(\frac{1}{3} + 1\right)\frac{1}{3}}{\sqrt{\frac{1}{3} - \frac{1}{9}}}$$

$$\Rightarrow y = \frac{1}{3}x \pm \frac{2\sqrt{2}}{3}$$

ILLUSTRATION 7.43

If the normal at $P(\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at G , then prove that $AG \cdot A'G = a^2(e^4 \sec^2 \theta - 1)$, where A and A' are the vertices of the hyperbola.

Sol. The equation of the normal at $P(a \sec \theta, b \tan \theta)$ to the given hyperbola is $ax \cos \theta + by \cot \theta = (a^2 + b^2)$.

This meets the transverse axis, i.e., the x -axis at G .

So, the coordinates of G are $\left(\frac{(a^2 + b^2)}{a} \sec \theta, 0\right)$.

The coordinates of the vertices A and A' are $(a, 0)$ and $(-a, 0)$, respectively.

$$\begin{aligned} \therefore AG \cdot A'G &= \left(-a + \frac{a^2 + b^2}{a} \sec \theta\right) \left(a + \frac{a^2 + b^2}{a} \sec \theta\right) \\ &= (-a + ae^2 \sec \theta)(a + ae^2 \sec \theta) \\ &= a^2(e^4 \sec^2 \theta - 1) \end{aligned}$$

ILLUSTRATION 7.44

Normals are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at points θ_1 and θ_2 meeting the conjugate axis at G_1 and G_2 , respectively.

If $\theta_1 + \theta_2 = \pi/2$, prove that $CG_1 \cdot CG_2 = \frac{a^2 e^4}{e^2 - 1}$, where C is the center of the hyperbola and e is its eccentricity.

Sol. The normal at point $P(a \sec \theta_1, b \tan \theta_1)$ is

$$ax \cos \theta_1 + by \cot \theta_1 = (a^2 + b^2)$$

It meets the conjugate axis at $G_1\left(0, \frac{a^2 + b^2}{b} \tan \theta_1\right)$.

The normal at point $Q(a \sec \theta_2, b \tan \theta_2)$ is

$$ax \cos \theta_2 + by \cot \theta_2 = (a^2 + b^2)$$

It meets the conjugate axis at $G_2\left(0, \frac{a^2 + b^2}{b} \tan \theta_2\right)$. Therefore,

$$\begin{aligned} CG_1 \cdot CG_2 &= \frac{(a^2 + b^2)^2}{b^2} \tan \theta_1 \tan \theta_2 \\ &= \frac{(a^2 + b^2)^2}{b^2} \quad \left(\because \theta_1 + \theta_2 = \frac{\pi}{2}\right) \\ &= \frac{a^4 \left(1 + \frac{b^2}{a^2}\right)^2}{b^2} \\ &= \frac{a^2 e^4}{e^2 - 1} \end{aligned}$$

ILLUSTRATION 7.45

Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at point P intersects the x -axis at $(9, 0)$, then find the eccentricity of the hyperbola.

Sol. Normal at $(6, 3)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{a^2 x}{6} + \frac{b^2 y}{3} = a^2 + b^2$$

It passes through $(9, 0)$.

$$\therefore \frac{9a^2}{6} = a^2 + b^2$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{1}{2}$$

$$\Rightarrow e^2 - 1 = \frac{1}{2}$$

$$\therefore e = \sqrt{\frac{3}{2}}$$

ILLUSTRATION 7.46

Prove that any hyperbola and its conjugate hyperbola cannot have common normal.

Sol. Consider hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation of normal to hyperbola at point $P(a \sec \theta, b \tan \theta)$ is

$$ax \cos \theta + by \cot \theta = a^2 + b^2 \quad (1)$$

Equation of normal to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at point $Q(a \tan \phi, b \sec \phi)$ is

$$ax \cot \phi + by \cos \phi = a^2 + b^2 \quad (2)$$

If Eqs. (1) and (2) represent the same straight line, then

$$\frac{\cot \phi}{\cos \theta} = \frac{\cos \phi}{\cot \theta} = 1$$

$$\Rightarrow \tan \phi = \sec \theta \text{ and } \sec \phi = \tan \theta$$

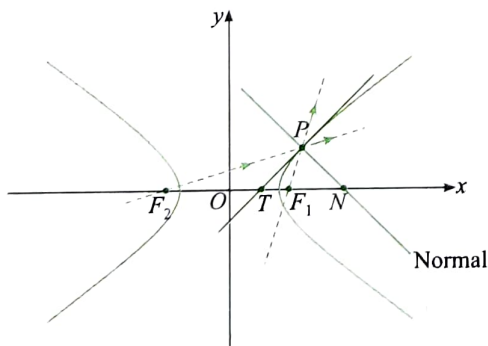
$$\Rightarrow \sec^2 \phi - \tan^2 \phi = \tan^2 \theta - \sec^2 \theta = -1, \text{ which is not possible.}$$

Thus, hyperbola and its conjugate hyperbola cannot have common normal.

IMPORTANT PROPERTY OF NORMAL

We know that tangent bisects one of the angles between two focal lengths of point of contact P .

Since two angle bisectors of pair of lines are perpendicular, normal at P will be another bisector of angle between these focal lengths.



This fact can be proved by showing that

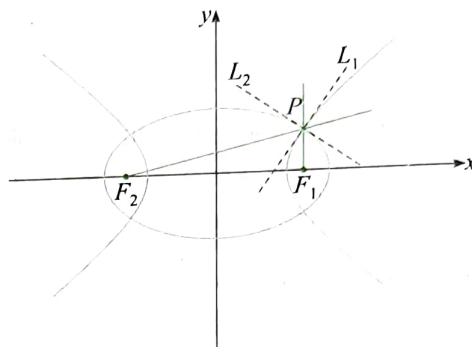
$$F_1N : F_2N = F_1P : F_2P$$

Also, this proves that incident ray from focus F_1 after getting reflected by hyperbola at point P , passes through another focus F_2 .

Orthogonal Intersection of Ellipse and Hyperbola

We have learned that in ellipse and hyperbola, tangent and normal are bisectors of angles between focal lengths of point on the curve.

Hence, if ellipse and hyperbola are confocal, then they intersect orthogonally. Conversely, if ellipse and hyperbola intersect orthogonally, then they are confocal.



As shown in the figure, ellipse and hyperbola are confocal having common foci at F_1 and F_2 . Therefore, they are intersecting orthogonally at point P , i.e., tangents to ellipse and hyperbola at point P are perpendicular to each other. Line L_1 is tangent to hyperbola and normal to ellipse. Line L_2 is tangent to ellipse and normal to hyperbola at point P .

ILLUSTRATION 7.47

A ray emanating from the point $(5, 0)$ is incident on the hyperbola $9x^2 - 16y^2 = 144$ at the point $P(8, 3\sqrt{3})$. Find the equation of the reflected ray after first reflection.

Sol. Given hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad (1)$$

Here, $a = 4$ and $b = 3$.

So, foci are $(\pm\sqrt{a^2 + b^2}, 0) \equiv (\pm 5, 0)$.

Incident ray through $F_1(5, 0)$ strikes the ellipse at point $P(8, 3\sqrt{3})$.

Therefore, reflected ray will go through another focus $F_2(-5, 0)$.

So, reflected ray is line through points F_2 and P , which is $3\sqrt{3}x - 13y + 15\sqrt{3} = 0$.

ILLUSTRATION 7.48

Normal to a rectangular hyperbola at P meets the transverse axis at N . If foci of hyperbola are S and S' , then find the value of $\frac{SN}{SP}$.

Sol. We know that PN is angle bisector of focal radii SP and $S'P$.

$$\therefore \frac{SP}{S'P} = \frac{SN}{S'N}$$

$$\therefore \frac{SN}{SP} = \frac{S'N}{S'P} = \frac{|SN - S'N|}{|SP - S'P|} = \frac{SS'}{|SP - S'P|} = e = \sqrt{2}$$

CONCEPT APPLICATION EXERCISE 7.5

1. If any line perpendicular to the transverse axis cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at points P and Q , respectively, then prove that normals at P and Q meet on the x -axis.
2. A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes at M and N and lines MP and NP are drawn perpendicular to the axes meeting at P . Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.
3. Prove that the locus of the point of intersection of the tangents at the ends of the normal chords of the hyperbola $x^2 - y^2 = a^2$ is $a^2(y^2 - x^2) = 4x^2y^2$.
4. Find the value of m , for which the line $y = mx + \frac{25\sqrt{3}}{3}$ is a normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
5. Normal is drawn at one of the extremities of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which meets the axes at points A and B . Then find the area of triangle OAB (O being the origin).

ANSWERS

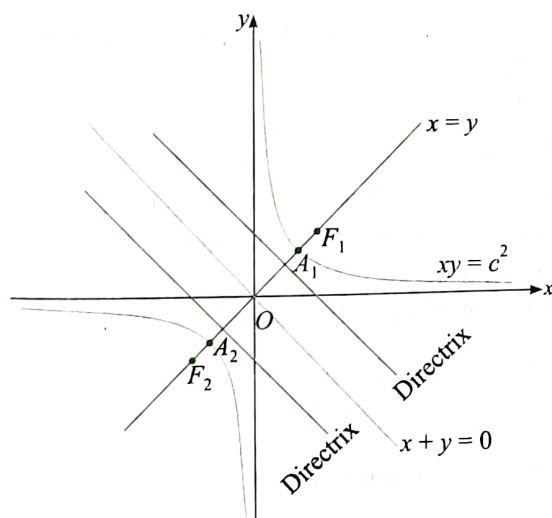
4. $m = -\frac{2}{\sqrt{3}}$

5. Area = $\frac{1}{2}a^2e^5$

HYPERBOLA HAVING ASYMPTOTES AS COORDINATE AXES

The equation of pair of coordinate axes is $xy = 0$.

Now, if this is the pair of asymptotes to hyperbola whose branches are lying in first and third quadrants, then the equation of the hyperbola will be of the form $xy = c^2$.



Clearly, this is rectangular hyperbola as asymptotes are perpendicular.

So, eccentricity of hyperbola is $\sqrt{2}$.

Axes of hyperbola are angle bisectors of asymptotes $y = 0$ and $x = 0$, which are $x - y = 0$ and $x + y = 0$.

For the above hyperbola, line $x - y = 0$ is transverse axis of hyperbola. So, vertices of hyperbola are $(\pm c, \pm c)$.

Centre of the hyperbola is $O(0, 0)$.

Distance of vertices $A(c, c)$ and $A'(-c, -c)$ from centre is $\sqrt{2}c$.

So, length of transverse axis is $2\sqrt{2}c (= 2a)$ which is same as length of conjugate axis.

Now, foci lie at distance $\sqrt{2}ce (= ae)$ or $2c$ from centre on the line $x - y = 0$.

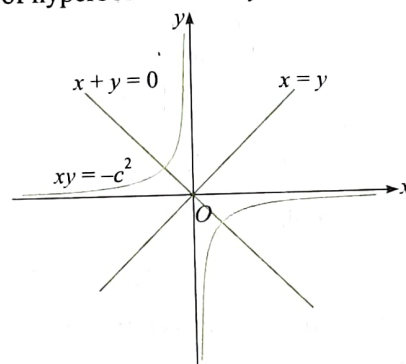
Therefore, foci are $F_1(\sqrt{2}c, \sqrt{2}c)$ and $F_2(-\sqrt{2}c, -\sqrt{2}c)$.

Directrices lie at distance $\frac{\sqrt{2}c}{e} (= \frac{a}{e})$ or c from the centre.

So, equations of two directrices are $x + y = \pm\sqrt{2}c$.

Parametric form of hyperbola is $x = ct, y = \frac{c}{t}$, where $t \in \mathbb{R} - \{0\}$.

If the branches of hyperbola lie in second and fourth quadrants, then equation of hyperbola will be $xy = -c^2$.

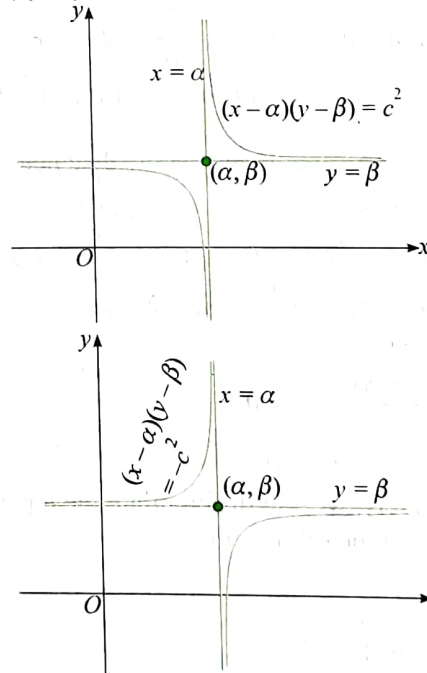


Transverse axis of this hyperbola is $x + y = 0$ and vertices are $(\pm c, \mp c)$.

Foci are $(\pm\sqrt{2}c, \mp\sqrt{2}c)$ and directrices are $x - y = \pm\sqrt{2}c$.

If the asymptotes of rectangular hyperbola are $x = \alpha$ and $y = \beta$, then equation of hyperbola will be

$$(x - \alpha)(y - \beta) = \pm c^2$$



TANGENT AND NORMAL TO HYPERBOLA

For hyperbola $xy - c^2 = 0$, equation of tangent at point $P(x_1, y_1)$ on it is

$$\frac{xy_1 + x_1y}{2} - c^2 = 0$$

$$\text{or } xy_1 + x_1y - 2c^2 = 9$$

$$\text{or } T = 0, \text{ where } T = xy_1 + x_1y - 2c^2$$

Equation of normal at point $P(x_1, y_1)$ will be $xx_1 - yy_1 = x_1^2 - y_1^2$.

Replacing x_1 by ct and y_1 by $\frac{c}{t}$, we can get the equation of tangent and normal in terms of parameter ' t '.

ILLUSTRATION 7.49

Consider hyperbola $xy = 16$ to find the following:

- Coordinates of vertices
- Length of transverse axis
- Coordinates of foci
- Length of latus rectum
- Equations of two directrices
- Equation of tangent at point $(2, 8)$
- Equation of normal at point $(2, 8)$
- Equation of chord of contact w.r.t. point $(2, 3)$
- Equation of chord which gets bisected at point $(5, 6)$
- Equation of tangent having slope -2
- Equation of normal having slope 2

Sol. We have hyperbola $xy = 16$.

- (i) Clearly, transverse axis is $x - y = 0$ and conjugate axis is $x + y = 0$.

Vertices are points of intersection of transverse axis $x - y = 0$ and hyperbola.

So, vertices are $A(4, 4)$ and $A'(-4, -4)$.

- (ii) Length of transverse axis is $AA' = 8\sqrt{2}$ ($= 2a$).

- (iii) Foci lie at distance ' ae ' from centre on line $x - y = 0$.

$$\text{Now, } ae = 4\sqrt{2}\sqrt{2} = 8$$

Therefore, foci are $F_1(4\sqrt{2}, 4\sqrt{2})$ and $F_2(-4\sqrt{2}, -4\sqrt{2})$.

- (iv) Length of latus rectum $= \frac{2b^2}{a} = 2a = 8\sqrt{2}$

(as $a = b$ for rectangular hyperbola)

- (v) Each of the directrices lies at distance $\frac{a}{e} (= 4)$ from centre and is parallel to the conjugate axis $x + y = 0$.

So, equation of two directrices are $x + y = \pm 4\sqrt{2}$.

- (vi) Equation of tangent at point $(2, 8)$ is

$$T = 0$$

$$\Rightarrow \frac{2y + 8x}{2} - 16 = 0$$

$$\Rightarrow 4x + y - 16 = 0$$

Alternatively, we can differentiate the equation of curve to get the slope of tangent.

Then, we get

$$y + x \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(2, 8)} = -\frac{8}{2} = -4$$

So, equation of tangent at $(2, 8)$ is

$$y - 8 = -4(x - 2)$$

$$\Rightarrow 4x + y - 16 = 0$$

- (vii) Equation of normal at $(2, 8)$ is $x - 4y + 30 = 0$.

- (viii) Equation of chord of contact w.r.t. point $(2, 3)$ is

$$T = 0$$

$$\Rightarrow \frac{2y + 3x}{2} - 16 = 0$$

$$\Rightarrow 3x + 2y - 32 = 0$$

- (ix) Equation of chord which gets bisected at point $(5, 6)$ is

$$T = S_1$$

$$\Rightarrow \frac{5y + 6x}{2} - 16 = (5)(6) - 16$$

$$\Rightarrow 6x + 5y = 60$$

- (x) To find the equation of tangent having slope -2 , we differentiate the curve and compare derivative to -2 .

Then, we get

$$\therefore \frac{dy}{dx} = -\frac{y}{x} = -2$$

$$\therefore y = 2x$$

Solving this with hyperbola, we get points $P(\sqrt{8}, 2\sqrt{8})$ and $Q(-\sqrt{8}, -2\sqrt{8})$ on the hyperbola where slope of tangent is -2 .

Equation of tangent at point P is

$$y - 2\sqrt{8} = -2(x - \sqrt{8})$$

$$\Rightarrow 2x + y = 4\sqrt{8}$$

Equation of tangent at point Q is

$$y + 2\sqrt{8} = -2(x + \sqrt{8})$$

$$\Rightarrow 2x + y = -4\sqrt{8}$$

- (xi) We have $\frac{dy}{dx} = -\frac{y}{x}$

Thus, slope of normal at any point on the curve is

$$-\frac{dx}{dy} = \frac{x}{y} = 2 \text{ (given)}$$

$$\therefore x = 2y$$

Solving this with hyperbola, we get points $A(2\sqrt{8}, \sqrt{8})$ and $B(-2\sqrt{8}, -\sqrt{8})$ on the hyperbola where slope of normal is 2 .

Equation of normal at point A is

$$y - \sqrt{8} = 2(x - 2\sqrt{8})$$

$$\Rightarrow 2x - y = 3\sqrt{8}$$

Equation of normal at point B is

$$y + \sqrt{8} = 2(x + 2\sqrt{8})$$

$$\Rightarrow 2x - y = -3\sqrt{8}$$

ILLUSTRATION 7.50

A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Sol. Consider hyperbola $xy = c^2$.

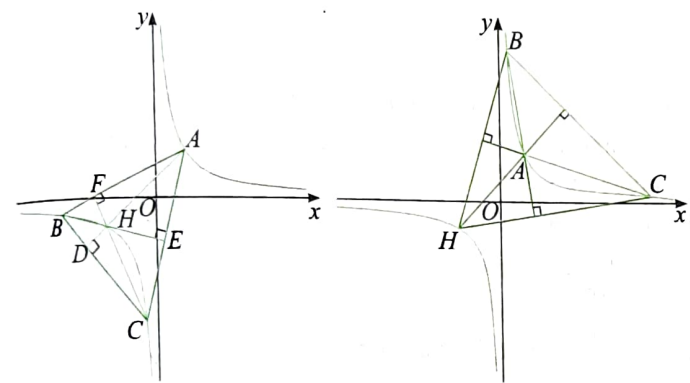
Let the vertices of triangle ABC which are lying on the hyperbola be

$$A\left(ct_1, \frac{c}{t_1}\right), B\left(ct_2, \frac{c}{t_2}\right) \text{ and } C\left(ct_3, \frac{c}{t_3}\right).$$

To get the orthocentre, we find the equations of two altitudes and solve.

$$\text{Slope of } BC = \frac{\frac{c}{t_3} - \frac{c}{t_2}}{ct_3 - ct_2} = -\frac{1}{t_2 t_3}$$

$$\therefore \text{Slope of altitude } AD = t_2 t_3$$



Equation of altitude AD is

$$y - \frac{c}{t_1} = t_2 t_3 (x - ct_1)$$

$$\text{or } t_1 y - c = t_1 t_2 t_3 x - ct_1^2 t_2 t_3 \quad (1)$$

Similarly, equation of altitude BE is

$$t_2 y - c = t_1 t_2 t_3 x - ct_1 t_2^2 t_3 \quad (2)$$

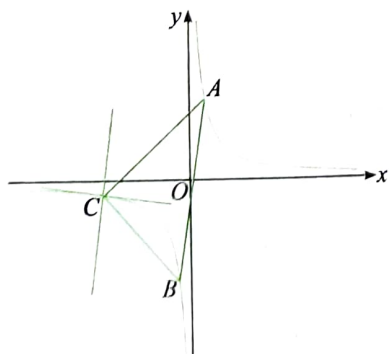
$$\text{Solving (1) and (2), we get the orthocentre } H\left(-\frac{c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right).$$

Clearly, orthocentre lies on the same hyperbola, i.e., on $xy = c^2$.

ILLUSTRATION 7.51

If A, B and C are three points on the hyperbola $xy = c^2$ such that AB subtends a right angle at C , then prove that AB is parallel to normal to hyperbola at point C .

Sol.



Let the coordinates of A, B and C be $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$, respectively.

$$\text{Slope of } CA = \frac{\frac{c}{t_3} - \frac{c}{t_1}}{ct_3 - ct_1} = -\frac{1}{t_3 t_1}$$

$$\text{Slope of } CB = -\frac{1}{t_2 t_3}$$

Given that $CA \perp CB$.

$$\therefore \left(-\frac{1}{t_1 t_3}\right) \times \left(-\frac{1}{t_2 t_3}\right) = -1$$

$$\Rightarrow \left(-\frac{1}{t_3^2}\right) \times \left(-\frac{1}{t_1 t_2}\right) = -1 \quad (1)$$

$$\text{Slope of } AB = -\frac{1}{t_1 t_2}$$

Also, differentiating the equation of curve, we get

$$\frac{dy}{dx} = -\frac{y}{x} = -\frac{c/t}{ct} = -\frac{1}{t^2}$$

$$\therefore \text{Slope of tangent to curve at point } C = -\frac{1}{t_3^2}$$

Hence, from Eq. (1), normal at C is parallel to AB .

ILLUSTRATION 7.52

Prove that product of parameters of four concyclic points on the hyperbola $xy = c^2$ is 1. Also, prove that the mean of these four concyclic points bisects the distance between the centres of the hyperbola and the circle.

Sol. Given equation of hyperbola is $xy = c^2$.

Let four concyclic points on the hyperbola be $(x_i, y_i); i = 1, 2, 3, 4$.

Let the equation of the circle through points A, B, C and D be

$$x^2 + y^2 + 2gx + 2fy + d = 0 \quad (1)$$

Solving circle and hyperbola, we get

$$x^2 + \frac{c^4}{x^2} + 2gx + 2f \cdot \frac{c^2}{x} + d = 0$$

$$\Rightarrow x^4 + 2gx^3 + dx^2 + 2fc^2x + c^4 = 0 \quad (2)$$

$$\therefore \text{Product of roots, } x_1 x_2 x_3 x_4 = c^4$$

$$\text{Now, } (x_i, y_i) \equiv \left(ct_i, \frac{c}{t_i}\right)$$

$$\therefore (ct_1)(ct_2)(ct_3)(ct_4) = c^4$$

$$\Rightarrow t_1 t_2 t_3 t_4 = 1$$

$$\text{Now, mean of points } (x_i, y_i); i = 1, 2, 3, 4, \text{ is } \left(\frac{\sum_{i=1}^4 x_i}{4}, \frac{\sum_{i=1}^4 y_i}{4}\right).$$

From Eq. (2),

$$x_1 + x_2 + x_3 + x_4 = -2g \quad (3)$$

$$\Rightarrow \frac{\sum_{i=1}^4 x_i}{4} = -\frac{g}{2}$$

$$\text{Also, } \sum_{i=1}^4 y_i = \sum_{i=1}^4 \frac{c^2}{x_i} = \frac{c^2 \sum_{i=1}^4 x_i x_2 x_3}{x_1 x_2 x_3 x_4} = \frac{c^2}{c^4} (-2fc^2) = -2f$$

$$\therefore \frac{\sum y_i}{4} = -\frac{f}{2}$$

$$\text{Thus, } \left(\frac{\sum_{i=1}^4 x_i}{4}, \frac{\sum_{i=1}^4 y_i}{4} \right) \equiv \left(-\frac{g}{2}, -\frac{f}{2} \right).$$

CONCEPT APPLICATION EXERCISE 7.6

- Find the asymptotes and axes of hyperbola having equation $xy - 3y - 4x + 7 = 0$.
- The chord PQ of the rectangular hyperbola $xy = a^2$ meets the x -axis at A . Point C is the midpoint of PQ and O is the origin. Prove that the triangle ACO is isosceles.
- If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, then prove that the orthocentre of triangle PQR is $(-x_4, -y_4)$.
- Prove that four normals can be drawn from a point to hyperbola $xy = c^2$. If the sum of the slopes of the normal from this point to the hyperbola is equal to λ ($\lambda \in \mathbb{R}^+$), then find the locus of point P .

ANSWERS

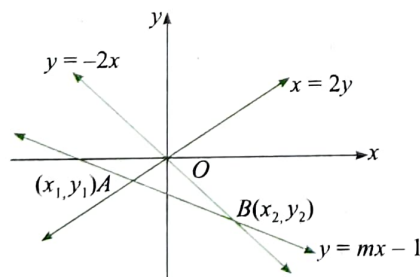
- Asymptotes: $x - 3 = 0$, $y - 4 = 0$
Transverse axis: $x - y + 1 = 0$
Conjugate axis: $x + y - 7 = 0$
- $x^2 = \lambda c^2$

Solved Examples

EXAMPLE 7.1

A variable line $y = mx - 1$ cuts the lines $x = 2y$ and $y = -2x$ at points A and B . Prove that the locus of the centroid of triangle OAB (O being the origin) is a hyperbola passing through the origin.

Sol.



Solving the variable line $y = mx - 1$ with $x = 2y$, we get

$$x_1 = \frac{2}{2m-1} \quad (1)$$

Solving with $y = -2x$, we get

$$x_2 = \frac{1}{m+2} \quad (2)$$

$$\text{Now, } y_1 + y_2 = m(x_1 + x_2) - 2$$

Let the centroid of triangle OAB be (h, k) . Then,

$$h = \frac{x_1 + x_2}{3}$$

$$\text{and } k = \frac{y_1 + y_2}{3} = \frac{m(x_1 + x_2) - 2}{3}$$

$$\text{or } m = \frac{3k+2}{3h}$$

$$\text{So, } 3h = x_1 + x_2 = \frac{2}{2\left(\frac{3k+2}{3h}\right) - 1} + \frac{1}{\left(\frac{3k+2}{3h}\right) + 2} \quad [\text{Using (1) and (2)}]$$

$$\text{or } \frac{2}{6k-3h+4} + \frac{1}{6h+3k+2} = 1$$

Simplifying, we get the final locus as $6x^2 - 9xy - 6y^2 - 3x - 4y = 0$ which is a hyperbola passing through the origin, as $h^2 > ab$ and $\Delta \neq 0$.

EXAMPLE 7.2

Let P be a point on the hyperbola $x^2 - y^2 = a^2$, where a is a parameter, such that P is nearest to the line $y = 2x$. Find the locus of P .

Sol. Consider any point $P(a \sec \theta, a \tan \theta)$ on $x^2 - y^2 = a^2$.

This point will be nearest to $y = 2x$ if the tangent at this point is parallel to $y = 2x$.

Differentiating $x^2 - y^2 = a^2$ w.r.t. x , we get

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\therefore \left[\frac{dy}{dx} \right] (a \sec \theta, a \tan \theta) = \operatorname{cosec} \theta$$

The slope of $y = 2x$ is 2. Therefore,

$$\operatorname{cosec} \theta = 2 \text{ or } \theta = \frac{\pi}{6}$$

$$\text{Thus, } P \equiv \left(a \sec \frac{\pi}{6}, a \tan \frac{\pi}{6} \right)$$

$$\equiv \left(\frac{2a}{\sqrt{3}}, \frac{a}{\sqrt{3}} \right) \equiv (h, k)$$

$$\text{i.e., } h = \frac{2a}{\sqrt{3}} \text{ and } k = \frac{a}{\sqrt{3}} \text{ or } k = \frac{h}{2}$$

So, the required locus is $2y - x = 0$.

EXAMPLE 7.3

Show that midpoints of focal chords of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on another hyperbola having same eccentricity.

Sol. Let the chord AB be bisected at point $P(h, k)$.

So, equation of chord AB is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \quad (\text{Using } T = S_1)$$

Let it pass through the focus $(ae, 0)$.

$$\therefore \frac{he}{a} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$$

Therefore, locus of point P is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$$

$$\Rightarrow \frac{1}{a^2}[x^2 - aex] - \frac{y^2}{b^2} = 0$$

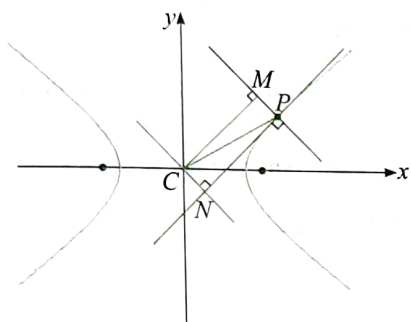
$$\Rightarrow \frac{\left(x - \frac{ae}{2}\right)^2}{a^2} - \frac{y^2}{b^2} = \frac{e^2}{4}$$

This is also hyperbola with eccentricity e .

EXAMPLE 7.4

From the centre C of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, perpendicular CN is drawn on any tangent to it at the point P in the first quadrant. Find the maximum area of triangle CPN .

Sol.



Equation of tangent at $P(4 \sec \theta, 3 \tan \theta)$ is

$$(3 \sec \theta)x - (4 \tan \theta)y - 12 = 0$$

$$\therefore CN = \frac{12}{\sqrt{9 \sec^2 \theta + 16 \tan^2 \theta}}$$

Equation of normal at P is

$$4x \cos \theta + 3y \cot \theta = 25$$

$$\therefore CM = \frac{25}{\sqrt{16 \cos^2 \theta + 9 \cot^2 \theta}}$$

$$\therefore \text{Area of triangle } CPN = \frac{1}{2} \times CM \times CN$$

$$= \frac{1}{2} \times \frac{25 \times 12}{\sqrt{144 + 81 \operatorname{cosec}^2 \theta + 256 \sin^2 \theta + 144}}$$

$$= \frac{150}{\sqrt{(9 \operatorname{cosec} \theta - 16 \sin \theta)^2 + 576}}$$

$$\leq \frac{150}{\sqrt{0 + 576}}$$

$$\therefore \text{Maximum area of triangle } CPN = \frac{150}{24} = \frac{25}{4} \text{ sq. units}$$

EXAMPLE 7.5

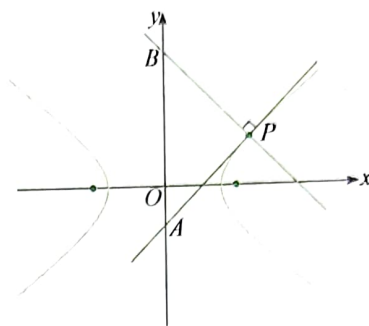
Semi transverse axis of hyperbola is 5. Tangent at point P and normal to this tangent meet conjugate axis at A and B , respectively. The circle on AB as diameter passes through two fixed points, the distance between which is 20. Find the eccentricity of hyperbola.

Sol. Consider hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Tangent to it at any point $P(a \sec \theta, b \tan \theta)$ is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

It meets y -axis at $A(0, -b \cot \theta)$.



Normal at point P is

$$ax \cos \theta + by \cot \theta = a^2 e^2$$

It meets y -axis at $B\left(0, \frac{a^2 e^2}{b} \tan \theta\right)$.

Now, circle with diameter as AB is

$$x^2 + (y + b \cot \theta)\left(y - \frac{a^2 e^2}{b} \tan \theta\right) = 0$$

$$\Rightarrow x^2 + y^2 - (a^2 e^2) + \left(b \cot \theta - \frac{a^2 e^2}{b} \tan \theta\right)y = 0$$

This circle passes through fixed points $(\pm ae, 0)$, distance between which is $2ae$.

$$2ae = 20$$

$$\therefore e = 2$$

(given)

EXAMPLE 7.6

Find the exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2 + \alpha^2 y^2 = \alpha^2$ such that the portion of the tangent intercepted by the hyperbola $\alpha^2 x^2 - y^2 = 1$ subtends a right angle at the centre of the curves.

Sol. Equation of tangent at point $P(\alpha \cos \theta, \sin \theta)$ is

$$\frac{x}{\alpha} \cos \theta + \frac{y}{1} \sin \theta = 1 \quad (1)$$

Let it cut the hyperbola at points P and Q .

Homogenising hyperbola $\alpha^2 x^2 - y^2 = 1$ w.r.t. line (1), we get

$$\alpha^2 x^2 - y^2 = \left(\frac{x}{\alpha} \cos \theta + y \sin \theta\right)^2$$

This is an equation of pair of straight lines OP and OQ .

Given that $\angle POQ = 90^\circ$.

$$\therefore \text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\Rightarrow \alpha^2 - \frac{\cos^2 \theta}{\alpha^2} - 1 - \sin^2 \theta = 0$$

$$\Rightarrow \alpha^2 - \frac{\cos^2 \theta}{\alpha^2} - 1 - 1 + \cos^2 \theta = 0$$

$$\Rightarrow \cos^2 \theta = \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1}$$

Now, $0 \leq \cos^2 \theta \leq 1$

$$\Rightarrow 0 \leq \frac{\alpha^2(2-\alpha^2)}{\alpha^2-1} \leq 1$$

Solving, we get

$$\alpha^2 \in \left[\frac{\sqrt{5}+1}{2}, 2 \right]$$

EXAMPLE 7.7

Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendicular drawn from the foci on the normal at the same point.

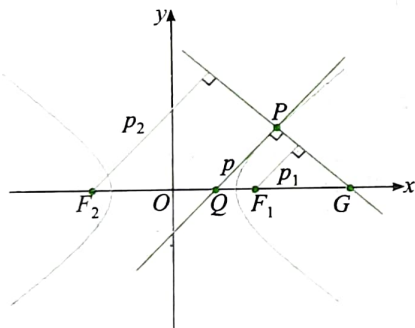
Sol. Equation of tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at point $P(a \sec \theta, b \tan \theta)$ is

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1 \quad (1)$$

Equation of normal at point P is

$$\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 e^2$$

Tangent meets x -axis at $Q(a \cos \theta, 0)$ and normal meets x -axis at $G(e^2 a \sec \theta, 0)$.



In the figure, $PQ = p$ and length of perpendiculars from foci F_1 and F_2 on the normal are p_1 and p_2 , respectively.

$$\begin{aligned} \text{Now, } \frac{p}{p_1} &= \frac{QG}{F_1 G} \\ &= \frac{e^2 a \sec \theta - a \cos \theta}{e^2 a \sec \theta - ae} \\ &= \frac{e^2 - \cos^2 \theta}{e^2 - e \cos \theta} = \frac{e + \cos \theta}{e} \end{aligned}$$

$$\therefore \frac{p}{p_1} = 1 + \frac{\cos \theta}{e} \quad (2)$$

Similarly, we get

$$\frac{p}{p_2} = 1 - \frac{\cos \theta}{e} \quad (3)$$

Adding (2) and (3), we get

$$\begin{aligned} \therefore \frac{p}{p_1} + \frac{p}{p_2} &= 2 \\ \Rightarrow \frac{2}{p} &= \frac{1}{p_1} + \frac{1}{p_2} \end{aligned}$$

Thus, p_1, p and p_2 are in H.P.

EXAMPLE 7.8

If one axis of varying hyperbola is fixed in its magnitude and position, then prove that the locus of the point of contact of a tangent drawn to it from a fixed point on the other axis is a parabola.

Sol. Consider hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a is fixed.

Let the coordinate of point P on the hyperbola be $(a \sec \theta, b \tan \theta) \equiv (h, k)$.

Equation of tangent to hyperbola at point P is

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1 \quad (1)$$

Let this tangent pass through fixed point $(0, c)$ on conjugate axis.

$$\therefore -c \tan \theta = b \quad (2)$$

From $a \sec \theta = h$ and $b \tan \theta = k$ and using (2), we get

$$\frac{k}{c} = -\tan^2 \theta \text{ and } \frac{h^2}{a^2} = \sec^2 \theta$$

Adding, we get

$$\frac{k}{c} + \frac{h^2}{a^2} = 1$$

$$\Rightarrow h^2 = \frac{a^2}{c}(c - k)$$

Hence, required locus is $x^2 = -\frac{a^2}{c}(y - c)$, which is a parabola.

EXAMPLE 7.9

If normal at P to a hyperbola of eccentricity 2 intersects its transverse and conjugate axes at Q and R , respectively, then prove that the locus of midpoint of QR is a hyperbola. Find the eccentricity of this hyperbola.

Sol. Consider hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Normal to any point $P(a \sec \theta, b \tan \theta)$ on the hyperbola is

$$\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2$$

It meets x -axis at $Q\left(\frac{(a^2 + b^2) \sec \theta}{a}, 0\right)$ and y -axis at $R\left(0, \frac{(a^2 + b^2) \tan \theta}{b}\right)$.

Let midpoint of QR be $T(h, k)$.

$$\text{Then } h = \frac{(a^2 + b^2) \sec \theta}{2a} \text{ and } k = \frac{(a^2 + b^2) \tan \theta}{2b}.$$

$$2h = ae^2 \sec \theta \text{ and } 2k = \frac{a^2 e^2}{b} \tan \theta$$

$$\frac{4h^2}{a^2 e^4} - \frac{4k^2}{\frac{a^4 e^4}{b^2}} = 1$$

$$\Rightarrow \frac{x^2}{\frac{a^2 e^4}{4}} - \frac{y^2}{\frac{a^4 e^4}{4b^2}} = 1, \text{ which is required locus.}$$

Clearly, it is hyperbola having eccentricity 'E' given by

$$E^2 = 1 + \frac{\frac{a^4 e^4}{4b^2}}{\frac{a^2 e^4}{4}} = 1 + \frac{a^2}{b^2}$$

$$= 1 + \frac{1}{e^2 - 1}$$

$$E = \sqrt{\frac{e^2}{e^2 - 1}} = \frac{2}{\sqrt{3}}$$

(as given that $e = 2$)

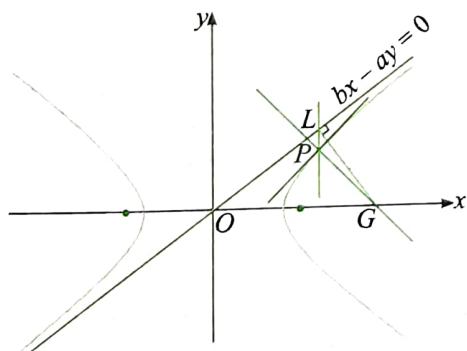
EXAMPLE 7.10

A normal is drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P which meets the transverse axis at G . If perpendicular from G on the asymptote meets it at L , then show that LP is parallel to conjugate axis.

Sol. Equation of normal at point $P(x_1, y_1)$ is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 e^2$$

It meets x -axis at $G(e^2 x_1, 0)$.



Now, GL is perpendicular to the asymptote $bx - ay = 0$.

So, equation of GL is

$$ax + by = ae^2 x_1 \quad (1)$$

Solving this line with asymptote $bx - ay = 0$, we get $x = x_1$ for point L .

Thus, abscissa of P and L are the same.

Therefore, LP is parallel to conjugate axis.

EXAMPLE 7.11

Curves $(x - 1)(y - 2) = 5$ and $(x - 1)^2 + (y + 2)^2 = r^2$ intersect at four points A, B, C and D . If centroid of $\triangle ABC$ lies on line $y = 3x - 4$, then find the locus of point D .

Sol. Given hyperbola is

$$(x - 1)(y - 2) = 5 \quad (1)$$

and circle is

$$(x - 1)^2 + (y + 2)^2 = r^2 \quad (2)$$

These curves intersect at four points A, B, C and D .

We know that mean of these four points is midpoint of centres of curves.

$$\sum_{i=1}^4 x_i = \frac{1+1}{2} = 1 \text{ and } \sum_{i=1}^4 y_i = \frac{2-2}{2} = 0,$$

where $(x_i, y_i); i = 1, 2, 3, 4$ are points A, B, C and D .

$$\therefore \sum_{i=1}^4 x_i = 4 \text{ and } \sum_{i=1}^4 y_i = 0$$

$$\therefore x_1 + x_2 + x_3 = 4 - x_4 \text{ and } y_1 + y_2 + y_3 = -y_4$$

So, centroid of triangle ABC is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right) \equiv \left(\frac{4 - x_4}{3}, \frac{-y_4}{3} \right)$$

Now, given that centroid lies on the line $y = 3x - 4$.

$$\therefore \frac{-y_4}{3} = 3 \left(\frac{4 - x_4}{3} \right) - 4$$

$$\therefore y_4 = 3x_4$$

Therefore, $y = 3x$, which is locus of point D .

Exercises

Single Correct Answer Type

- If the distance between the foci and the distance between the two directrices of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are in the ratio 3 : 2, then $b : a$ is
 (1) 1 : $\sqrt{2}$ (2) $\sqrt{3} : \sqrt{2}$
 (3) 1 : 2 (4) 2 : 1
- There is a point P on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ such that its distance from the right directrix is the average of its distances from the two foci. Then the x -coordinate of P is
 (1) $-64/5$ (2) $-32/9$
 (3) $-64/9$ (4) none of these
- The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents
 (1) no locus if $k > 0$ (2) an ellipse if $k < 0$
 (3) a point if $k = 0$ (4) a hyperbola if $k > 0$
- Let a and b be nonzero real numbers. Then the equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents
 (1) four straight lines, when $c = 0$ and a, b are of the same sign
 (2) two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a
 (3) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a
 (4) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a
- Which of the following is independent of α in the hyperbola $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1$, ($0 < \alpha < \pi/2$)?
 (1) Eccentricity (2) Abscissa of foci
 (3) Directrix (4) Vertex
- Which of the following pairs may represent the eccentricities of two conjugate hyperbolas, for all $\alpha \in (0, \pi/2)$?
 (1) $\sin \theta, \cos \theta$ (2) $\tan \theta, \cot \theta$
 (3) $\sec \theta, \operatorname{cosec} \theta$ (4) $1 + \sin \theta, 1 + \cos \theta$
- If a variable line has its intercepts on the coordinate axes e and e' , where $e/2$ and $e'/2$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where $r =$
 (1) 1 (2) 2
 (3) 3 (4) cannot be decided
- A hyperbola having the transverse axis of length $2 \sin \theta$ is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is
 (1) $x^2 \operatorname{cosec}^2 \theta - y^2 \sec^2 \theta = 1$
 (2) $x^2 \sec^2 \theta - y^2 \operatorname{cosec}^2 \theta = 1$
 (3) $x^2 \sin^2 \theta - y^2 \cos^2 \theta = 1$
 (4) $x^2 \cos^2 \theta - y^2 \sin^2 \theta = 1$
- If the distances of one focus of hyperbola from its directrices are 5 and 3, then its eccentricity is
 (1) $\sqrt{2}$ (2) 2 (3) 4 (4) 8
- Let the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ be confocal ($a > A$ and $a > b$) having the foci at S_1 and S_2 . Also, let P be their point of intersection. Then S_1P and S_2P are the roots of quadratic equation
 (1) $x^2 + 2ax + (a^2 - A^2) = 0$
 (2) $x^2 - 2Ax + (a^2 + A^2) = 0$
 (3) $x^2 - 2Ax + (a^2 - A^2) = 0$
 (4) $x^2 - 2ax + (a^2 - A^2) = 0$
- Two tangents are drawn from a point on hyperbola $x^2 - y^2 = 5$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. If they make angles α and β with x -axis, then
 (1) $\alpha - \beta = \pm \frac{\pi}{2}$ (2) $\alpha + \beta = \frac{\pi}{2}$
 (3) $\alpha + \beta = \pi$ (4) $\alpha + \beta = 0$
- Equation of the rectangular hyperbola whose focus is $(1, -1)$ and the corresponding directrix is $x - y + 1 = 0$ is
 (1) $x^2 - y^2 = 1$ (2) $xy = 1$
 (3) $2xy - 4x + 4y + 1 = 0$ (4) $2xy + 4x - 4y - 1 = 0$
- If two circles $(x + 4)^2 + y^2 = 1$ and $(x - 4)^2 + y^2 = 9$ are touched externally by a variable circle, then locus of centre of variable circle is
 (1) $\frac{x^2}{15} - \frac{y^2}{1} = 1$ (2) $\frac{x^2}{4} - \frac{y^2}{12} = 1$
 (3) $\frac{x^2}{1} - \frac{y^2}{15} = 1$ (4) $\frac{x^2}{12} - \frac{y^2}{4} = 1$
- If the vertex of a hyperbola bisects the distance between its center and the corresponding focus, then the ratio of the square of its conjugate axis to the square of its transverse axis is
 (1) 2 (2) 4 (3) 6 (4) 3
- The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is
 (1) $4/3$ (2) $4/\sqrt{3}$
 (3) $2/\sqrt{3}$ (4) none of these
- Let LL' be the latus rectum through the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and A' be the farther vertex. If $\Delta A'LL'$ is equilateral, then the eccentricity of the hyperbola is
 (1) $\sqrt{3}$ (2) $\sqrt{3} + 1$
 (3) $(\sqrt{3} + 1)/\sqrt{2}$ (4) $(\sqrt{3} + 1)/\sqrt{3}$
- The eccentricity of the conjugate hyperbola of the hyperbola $x^2 - 3y^2 = 1$ is
 (1) 2 (2) $2/\sqrt{3}$ (3) 4 (4) $4/5$

18. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}t = 0$ and $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is

- (1) $\sqrt{3}$ (2) 2
(3) $2/\sqrt{3}$ (4) $4/3$

19. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then a value of α is

- (1) $\pi/6$ (2) $\pi/4$
(3) $\pi/3$ (4) $\pi/2$

20. The equations of the transverse and conjugate axes of a hyperbola are, respectively, $x + 2y - 3 = 0$ and $2x - y + 4 = 0$, and their respective lengths are $\sqrt{2}$ and $2/\sqrt{3}$. The equation of the hyperbola is

- (1) $\frac{2}{5}(x + 2y - 3)^2 - \frac{3}{5}(2x - y + 4)^2 = 1$
(2) $\frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$
(3) $2(2x - y + 4)^2 - 3(x + 2y - 3)^2 = 1$
(4) $2(x + 2y - 3)^2 - 3(2x - y + 4)^2 = 1$

21. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A . Let B be one of the endpoints of its latus rectum. If C is the focus of the hyperbola nearest to the point A , then the area of triangle ABC is

- (1) $1 - \sqrt{2/3}$ (2) $\sqrt{3/2} - 1$
(3) $1 + \sqrt{2/3}$ (4) $\sqrt{3/2} + 1$

22. Let two points P and Q lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose center C be such that CP is perpendicular to CQ , $a < b$. Then the value of $\frac{1}{CP^2} + \frac{1}{CQ^2}$ is

- (1) $\frac{b^2 - a^2}{2ab}$ (2) $\frac{1}{a^2} + \frac{1}{b^2}$
(3) $\frac{2ab}{b^2 - a^2}$ (4) $\frac{1}{a^2} - \frac{1}{b^2}$

23. The angle between the lines joining the origin to the points of intersection of the line $\sqrt{3}x + y = 2$ and the curve $y^2 - x^2 = 4$ is

- (1) $\tan^{-1}(2/\sqrt{3})$ (2) $\pi/6$
(3) $\tan^{-1}(\sqrt{3}/2)$ (4) $\pi/2$

24. A variable chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, ($b > a$), subtends a right angle at the center of the hyperbola if this chord touches

- (1) a fixed circle concentric with the hyperbola
(2) a fixed ellipse concentric with the hyperbola
(3) a fixed hyperbola concentric with the hyperbola
(4) a fixed parabola having vertex at $(0, 0)$

25. If the distance between two parallel tangents drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{49} = 1$ is 2, then their slope is equal to

- (1) $\pm 5/2$ (2) $\pm 4/5$
(3) $\pm 7/2$ (4) none of these

26. If $ax + by = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $a^2 - b^2$ is equal to

- (1) $1/a^2 e^2$ (2) $a^2 e^2$
(3) $b^2 e^2$ (4) none of these

27. A tangent drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(\frac{\pi}{6})$ forms a triangle of area $3a^2$ sq. units with the coordinate axes. Then the square of its eccentricity is

- (1) 15 (2) 24 (3) 17 (4) 14

28. If the values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of the equation $x^2 - (a + b)x - 4 = 0$, then the value of $(a + b)$ is equal to

- (1) 2 (2) 4
(3) zero (4) none of these

29. The locus of a point whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$ is a/an

- (1) ellipse (2) circle
(3) hyperbola (4) parabola

30. The sides AC and AB of a $\triangle ABC$ touch the conjugate hyperbola of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the vertex A lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the side BC must touch

- (1) parabola (2) circle
(3) hyperbola (4) ellipse

31. The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$, is

- (1) zero (2) 1 (3) 2 (4) 4

32. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point $(0, -b)$ and the normal at P passes through the point $(2a\sqrt{2}, 0)$. Then the eccentricity of the hyperbola is

- (1) 2 (2) $\sqrt{2}$ (3) 3 (4) $\sqrt{3}$

33. The locus of the feet of the perpendiculars drawn from either focus on a variable tangent to the hyperbola $16x^2 - 9y^2 = 1$ is

- (1) $x^2 + y^2 = 9$ (2) $x^2 + y^2 = 1/9$
(3) $x^2 + y^2 = 7/144$ (4) $x^2 + y^2 = 1/16$

34. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T . If O is the center of the hyperbola, then, $OT \cdot ON$ is equal to

- (1) e^2 (2) a^2 (3) b^2 (4) b^2/a^2

35. The coordinates of a point on the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$ which is nearest to the line $3x + 2y + 1 = 0$ are
 (1) (6, 3) (2) (-6, -3)
 (3) (-6, 3) (4) (6, -3)
36. The tangent at a point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets one of the directrix at F . If PF subtends an angle θ at the corresponding focus, then $\theta =$
 (1) $\pi/4$ (2) $\pi/2$
 (3) $3\pi/4$ (4) π
37. The locus of a point, from where the tangents to the rectangular hyperbola $x^2 - y^2 = a^2$ contain an angle of 45° , is
 (1) $(x^2 + y^2)^2 + a^2(x^2 - y^2) = 4a^2$
 (2) $2(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^2$
 (3) $(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^4$
 (4) $(x^2 + y^2)^2 + a^2(x^2 - y^2) = a^4$
38. If tangents PQ and PR are drawn from a variable point P to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, ($a > b$), so that the fourth vertex S of parallelogram $PQSR$ lies on the circumcircle of triangle PQR , then the locus of P is
 (1) $x^2 + y^2 = b^2$ (2) $x^2 + y^2 = a^2$
 (3) $x^2 + y^2 = a^2 - b^2$ (4) none of these
39. The number of points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 3$ from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$ is/are
 (1) 0 (2) 2
 (3) 3 (4) 4
40. If a ray of light incident along the line $3x + (5 - 4\sqrt{2})y = 15$ gets reflected from the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then its reflected ray goes along the line
 (1) $x\sqrt{2} - y + 5 = 0$ (2) $\sqrt{2}y - x + 5 = 0$
 (3) $\sqrt{2}y - x - 5 = 0$ (4) none of these
41. The chords of contact of a point P w.r.t. a hyperbola and its auxiliary circle are at right angle. Then the point P lies on
 (1) conjugate hyperbola (2) one of the directrix
 (3) asymptotes (4) none of these
42. The ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $a^2x^2 - y^2 = 4$ intersect at right angles. Then the equation of the circle through the points of intersection of two conics is
 (1) $x^2 + y^2 = 5$
 (2) $\sqrt{5}(x^2 + y^2) - 3x - 4y = 0$
 (3) $\sqrt{5}(x^2 + y^2) + 3x + 4y = 0$
 (4) $x^2 + y^2 = 25$
43. The locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ forms a triangle of constant area with the coordinate axes is
 (1) a straight line (2) a hyperbola
 (3) an ellipse (4) a circle
44. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is
 (1) $9x^2 - 8y^2 + 18x - 9 = 0$
 (2) $9x^2 - 8y^2 - 18x + 9 = 0$
 (3) $9x^2 - 8y^2 - 18x - 9 = 0$
 (4) $9x^2 - 8y^2 + 18x + 9 = 0$
45. If the tangent at point $P(h, k)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ cuts the circle $x^2 + y^2 = a^2$ at points $Q(x_1, y_1)$ and $R(x_2, y_2)$, then the value of $\frac{1}{y_1} + \frac{1}{y_2}$ is
 (1) $\frac{1}{k}$ (2) $\frac{2}{k}$
 (3) $\frac{ab}{k}$ (4) $\frac{a+b}{k}$
46. Let $P(a \sec \theta, b \tan \theta)$ and $Q(a \sec \phi, b \tan \phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q , then k is equal to
 (1) $\frac{a^2 + b^2}{a}$ (2) $-\left(\frac{a^2 + b^2}{a}\right)$
 (3) $\frac{a^2 + b^2}{b}$ (4) $-\left(\frac{a^2 + b^2}{b}\right)$
47. A normal to the hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$ has equal intercepts on the positive x - and y -axes. If this normal touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $a^2 + b^2$ is equal to
 (1) 5 (2) 25
 (3) 16 (4) none of these
48. The portion of the asymptote of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (between the center and the tangent at vertex) in the first quadrant is cut by the line $y + \lambda(x - a) = 0$ (λ is a parameter). Then,
 (1) $\lambda \in R$ (2) $\lambda \in (0, \infty)$
 (3) $\lambda \in (-\infty, 0)$ (4) none of these
49. If the angle between the asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 120° and the product of perpendiculars drawn from the foci upon its any tangent is 9, then the locus of the point of intersection of perpendicular tangents of the hyperbola can be
 (1) $x^2 + y^2 = 6$ (2) $x^2 + y^2 = 9$
 (3) $x^2 + y^2 = 3$ (4) $x^2 + y^2 = 18$
50. Let any double ordinate PNP' of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ be produced on both sides to meet the asymptotes in Q and Q' . Then $PQ \cdot P'Q$ is equal to
 (1) 25 (2) 16
 (3) 41 (4) none of these

51. For a hyperbola whose center is at $(1, 2)$ and the asymptotes are parallel to lines $2x + 3y = 0$ and $x + 2y = 1$, the equation of the hyperbola passing through $(2, 4)$ is

- (1) $(2x + 3y - 5)(x + 2y - 8) = 40$
- (2) $(2x + 3y - 8)(x + 2y - 5) = 40$
- (3) $(2x + 3y - 8)(x + 2y - 5) = 30$
- (4) none of these

52. The asymptotes of the hyperbola $\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$ and $\frac{x^2}{a_2^2} - \frac{y^2}{b_2^2} = 1$ are perpendicular to each other. Then,

- (1) $a_1/a_2 = b_1/b_2$
- (2) $a_1a_2 = b_1b_2$
- (3) $a_1a_2 + b_1b_2 = 0$
- (4) $a_1 - a_2 = b_1 - b_2$

53. If $S = 0$ is the equation of the hyperbola $x^2 + 4xy + 3y^2 - 4x + 2y + 1 = 0$, then the value of k for which $S + K = 0$ represents its asymptotes is

- (1) 20
- (2) -16
- (3) -22
- (4) 18

54. If two distinct tangents can be drawn from the point $(\alpha, 2)$

on different branches of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$, then

- (1) $|\alpha| < 3/2$
- (2) $|\alpha| > 2/3$
- (3) $|\alpha| > 3$
- (4) none of these

55. A hyperbola passes through $(2, 3)$ and has asymptotes $3x - 4y + 5 = 0$ and $12x + 5y - 40 = 0$. Then, the equation of its transverse axis is

- (1) $77x - 21y - 265 = 0$
- (2) $21x - 77y + 265 = 0$
- (3) $21x - 77y - 265 = 0$
- (4) $21x + 77y - 265 = 0$

56. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. The area cut-off by the chord of contact on the asymptotes is equal to

- (1) $a/2$
- (2) ab
- (3) $2ab$
- (4) $4ab$

57. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is

- (1) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$
- (2) $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$
- (3) $2x^2 + 5xy + 2y^2 = 0$
- (4) none of these

58. The asymptotes of the hyperbola $xy = hx + ky$ are

- (1) $x - k = 0$ and $y - h = 0$
- (2) $x + h = 0$ and $y + k = 0$
- (3) $x - k = 0$ and $y + h = 0$
- (4) $x + k = 0$ and $y - h = 0$

59. The center of a rectangular hyperbola lies on the line $y = 2x$. If one of the asymptotes is $x + y + c = 0$, then the other asymptote is

$$(1) 6x + 3y - 4c = 0$$

$$(2) 3x + 6y - 5c = 0$$

$$(3) 3x - 6y - c = 0$$

$$(4) \text{ none of these}$$

60. The equation of a rectangular hyperbola whose asymptotes are $x = 3$ and $y = 5$ and passing through $(7, 8)$ is

- (1) $xy - 3y + 5x + 3 = 0$
- (2) $xy + 3y + 4x + 3 = 0$
- (3) $xy - 3y + 5x - 3 = 0$
- (4) $xy - 3y - 5x + 3 = 0$

61. If tangents OQ and OR are drawn to variable circles having radius r and the center lying on the rectangular hyperbola $xy = 1$, then the locus of the circumcenter of triangle OQR is (O being the origin)

- (1) $xy = 4$
- (2) $xy = 1/4$
- (3) $xy = 1$
- (4) none of these

62. Four points are such that the line joining any two points is perpendicular to the line joining other two points. If three points out of these lie on a rectangular hyperbola, then the fourth point will lie on

- (1) the same hyperbola
- (2) the conjugate hyperbola
- (3) one of the directrix
- (4) one of the asymptotes

63. If S_1 and S_2 are the foci of the hyperbola whose length of the transverse axis is 4 and that of the conjugate axis is 6, and S_3 and S_4 are the foci of the conjugate hyperbola, then the area of quadrilateral $S_1S_3S_2S_4$ is

- (1) 24
- (2) 26
- (3) 22
- (4) none of these

64. Suppose the circle having equation $x^2 + y^2 = 3$ intersects the rectangular hyperbola $xy = 1$ at points A, B, C , and D . The equation $x^2 + y^2 - 3 + \lambda(xy - 1) = 0$, $\lambda \in R$, represents

- (1) a pair of lines through the origin for $\lambda = -3$
- (2) an ellipse through A, B, C , and D for $\lambda = -3$
- (3) a parabola through A, B, C , and D for $\lambda = -3$
- (4) a circle for any $\lambda \in R$

65. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is

- (1) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$
- (2) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$
- (3) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$
- (4) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

66. The locus of the foot of the perpendicular from the center of the hyperbola $xy = 1$ on a variable tangent is

- (1) $(x^2 - y^2)^2 = 4xy$
- (2) $(x^2 + y^2)^2 = 2xy$
- (3) $(x^2 + y^2) = 4xy$
- (4) $(x^2 + y^2)^2 = 4xy$

67. The curve $xy = c$, ($c > 0$), and the circle $x^2 + y^2 = 1$ touch at two points. Then the distance between the points of contact is

- (1) 1
- (2) 2
- (3) $2\sqrt{2}$
- (4) none of these

68. Let C be a curve which is the locus of the point of intersection of lines $x = 2 + m$ and $my = 4 - m$. A circle $s \equiv (x - 2)^2 + (y + 1)^2 = 25$ intersects the curve C at four points: P, Q, R , and S . If O is center of the curve C , then $OP^2 + OQ^2 + OR^2 + OS^2$ is
- (1) 50 (2) 100 (3) 25 (4) 25/2

Multiple Correct Answers Type

- If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ at four points $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$, and $S(x_4, y_4)$, then
 - $x_1 + x_2 + x_3 + x_4 = 0$
 - $y_1 + y_2 + y_3 + y_4 = 0$
 - $x_1 x_2 x_3 x_4 = c^4$
 - $y_1 y_2 y_3 y_4 = c^4$
- The equation $(x - \alpha)^2 + (y - \beta)^2 = k(lx + my + n)^2$ represents
 - a parabola for $k < (l^2 + m^2)^{-1}$
 - an ellipse for $0 < k < (l^2 + m^2)^{-1}$
 - a hyperbola for $k > (l^2 + m^2)^{-1}$
 - a point circle for $k = 0$
- If $(5, 12)$ and $(24, 7)$ are the foci of a hyperbola passing through the origin, then (where e is eccentricity and LR is Latus Rectum)
 - $e = \sqrt{386}/12$
 - $e = \sqrt{386}/13$
 - LR = 121/6
 - LR = 121/3
- For the hyperbola $9x^2 - 16y^2 - 18x + 32y - 151 = 0$,
 - one of the directrix is $x = 21/5$
 - the length of latus rectum is 9/2
 - foci are $(6, 1)$ and $(-4, 1)$
 - the eccentricity is 5/4
- Let a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse. Also, the product of the eccentricities of the given ellipse and hyperbola is 1. Then,
 - the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
 - the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
 - the focus of the hyperbola is $(5, 0)$
 - the vertex of the hyperbola is $(5\sqrt{3}, 0)$
- If the foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ coincide with the foci of $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and the eccentricity of the hyperbola is 2, then
 - $a^2 + b^2 = 16$
 - there is no director circle to the hyperbola
 - the center of the director circle is $(0, 0)$
 - the length of latus rectum of the hyperbola is 12
- The differential equation $dy/dx = 3y/2x$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity
 - $\sqrt{3/5}$
 - $\sqrt{5/3}$
 - $\sqrt{2/5}$
 - $\sqrt{5/2}$
- If P is a point on a hyperbola, then
 - the locus of the excenter of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci) is tangent at vertex
 - the locus of the excenter of the circle described opposite to $\angle S'$ is a hyperbola
 - the locus of the excenter of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci) is a hyperbola
 - the locus of the excenter of the circle described opposite to $\angle S'$ is tangent at vertex
- If the ellipse $x^2 + 2y^2 = 4$ and the hyperbola $S = 0$ have same end points of the latus rectum, then the eccentricity of the hyperbola can be
 - $\operatorname{cosec} \frac{\pi}{4}$
 - $\operatorname{cosec} \frac{\pi}{3}$
 - $2 \sin \frac{\pi}{3} + \sin \frac{\pi}{4}$
 - $\sqrt{2} \sin \frac{\pi}{3} + \sin \frac{\pi}{4}$
- For which of the hyperbolas, can we have more than one pair of perpendicular tangents?
 - $\frac{x^2}{4} - \frac{y^2}{9} = 1$
 - $\frac{x^2}{4} - \frac{y^2}{9} = -1$
 - $x^2 - y^2 = 4$
 - $xy = 44$
- The lines parallel to the normal to the curve $xy = 1$ is/are
 - $3x + 4y + 5 = 0$
 - $3x - 4y + 5 = 0$
 - $4x + 3y + 5 = 0$
 - $3y - 4x + 5 = 0$
- From the point $(2, 2)$, tangents are drawn to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Then the point of contact lies in the
 - first quadrant
 - second quadrant
 - third quadrant
 - fourth quadrant
- For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn.
 - If $n = 1$, then $e = \sqrt{2}$.
 - If $n > 1$, then $0 < e < \sqrt{2}$.
 - If $n = 0$, then $e > \sqrt{2}$.
 - None of these.
- If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes at G and g and C is the center of the hyperbola, then
 - $PG = PC$
 - $Pg = PC$
 - $PG = Pg$
 - $Gg = 2PC$

15. A tangent to the hyperbola $y = \frac{x+9}{x+5}$ passing through the origin is

- (1) $x + 25y = 0$ (2) $x + y = 0$
(3) $5x - y = 0$ (4) $x - 25y = 0$

16. Tangents which are parallel to the line $2x + y + 8 = 0$ are drawn to hyperbola $x^2 - y^2 = 3$. The points of contact of these tangents is/are

- (1) (2, 1) (2) (2, -1)
(3) (-2, -1) (4) (-2, 1)

17. If the tangents to the hyperbola $x^2 - 9y^2 = 9$ are drawn from point (3, 2), then

- (1) equation of one of the tangents is $x = 3$
(2) equation of one of the tangents is $5x - 12y + 9 = 0$
(3) the area of triangle that these tangents form with their chord of contact is 12 sq. units
(4) the area of triangle that these tangents form with their chord of contact is 8 sq. units

18. Circles are drawn on the chords of the rectangular hyperbola $xy = 4$ parallel to the line $y = x$ as diameters. All such circles pass through two fixed points whose coordinates are

- (1) (2, 2) (2) (2, -2)
(3) (-2, 2) (4) (-2, -2)

Linked Comprehension Type

For Problems 1-3

Consider an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$). A hyperbola has

its vertices at the extremities of minor axis of the ellipse and the length of major axis of the ellipse is equal to the distance between the foci of hyperbola. Let e_1 and e_2 be the eccentricities of ellipse and hyperbola, respectively. Also, let A_1 be the area of the quadrilateral formed by joining all the foci and A_2 be the area of the quadrilateral formed by all the directrices.

1. The relation between e_1 and e_2 is given by

- (1) $e_1 e_2 = 1$ (2) $e_2^2 (1 - e_1^2) = 1$
(3) $e_1^2 (e_2^2 - 1) = 1$ (4) $e_1 e_2 (1 - e_1^2) = 1$

2. If the tangent drawn at a point P on ellipse passes through the focus of hyperbola, then the eccentric angle of point P is (P lies in 1st quadrant)

- (1) $\tan^{-1} \left(\frac{1}{\sqrt{1 - e_1^2}} \right)$ (2) $\tan^{-1} \left(\frac{e_1}{\sqrt{1 - e_1^2}} \right)$
(3) $\tan^{-1} \left(\frac{1}{\sqrt{e_2^2 - 1}} \right)$ (4) $\tan^{-1} \sqrt{1 - e_1^2}$

3. If $A_1 : A_2 = 2 : 1$, then $e_2 : e_1$ is equal to

- (1) 2 : 1 (2) 3 : 2
(3) $\sqrt{2} : 1$ (4) 5 : 2

For Problems 4-6

Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{a^2} = 1$ and the circle $x^2 + (y - 3)^2 = 9$.

Also, the given hyperbola and the ellipse $\frac{x^2}{41} + \frac{y^2}{16} = 1$ are orthogonal to each other.

4. Combined equation of pair of common tangents between the hyperbola and the circle is given by

- (1) $x^2 - y^2 = 0$ (2) $x^2 - 9 = 0$
(3) $9y^2 - 16x^2 = 0$ (4) No common tangent

5. The number of points on the hyperbola and the circle from which tangents drawn to the circle and the hyperbola, respectively, are perpendicular to each other is

- (1) 0 (2) 2 (3) 4 (4) 6

6. A variable line cuts the circle at points A and B and it cuts the hyperbola at points C and D . The locus of midpoint of AB such that tangents at points C and D always intersect each other at the directrix of the hyperbola, is

- (1) $x^2 + y^2 \pm 5x - 3y = 0$ (2) $x^2 + y^2 + 5x \pm 3y = 0$
(3) $x^2 - y^2 \pm 5x - 3y = 0$ (4) $x^2 - y^2 + 5x \pm 3y = 0$

For Problems 7-9

The locus of the foot of perpendicular from any focus of a hyperbola upon any tangent to the hyperbola is the auxiliary circle of the hyperbola. Consider the foci of a hyperbola as $(-3, -2)$ and $(5, 6)$ and the foot of perpendicular from the focus $(5, 6)$ upon a tangent to the hyperbola as $(2, 5)$.

7. The conjugate axis of the hyperbola is

- (1) $4\sqrt{11}$ (2) $2\sqrt{11}$ (3) $4\sqrt{22}$ (4) $2\sqrt{22}$

8. The directrix of the hyperbola corresponding to the focus $(5, 6)$ is

- (1) $2x + 2y - 1 = 0$ (2) $2x + 2y - 11 = 0$
(3) $2x + 2y - 7 = 0$ (4) $2x + 2y - 9 = 0$

9. The point of contact of the tangent with the hyperbola is

- (1) $(2/9, 31/3)$ (2) $(7/4, 23/4)$
(3) $(2/3, 9)$ (4) $(7/9, 7)$

For Problems 10-12

Let $P(x, y)$ be a variable point such that

$$|\sqrt{(x-1)^2 + (y-2)^2} - \sqrt{(x-5)^2 + (y-5)^2}| = 3$$

which represents a hyperbola.

10. The eccentricity e' of the corresponding conjugate hyperbola is

- (1) 5/3 (2) 4/3 (3) 5/4 (4) $3/\sqrt{7}$

11. The locus of the intersection of two perpendicular tangents to the hyperbola is

- (1) $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{55}{4}$
 (2) $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{25}{4}$
 (3) $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{7}{4}$
 (4) none of these

12. If the origin is shifted to the point $(3, 7/2)$ and the axes are rotated through an angle θ in clockwise sense so that the equation of the given hyperbola changes to the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then θ is

- (1) $\tan^{-1}(4/3)$ (2) $\tan^{-1}(3/4)$
 (3) $\tan^{-1}(5/3)$ (4) $\tan^{-1}(3/5)$

For Problems 13–15

In a hyperbola, the portion of the tangent intercepted between the asymptotes is bisected at the point of contact. Consider a hyperbola whose center is at the origin. A line $x + y = 2$ touches this hyperbola at $P(1, 1)$ and intersects the asymptotes at A and B such that $AB = 6\sqrt{2}$ units.

13. The equation of the pair of asymptotes is

- (1) $5xy + 2x^2 + 2y^2 = 0$ (2) $3x^2 + 4y^2 + 6xy = 0$
 (3) $2x^2 + 2y^2 - 5xy = 0$ (4) none of these

14. The angle subtended by AB at the center of the hyperbola is

- (1) $\sin^{-1} \frac{4}{5}$ (2) $\sin^{-1} \frac{2}{5}$
 (3) $\sin^{-1} \frac{3}{5}$ (4) none of these

15. The equation of the tangent to the hyperbola at $(-1, 7/2)$ is

- (1) $5x + 2y = 2$ (2) $3x + 2y = 4$
 (3) $3x + 4y = 11$ (4) none of these

For Problems 16–18

A point P moves such that the sum of the slopes of the normals drawn from it to the hyperbola $xy = 16$ is equal to the sum of ordinates of feet of normals. The locus of P is a curve C .

16. The equation of the curve C is

- (1) $x^2 = 4y$ (2) $x^2 = 16y$
 (3) $x^2 = 12y$ (4) $y^2 = 8x$

17. If the tangent to the curve C cuts the coordinate axes at A and B , then the locus of the middle point of AB is

- (1) $x^2 = 4y$ (2) $x^2 = 2y$
 (3) $x^2 + 2y = 0$ (4) $x^2 + 4y = 0$

18. The area of the equilateral triangle inscribed in the curve C having one vertex as the vertex of curve C is

- (1) $772\sqrt{3}$ sq. units (2) $776\sqrt{3}$ sq. units
 (3) $760\sqrt{3}$ sq. units (4) $768\sqrt{3}$ sq. units

For Problems 19–21

The vertices of $\triangle ABC$ lie on a rectangular hyperbola such that the orthocenter of the triangle is $(3, 2)$ and the asymptotes of the rectangular hyperbola are parallel to the coordinate axes. The two perpendicular tangents of the hyperbola intersect at the point $(1, 1)$.

19. The equation of the pair of asymptotes is

- (1) $xy - 1 = x - y$ (2) $xy + 1 = x + y$
 (3) $2xy = x + y$ (4) none of these

20. The equation of the rectangular hyperbola is

- (1) $xy = 2x + y - 2$ (2) $2xy = x + 2y + 5$
 (3) $xy = x + y + 1$ (4) none of these

21. The number of real tangents that can be drawn from the point $(1, 1)$ to the rectangular hyperbola is

- (1) 4 (2) 0 (3) 3 (4) 2

Matrix Match Type

1. Let the foci of the hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ be the vertices of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the foci of the ellipse be the vertices of the hyperbola. Let the eccentricities of the ellipse and hyperbola be e_E and e_H , respectively. Then match the following lists.

List I	List II
a. b/B is equal to	p. 1
b. $e_H + e_E$ can be	q. 2
c. If the angle between the asymptotes of the hyperbola is $2\pi/3$, then $4e_E$ is equal to	r. 3
d. If $e_E^2 = 1/2$ and (x, y) is the point of intersection of the ellipse and the hyperbola, then $3x^2/2y^2$ is	s. 4

2. Match the following lists:

List I	List II
a. The points common to the hyperbola $x^2 - y^2 = 9$ and the circle $x^2 + y^2 = 41$ are	p. $(-5, -4)$
b. Tangents are drawn from the point $(0, -9/4)$ to the hyperbola $x^2 - y^2 = 9$. Then the point of tangency may have coordinates	q. $(5, 4)$
c. The point which is diametrically opposite to point $(5, 4)$ with respect to the hyperbola $x^2 - y^2 = 9$ is	r. $(-5, 4)$
d. If P and Q lie on the hyperbola $x^2 - y^2 = 9$ such that the area of the isosceles triangle PQR , where $PR = QR$ is 10 sq. units and $R \equiv (0, -6)$, then P can have the co-ordinates	s. $(5, -4)$

1. $A(-2, 0)$ and $B(2, 0)$ are two fixed points and P is a point such that $PA - PB = 2$. Let S be the circle $x^2 + y^2 = r^2$. Then match the following lists:

List I	List II
a. If $r = 2$, then the number of points P satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	p. 2
b. If $r = 1$, then the number of points satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	q. 4
c. For $r = 2$, then the number of common tangents is	r. 0
d. For $r = 1/2$, then the number of common tangents is	s. 1

4. Match the following lists:

List I	List II
a. If z is a complex number such that $\text{Im}(z^2) = 3$, then the eccentricity of the locus is	p. $\sqrt{3}$
b. If the latus rectum of a hyperbola through one focus subtends an angle of 60° at the other focus, then its eccentricity is	q. 2
c. If $A \equiv (3, 0)$ and $B \equiv (-3, 0)$ and $PA - PB = 4$, then the eccentricity of conjugate hyperbola is	r. $\sqrt{2}$
d. If the angle between the asymptotes of a hyperbola is $\pi/3$, then the eccentricity of its conjugate hyperbola is	s. $\frac{3}{\sqrt{5}}$

5. If the ellipse $x^2 + k^2 y^2 = k^2 a^2$ is confocal with the hyperbola $x^2 - y^2 = a^2$, then match the following lists and choose the correct code.

List I	List II
a. Square of the ratio of eccentricities of hyperbola and ellipse is	p. 2
b. Ratio of major axis of ellipse and transverse axis of hyperbola is	q. 3
c. If ellipse and hyperbola cut each other at an angle θ , then the value of $2 \csc \theta$ is	r. $\frac{1}{\sqrt{3}}$
d. Ratio of length of latus rectum of ellipse and hyperbola is	s. $\sqrt{3}$

Codes:

a	b	c	d
(1) q	r	s	p
(2) r	p	s	q
(3) q	s	p	r
(4) s	r	q	p

Numerical Value Type

1. The eccentricity of the hyperbola

$$\left| \sqrt{(x-3)^2 + (y-2)^2} - \sqrt{(x+1)^2 + (y+1)^2} \right| = 1 \text{ is}$$

2. If $y = mx + c$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, having eccentricity 5, then the least positive integral value of m is _____.

3. Consider the graphs of $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$, where A is a positive constant and $x, y \in R$. The number of points in which the two graphs intersect is _____.

4. If $4(x - \sqrt{2})^2 + \lambda(y - \sqrt{3})^2 = 45$ and $(x - \sqrt{2})^2 - 4(y - \sqrt{3})^2 = 5$ cut orthogonally, then integral value of λ is _____.

5. If the hyperbola $x^2 - y^2 = 4$ is rotated by 45° in the anticlockwise direction about its center keeping the axis intact, then the equation of the hyperbola is $xy = a^2$, where a^2 is equal to _____.

6. Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x -axis. If $\tan \theta \cdot \tan \phi = 2$, then the value of $2\alpha^2 - \beta^2$ is _____.

7. The area of triangle formed by the tangents from the point $(3, 2)$ to the hyperbola $x^2 - 9y^2 = 9$ and the chord of contact w.r.t. the point $(3, 2)$ is _____.

8. Number of integral values of b for which tangent parallel to line $y = x + 1$ can be drawn to hyperbola $\frac{x^2}{5} - \frac{y^2}{b^2} = 1$ is _____.

9. If tangents drawn from the point $(a, 2)$ to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are perpendicular, then the value of a^2 is _____.

10. Radii of director circles of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $2r$ and r , respectively. If e_1 and e_2 are the eccentricities of ellipse and hyperbola, respectively, then $4e_2^2 - e_1^2 =$ _____.

11. If L is the length of the latus rectum of the hyperbola for which $x = 3$ and $y = 2$ are the equations of asymptotes and which passes through the point $(4, 6)$, then the value of $L/\sqrt{2}$ is _____.

12. If the angle between the asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{\pi}{3}$, then the eccentricity of conjugate hyperbola is _____.

13. If the chord $x \cos \alpha + y \sin \alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the center, and the diameter of the circle, concentric with the hyperbola, to which the given chord is a tangent is d , then the value of d is _____.

JEE MAIN

Single Correct Answer Type

1. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is

- (1) $4/\sqrt{3}$ (2) $2/\sqrt{3}$
(3) $\sqrt{3}$ (4) $4/3$ (JEE Main 2016)

2. A hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the tangent to this hyperbola at P also passes through the point

- (1) $(-\sqrt{2}, -\sqrt{3})$ (2) $(3\sqrt{2}, 2\sqrt{3})$
(3) $(2\sqrt{2}, 3\sqrt{3})$ (4) $(\sqrt{3}, \sqrt{2})$ (JEE Main 2017)

3. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q . If these tangents intersect at the point $T(0, 3)$ then the area (in sq. units) of ΔPTQ is

- (1) $36\sqrt{5}$ (2) $45\sqrt{5}$
(3) $54\sqrt{3}$ (4) $60\sqrt{3}$ (JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x -axis at $(9, 0)$, then the eccentricity of the hyperbola is

- (1) $\sqrt{5/2}$ (2) $\sqrt{3/2}$
(3) $\sqrt{2}$ (4) $\sqrt{3}$ (IIT-JEE 2011)

Multiple Correct Answers Type

1. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then

- (1) the equation of the ellipse is $x^2 + 2y^2 = 2$
(2) the foci of the ellipse are $(\pm 1, 0)$
(3) the equation of the ellipse is $x^2 + 2y^2 = 4$
(4) the foci of the ellipse are $(\pm\sqrt{2}, 0)$ (IIT-JEE 2009)

2. Let the eccentricity of the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

(1) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(2) a focus of the hyperbola is $(2, 0)$

(3) the eccentricity of the hyperbola is $\frac{2}{\sqrt{3}}$

(4) the equation of the hyperbola is $x^2 - 3y^2 = 3$ (IIT-JEE 2011)

3. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are

(1) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (2) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(3) $(3\sqrt{3}, -2\sqrt{2})$ (4) $(3\sqrt{3}, 2\sqrt{2})$ (IIT-JEE 2012)

4. Consider the hyperbola $H: x^2 - y^2 = 1$ and a circle S with centre $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x -axis at point M . If (l, m) is the centroid of the triangle PMN , then the correct expression(s) is (are)

(1) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(2) $\frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$ for $x_1 > 1$

(3) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(4) $\frac{dm}{dy_1} = \frac{1}{3}$ for $x_1 > 0$ (JEE Advanced 2015)

5. If $2x - y + 1 = 0$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right angled triangle?

- (1) $2a, 4, 1$ (2) $2a, 8, 1$
(3) $a, 4, 1$ (4) $a, 4, 2$

(JEE Advanced 2017)

Linked Comprehension Type

For Problems 1 and 2

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at points A and B . (IIT-JEE 2010)

1. The equation of a common tangent with positive slope to the circle as well as to the hyperbola is

- (1) $2x - \sqrt{5}y - 20 = 0$ (2) $2x - \sqrt{5}y + 4 = 0$
 (3) $3x - 4y + 8 = 0$ (4) $4x - 3y + 4 = 0$

2. The equation of the circle with AB as its diameter is

- (1) $x^2 + y^2 - 12x + 24 = 0$
 (2) $x^2 + y^2 + 12x + 24 = 0$
 (3) $x^2 + y^2 + 24x - 12 = 0$
 (4) $x^2 + y^2 - 24x - 12 = 0$

Matrix Match Type

1. Match the conic in List I with the statements/expressions in List II. (IIT-JEE 2009)

List I	List II
a. Circle	p. The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
b. Parabola	q. Point z in the complex plane satisfying $ z + 2 - z - 2 = \pm 3$
c. Ellipse	r. Points of the conic have parametric representation $x = \sqrt{3} \left(\frac{1 - t^2}{1 + t^2} \right), y = \frac{2t}{1 + t^2}$
d. Hyperbola	s. The eccentricity of the conic lies in the interval $1 \leq e < \infty$
	t. Points z in the complex plane satisfying $\operatorname{Re}(z + 1)^2 = z ^2 + 1$

For problems 2-4

Answer Q.2, Q.3, and Q.4 by appropriately matching the information given in the three columns of the following table.

Lists I, II and III contain conics, equation of tangents to the conics and points of contact, respectively.

List I	List II	List III
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m} \right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2 + 1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2 + 1}}, \frac{a}{\sqrt{m^2 + 1}} \right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2 - 1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2 + 1}}, \frac{1}{\sqrt{a^2m^2 + 1}} \right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2 + 1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2 - 1}}, \frac{-1}{\sqrt{a^2m^2 - 1}} \right)$

(JEE Advanced 2017)

2. If the tangent to a suitable conic (List I) at $\left(\sqrt{3}, \frac{1}{2} \right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only CORRECT combination?

- (1) (II) (iii) (R) (2) (IV) (iv) (S)
 (3) (IV) (iii) (S) (4) (II) (iv) (R)

3. If a tangent to a suitable conic (List I) is found to be $y = x + 8$ and its point of contact is $(8, 16)$, then which of the following options is the only CORRECT combination?

- (1) (III) (i) (P) (2) (III) (ii) (Q)
 (3) (II) (iv) (R) (4) (I) (ii) (Q)

4. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (List I) at the point of contact $(-1, 1)$, then which of the following options is the only CORRECT combination for obtaining its equation?

- (1) (II) (ii) (Q) (2) (III) (i) (P)
 (3) (I) (i) (P) (4) (I) (ii) (Q)

5. Let $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the xy -plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N . Let the area of the triangle LMN be $4\sqrt{3}$.

List I	List II
P. The length of the conjugate axis of H is	I. 8
Q. The eccentricity of H is	II. $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	III. $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	IV. 4

The correct option is:

- (1) $P \rightarrow IV; Q \rightarrow II; R \rightarrow I; S \rightarrow III$
 (2) $P \rightarrow IV; Q \rightarrow III; R \rightarrow I; S \rightarrow II$
 (3) $P \rightarrow IV; Q \rightarrow I; R \rightarrow III; S \rightarrow II$
 (4) $P \rightarrow III; Q \rightarrow IV; R \rightarrow II; S \rightarrow I$

Numerical Value Type

1. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x -axis, then the eccentricity of the hyperbola is _____. (IIT-JEE 2010)

Answers Key

EXERCISES

Single Correct Answer Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (1) | 2. (1) | 3. (3) | 4. (2) | 5. (2) |
| 6. (3) | 7. (2) | 8. (1) | 9. (2) | 10. (4) |
| 11. (2) | 12. (3) | 13. (3) | 14. (3) | 15. (3) |
| 16. (4) | 17. (1) | 18. (2) | 19. (2) | 20. (2) |
| 21. (2) | 22. (4) | 23. (3) | 24. (1) | 25. (1) |
| 26. (1) | 27. (3) | 28. (3) | 29. (3) | 30. (4) |
| 31. (1) | 32. (2) | 33. (4) | 34. (2) | 35. (3) |
| 36. (2) | 37. (3) | 38. (3) | 39. (1) | 40. (4) |
| 41. (3) | 42. (1) | 43. (2) | 44. (2) | 45. (2) |
| 46. (4) | 47. (4) | 48. (2) | 49. (4) | 50. (2) |
| 51. (2) | 52. (3) | 53. (3) | 54. (1) | 55. (4) |
| 56. (4) | 57. (1) | 58. (1) | 59. (4) | 60. (4) |
| 61. (2) | 62. (1) | 63. (2) | 64. (1) | 65. (1) |
| 66. (4) | 67. (2) | 68. (2) | | |

Multiple Correct Answers Type

- | | |
|-----------------------|------------------------|
| 1. (1), (2), (3), (4) | 2. (2), (3), (4) |
| 3. (1), (3) | 4. (1), (2), (3), (4) |
| 5. (1), (3) | 6. (1), (2), (4) |
| 7. (2), (4) | 8. (1), (2) |
| 9. (1), (4) | 10. (2) |
| 11. (2), (4) | 12. (3), (4) |
| 13. (1), (2), (3) | 14. (1), (2), (3), (4) |
| 15. (1), (2) | 16. (2), (4) |
| 17. (1), (2), (4) | 18. (1), (4) |

Linked Comprehension Type

- | | | | | |
|---------|---------|---------|---------|---------|
| 1. (2) | 2. (3) | 3. (4) | 4. (2) | 5. (3) |
| 6. (1) | 7. (4) | 8. (2) | 9. (3) | 10. (3) |
| 11. (4) | 12. (2) | 13. (1) | 14. (3) | 15. (2) |
| 16. (2) | 17. (3) | 18. (4) | 19. (2) | 20. (3) |
| 21. (4) | | | | |

Matrix Match Type

- $a \rightarrow p; b \rightarrow r, s; c \rightarrow q; d \rightarrow s$
- $a \rightarrow p, q, r, s; b \rightarrow q, r; c \rightarrow p; d \rightarrow p, s$
- $a \rightarrow p; b \rightarrow s; c \rightarrow r; d \rightarrow p$
- $a \rightarrow r; b \rightarrow p; c \rightarrow s; d \rightarrow q$
- (3)

Numerical Value Type

- | | | | | |
|---------|---------|----------|--------|---------|
| 1. (5) | 2. (5) | 3. (4) | 4. (9) | 5. (3) |
| 6. (7) | 7. (8) | 8. (2) | 9. (3) | 10. (6) |
| 11. (4) | 12. (2) | 13. (24) | | |

ARCHIVES

JEE MAIN

Single Correct Answer Type

- (2)
- (3)
- (2)

JEE ADVANCED

Single Correct Answer Type

- (2)

Multiple Correct Answers Type

- | | |
|------------------|------------------|
| 1. (1), (2) | 2. (2), (4) |
| 3. (1), (2) | 4. (1), (2), (4) |
| 5. (2), (3), (4) | |

Linked Comprehension Type

- (2)
- (1)

Matrix Match Type

- $d \rightarrow q, s$
- (4)
- (1)
- (4)
- (2)

Numerical Value Type

- (2)